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THE VARIETY OF RESIDUATED LATTICES IS GENERATED BY ITS FINITE SIMPLE MEMBERS

A b s t r a c t. We show that the variety of residuated lattices is generated by its finite simple members, improving upon a finite model property result of Okada and Terui. The reasoning is a blend of proof-theoretic and algebraic arguments.

1. Introduction

In this paper, we will show that the variety of residuated lattices is generated by *finite simple* residuated lattices. The "simplicity" part of the proof is based on Grišin's idea from [5], whereas the "finiteness" part employs a kind of algebraic filtration argument. Since the set of formulas valid in all residuated lattices is equal to the set of formulas provable in the propositional logic $\mathbf{FL_{ew}}$, the propositional logic obtained from the

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intuitionistic logic by deleting the *contraction rule* (see [8], for instance), our result can be restated as follows: for any formula A, A is provable in $\mathbf{FL_{ew}}$ if and only if it is valid in any finite simple residuated lattice.

Our result, on the one hand, strengthens the *finite model property* result for $\mathbf{FL_{ew}}$, from [6], and, on the other hand, makes a remarkable contrast with the variety of Heyting algebras, i.e., residuated lattices in which $x \leq x^2$ holds, as any simple Heyting algebra is a two-valued Boolean algebra. Also, it may be interesting to see how nicely proof-theoretic methods work to bring about a purely algebraic consequence, though the credit, of course, should go to Grišin.

Another consequence is of logical nature. Namely, we identify the limit of a certain sequence of logics without contraction, which was proposed as an open question in [8]. Let E_k and EM_k be axioms $p^k \supset p^{k+1}$ and $p \vee \neg p^k$, respectively, and let $\mathbf{FL_{ew}}[\mathrm{E}_k]$ (and $\mathbf{FL_{ew}}[\mathrm{EM}_k]$) be the logic obtained from $\mathbf{FL_{ew}}$ by adding the axiom E_k (and EM_k, respectively) to it. In [8], it was shown that the logic $\mathbf{FL_{ew}}[\mathrm{E}_k]$ (and $\mathbf{FL_{ew}}[\mathrm{EM}_k]$) is determined by the class of all residuated lattices (and simple residuated lattices, respectively) satisfying $x^k = x^{k+1}$. Both sequences of logics $\{\mathbf{FL_{ew}}[\mathrm{E}_k]\}_k$ and $\{\mathbf{FL_{ew}}[\mathrm{EM}_k]\}_k$ are monotone decreasing sequences of logics (as sets of formulas), each of which starts either from intuitionistic logic or from classical logic. A natural question is what do they converge to, i.e., what is the intersection of each. In [8], it was proved that the intersection of $\mathbf{FL_{ew}}[\mathrm{E}_k]$ for $k < \omega$ is $\mathbf{FL_{ew}}$. We will show in the present paper that the intersection of $\mathbf{FL_{ew}}[\mathrm{EM}_k]$ for $k < \omega$ is also equal to $\mathbf{FL_{ew}}$.

An algebra $\mathbf{M} = \langle M; \cap, \cup, \cdot, \rightarrow, 0, 1\rangle$ is called a *residuated lattice* if it satisfies the following:

1. $\langle M, \cap, \cup, 0, 1\rangle$ is a bounded lattice with the greatest element 1 and the least element 0,
2. $\langle M, \cdot, 1\rangle$ is a commutative monoid,
3. for $x, y \in M$, $\quad x \cdot y \leq z$ if and only if $x \leq y \rightarrow z$.

For general information on residuated lattices, and their connection with logics without contraction, we refer the reader to [8]. In the following, we will consider only *non-degenerate* residuated lattices, i.e., residuated lattices satisfying $0 \neq 1$. We define $\sim x$ by $\sim x = x \to 0$. Following [8], we say that a residuated lattice is *classical* when:

$$(DN) \qquad\qquad \sim\sim x = x, \text{ for any } x$$

holds in it.

A residuated lattice $\mathbf{M}$ is *simple* if it is non-degenerate and has only two (congruence) filters: $\{1\}$ and M itself. It is easy to see that for any filter F of a given residuated lattice $\mathbf{M}^*$ the quotient algebra $\mathbf{M}^*/F$ is simple if and only if F is a maximal filter. We can show the following lemma, where x^m denotes the m-fold product of x with itself.

Lemma 1.1. *A residuated lattice $\mathbf{M}$ is simple iff for any $x < 1$ in M there exists a positive integer m such that $x^m = 0$.*

A residuated lattice $\mathbf{M}$ is *semisimple* if $\mathbf{M}$ can be represented by a subdirect product of simple residuated lattices. Let Φ_M be the set of all maximal filters of $\mathbf{M}$. Define the *radical* Rad_M of $\mathbf{M}$ by $\mathrm{Rad}_M = \bigcap_{F \in \Phi_M} F$. Then, without difficulty we can show that a residuated lattice $\mathbf{M}$ is semisimple if and only if $\mathrm{Rad}_M = \{1\}$. In [5], Grišin proved the following.

Proposition 1.2. *Every free classical residuated lattice is semisimple.*

2. Simplicity

The bulk of the present paper (sections 2, 3 and 4) is devoted to proving the theorem below. As will be clear from what follows, the proof goes essentially the same way as that of Proposition 1.2.

Theorem 2.1. *Every free residuated lattice is semisimple.*

By our theorem, we have the following corollary immediately.

Corollary 2.2. *The variety of all residuated lattices is generated by its simple members.*

For any element x in a given residuated lattice and a positive integer m, $\tilde{m}x$ denotes $\sim(\sim x)^m$. Note that $\tilde{1}x = \sim\sim x$. Similarly to a result in [5], we can show the following. (See [8] for the proof.)

Theorem 2.3. *For any x in a given residuated lattice $\mathbf{M}$, $x \in \mathrm{Rad}_M$ if and only if for any $n \geq 1$ there exists $m \geq 1$ such that $\tilde{m}(x^n) = 1$.*

To prove our Theorem 2.1 by using Theorem 2.3, we will introduce the propositional logic $\mathbf{FL_{ew}}$, which is determined by the class of all residuated lattices. That is, for any formula A, A is provable in $\mathbf{FL_{ew}}$ if and only if it is valid in any residuated lattice. As shown in the next section, $\mathbf{FL_{ew}}$ is formalized as a sequent calculus, which is obtained from Gentzen's sequent calculus **LJ** for the intuitionistic logic by deleting the contraction rule. The language of $\mathbf{FL_{ew}}$ consists of a logical constant $\bot$, logical connectives $\supset, \wedge, \vee$ and $*$ (called *fusion*). They correspond to the constant 0 and the operations $\rightarrow, \cap, \cup, \cdot$ of residuated lattices, respectively. The negation $\neg A$ of a formula A is defined as an abbreviation of $A \supset \bot$.

Similarly to the corresponding algebraic expressions, we will define also A^n and $\tilde{m}B$ to be the formula $A * \cdots * A$ with n times A, and the formula $\neg(\neg B)^m$.

Now, since any free residuated lattice can be regarded as a Lindenbaum algebra of the logic $\mathbf{FL_{ew}}$ (with a certain set of propositional variables), to get Theorem 2.1 by the help of Theorem 2.3, it suffices to show the next lemma, whose proof will be presented in the next two sections.

Lemma 2.4. (main lemma) *If a formula A is not provable in $\mathbf{FL_{ew}}$, there exists a number $N(\geq 1)$ such that $\tilde{m}(A^N)$ is not provable in $\mathbf{FL_{ew}}$ for any $m \geq 1$.*

We close this section with the following immediate logical consequence of Corollary 2.2.

Corollary 2.5. *The logic* $\mathbf{FL_{ew}}$ *is determined by the class of all simple residuated lattices.*

3. Sequent calculi for $\mathbf{FL_{ew}}$

We will introduce first a sequent calculus $\mathbf{SFL_{ew}}$ for the logic $\mathbf{FL_{ew}}$, which is essentially the same system as one introduced in [8], but in a slightly modified form. We will both restrict and generalize the form of initial sequents, and also generalize the form of rules for fusion. This will allow us to remove the weakening rule. In our new rules for fusion, we will allow formulas of the form $A_1 * \cdots * A_m$ for any $m \geq 2$. The lack of parentheses in formulas of this form is harmless, since the associativity of $*$ is provable in $\mathbf{FL_{ew}}$.

Now, a *sequent* of $\mathbf{SFL_{ew}}$ is of the form $\Gamma \to B$, where Γ is a possibly empty *multiset* of formulas. (Note that the right-hand side of $\to$ must always exist.) As usual, when Γ is a multiset $\{A_1, \ldots, A_m\}$, the sequent $\Gamma \to B$ is also expressed as $A_1, \ldots, A_m \to B$. Also, the multiset union of multisets Γ and Δ (and of $\{A\}$ and Δ) is denoted by Γ, Δ (and A, Δ, respectively). In the following, both Γ and Δ denote arbitrary multisets of formulas. The system $\mathbf{SFL_{ew}}$ consists of the following initial sequents:

1. $p, \Gamma \to p$ for any propositional variable p
2. $\bot, \Gamma \to C$

and the following rules of inference:

Cut rule:

$$\frac{\Gamma \to A \quad A, \Delta \to C}{\Gamma, \Delta \to C}$$

Rules for logical connectives:

$$\frac{A, \Gamma \to B}{\Gamma \to A \supset B}\ (\to\supset) \qquad\qquad \frac{\Gamma \to A \quad B, \Delta \to C}{A \supset B, \Gamma, \Delta \to C}\ (\supset\to)$$

$$\frac{\Gamma \to A}{\Gamma \to A \vee B}\ (\to \vee 1) \qquad \frac{\Gamma \to B}{\Gamma \to A \vee B}\ (\to \vee 2)$$

$$\frac{A, \Gamma \to C \quad B, \Gamma \to C}{A \vee B, \Gamma \to C}\ (\vee \to)$$

$$\frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \wedge B}\ (\to \wedge)$$

$$\frac{A, \Gamma \to C}{A \wedge B, \Gamma \to C}\ (\wedge 1 \to) \qquad \frac{B, \Gamma \to C}{A \wedge B, \Gamma \to C}\ (\wedge 2 \to)$$

$$\frac{\Gamma_1 \to A_1 \quad \cdots \quad \Gamma_m \to A_m}{\Gamma_1, \ldots, \Gamma_m \to A_1 * \cdots * A_m}\ (\to *) \qquad \frac{A_1, \ldots, A_m, \Gamma \to C}{A_1 * \cdots * A_m, \Gamma \to C}\ (* \to)$$

In each rule I, the formula introduced by it is called the *principal* formula of I, and formulas in upper sequent(s) of I which become components of the principal formula are called the *auxiliary* formulas. Other formulas, i.e. the formula C and formulas in Γ, Γ_i, Δ, are called the *side* formulas.

As usual, we say that a formula A is provable in $\mathbf{SFL_{ew}}$ if the sequent $\to A$ is provable in it. Since the cut elimination theorem holds for the calculus $\mathbf{SFL_{ew}}$ (the proof from [9] works, with obvious modifications), the cut rule is redundant. Next, we will introduce another sequent calculus $\mathbf{SFL_{ew}}^+$, which is shown to be equivalent to $\mathbf{SFL_{ew}}$. Here, we say that A is a $*$-*formula* if the outermost logical connective of A is $*$. Now, $\mathbf{SFL_{ew}}^+$ is the system obtained from $\mathbf{SFL_{ew}}$ by deleting the cut rule and by adding the following condition (+) in any application of the rules $(\to *)$ and $(* \to)$:

$$\textit{none of } A_i s \textit{ are } *\textit{-formulas.} \qquad (+)$$

We note here that when we show the equivalence of $\mathbf{SFL_{ew}}^+$ to $\mathbf{SFL_{ew}}$ in the following, we will identify each formula of the form $A_1 * \cdots * A_m$ of $\mathbf{SFL_{ew}}^+$ with any formula of $\mathbf{SFL_{ew}}$, which is obtained from $A_1 * \cdots * A_m$ by introducing proper bracketing in it. This doesn't cause any confusion. The rest of this section is devoted to the proof of the following lemma.

Lemma 3.1. *For any sequent $\Gamma \to D$, $\Gamma \to D$ is provable in* $\mathbf{SFL_{ew}}$ *if and only if it is provable in* $\mathbf{SFL_{ew}}^+$.

Proof. It is enough to show that if a sequent $\Gamma \to D$ is provable in $\mathbf{SFL_{ew}}$ then it is provable in $\mathbf{SFL_{ew}}^+$. In other words, it suffices to show that any application of $(\to *)$ and $(* \to)$ not satisfying the condition $(+)$ in a given cut-free proof P of $\Gamma \to D$ can be replaced by one with $(+)$. Let B be any formula of the form $B_1 * \cdots * B_k$ $(k > 0)$, where none of B_js are $*$-formulas. Then we define the *degree* $d(B)$ of B by $d(B) = k - 1$, i.e. the number of outermost $*$ in B. Next, when I is either an application of $(* \to)$ of the form

$$\frac{A_1, \ldots, A_m, \Gamma \to C}{A_1 * \cdots * A_m, \Gamma \to C}$$

or an application of $(\to *)$ of the form

$$\frac{\Gamma_1 \to A_1 \quad \cdots \quad \Gamma_m \to A_m}{\Gamma_1, \ldots, \Gamma_m \to A_1 * \cdots * A_m}$$

we define the *degree* $d(I)$ of I by $d(I) = d(A_1) + \cdots + d(A_m)$. It is obvious that an application I of either $(* \to)$ or $(\to *)$ satisfies the condition $(+)$ if and only if $d(I) = 0$.

Assume first that there exists an application of $(* \to)$ not satisfying the condition $(+)$ in P. Let us take one of the uppermost applications among them, which we call J. Obviously, $d(J) > 0$. We suppose that the lower sequent of J is $A_1 * \cdots * A_m, \Gamma \to C$ with the principal formula $A_1 * \cdots * A_m$. We will show by induction on the degree $d(J)$ that $A_1 * \cdots * A_m, \Gamma \to C$ has a cut-free proof in which every application of $(* \to)$ satisfies $(+)$.

By our assumption, the degree of one of A_is must be nonzero. Without loss of generality, we can assume that A_1 is of the form $D_1 * \cdots * D_s$ for $s > 1$. Let Q be the proof of $A_1 * \cdots * A_m, \Gamma \to C$ which is a subproof of P. We will trace back ancestors of the auxiliary formula A_1 of J in all branches of Q. Then, we can see that in every sequent in these branches, A_1 is introduced either as the principal formula of an application of $(* \to)$ (with $(+)$), or as a side formula of an initial sequent. Now, we replace any ancestor A_1 by the multiset $D_1, \ldots, D_s$. If one of such A_1 is introduced by an application of $(* \rightarrow)$ then its lower sequent becomes identical with the upper sequent by this replacement. In such a case, we eliminate this $(* \rightarrow)$. On the other hand, if it is introduced as a side formula of an initial sequent, the sequent obtained by this replacement remains still an initial sequent. Hence, the figure Q' obtained from Q by this replacement remains a correct proof of $A_1 * \cdots * A_m, \Gamma \rightarrow C$ whose last inference is an application J' of $(* \rightarrow)$, with the upper sequent $D_1, \ldots, D_s, A_2, \ldots, A_n, \Gamma \rightarrow C$. Since $d(J') = d(J) - d(A_1) < d(J)$, by the hypothesis of induction, $A_1 * \cdots * A_n, \Gamma \rightarrow C$ has a cut-free proof in which every application of $(* \rightarrow)$ satisfies $(+)$. In this way, we have a cut-free proof P$'$ of $\Gamma \rightarrow D$ where every application of $(* \rightarrow)$ satisfies the condition $(+)$.

For example, consider the following (sub)proof whose last inference is an application of $(* \rightarrow)$ without the condition $(+)$.

$$\dfrac{\dfrac{r * s, p, p \supset (r \supset q) \rightarrow p \qquad \dfrac{r, s, p, p \supset (r \supset q) \rightarrow q}{r * s, p, p \supset (r \supset q) \rightarrow q}}{r * s, p, p \supset (r \supset q) \rightarrow p \wedge q}}{r * s * p, p \supset (r \supset q) \rightarrow p \wedge q}$$

The replacement mentioned above will change the above proof into the following one.

$$\dfrac{\dfrac{r, s, p, p \supset (r \supset q) \rightarrow p \qquad r, s, p, p \supset (r \supset q) \rightarrow q}{r, s, p, p \supset (r \supset q) \rightarrow p \wedge q}}{r * s * p, p \supset (r \supset q) \rightarrow p \wedge q}$$

Next, we will remove any application of $(\rightarrow *)$ not satisfying the condition $(+)$ in P'. Suppose that there exists such an application. Similarly to the above, take one of the uppermost applications of $(\rightarrow *)$ not satisfying $(+)$, called J, which is of the form given above. We will show by induction on $d(J)$ that the lower sequent $\Gamma_1, \ldots, \Gamma_m \rightarrow A_1 * \cdots * A_m$ has a cut-free proof in which every application of $(\rightarrow *)$ satisfies $(+)$. Without loss of generality, we can assume that $d(A_1) = k > 0$ and that A_1 is of the form $D_1 * \cdots * D_k$, where none of D_js are $*$-formulas. Let R be the proof of

$\Gamma_1 \to A_1$, which is a subproof of P'. We will trace back the branches in R which consist of sequents having A_1 in the conclusion (such an A_1 must clearly be an ancestor of the A_1 from $\Gamma_1 \to A_1$) to the places where this A_1 is introduced. There are two possibilities. It is introduced either as an initial sequent of the form: $\bot, \Delta \to A_1$, or as the principal formula of an application of $(\to *)$ of the form:

$$\frac{\Delta_1 \to D_1 \quad \cdots \quad \Delta_k \to D_k}{\Delta_1, \ldots, \Delta_k \to A_1}$$

We will modify the proof R as follows. If the first case happens, we replace the above sequent by $\bot, \Delta, \Gamma_2, \ldots, \Gamma_m \to A_1 * \cdots * A_m$, which is still an initial sequent. For the second case, we replace it by:

$$\frac{\Delta_1 \to D_1 \quad \cdots \quad \Delta_k \to D_k \quad \Gamma_2 \to A_2 \quad \cdots \quad \Gamma_m \to A_m}{\Delta_1, \ldots, \Delta_k, \Gamma_2, \ldots, \Gamma_m \to A_1 * A_2 * \cdots * A_m}$$

and put the subproof of each $\Gamma_i \to A_i$ in R over it for each $i = 2, \ldots, m$. Note that the degree of the above application of $(\to *)$ is $d(J) - d(A_1)$, which is smaller than $d(J)$. Therefore, by the hypothesis of induction the lower sequent of this inference has a cut-free proof in which every application of $(\to *)$ satisfies $(+)$. Finally, we replace every sequent $\Sigma \to A_1$ in a branch which we have traced, by the sequent $\Sigma, \Gamma_2, \ldots, \Gamma_m \to A_1 * \cdots * A_m$. Then, after this replacement, we get a proof R' of $\Gamma_1, \Gamma_2, \ldots, \Gamma_m \to A_1 * \cdots * A_m$ in which every application of $(\to *)$ satisfies $(+)$. (Note that when $(\supset\to)$ is used somewhere in these branches, this replacement will be done only for the right upper sequent. Thus, this replacement never cause unnecessary duplications of $\Gamma_2, \ldots, \Gamma_m$.) By repeating this, we have a proof of $\Gamma \to D$ in $\mathbf{SFL_{ew}}^+$.

4. Proof of main lemma

We will give a proof of our main lemma, Lemma 2.4, in this section. The proof is obtained by modifying the proof given by Grišin slightly. For each formula A, $l(A)$ denotes the length of A (as a sequence of symbols).

For a sequence Γ of formulas $A_1, \dots, A_m$, the length $l(\Gamma)$ of Γ is defined by $l(\Gamma) = l(A_1) + \cdots + l(A_m)$. We will prove the following stronger form of Lemma 2.4. In the following, we will express the multiset $\neg A^N, \dots, \neg A^N$ with m times $\neg A^N$ as $\{\neg A^N\}^m$.

Lemma 4.1. *Suppose that a formula A is not provable in $\mathbf{SFL_{ew}}^+$ and that N is any positive integer greater than $l(A)$. Then, for any sequent $\Gamma \to C$ such that $l(\Gamma, C) \leq l(A)$ and any positive integer m, if $\{\neg A^N\}^m, \Gamma \to C$ is provable in $\mathbf{SFL_{ew}}^+$ then $\Gamma \to C$ is provable in $\mathbf{SFL_{ew}}^+$.*

Since $\to \bot$ is not provable in $\mathbf{SFL_{ew}}^+$, Lemma 2.4 follows immediately from Lemma 4.1 by taking the sequent $\to \bot$ for $\Gamma \to C$. We will give a proof of Lemma 4.1 in the rest of this section.

Proof of Lemma 4.1. The proof will proceed by double induction on $(m, l(\Gamma, C))$. So, we assume that our Lemma holds for $m' < m$ and it also holds for $(m, l(\Delta, D))$, whenever $l(\Delta, D) < l(\Gamma, C)$. We suppose that A is not provable but $\{\neg A^N\}^m, \Gamma \to C$ is provable in $\mathbf{SFL_{ew}}^+$. Suppose first that $\{\neg A^N\}^m, \Gamma \to C$ is an initial sequent. Then, either C is a propositional variable which occurs also in Γ, or $\bot$ occurs in Γ. It is clear that $\Gamma \to C$ is provable in either case.

Next, suppose that the sequent $\{\neg A^N\}^m, \Gamma \to C$ is the lower sequent of an inference rule I. We first assume that the principal formula of I is either in Γ or in C. Then, (each of) upper sequent(s) of I is of the form $\{\neg A^N\}^{m_i}, \Delta_i \to D_i$ with $m_i \leq m$ and $l(\Delta_i, D_i) < l(\Gamma, C)$ (by the *subformula property* and the fact that $\mathbf{SFL_{ew}}^+$ has no contraction rule). Thus, by the hypothesis of induction, (each) $\Delta_i \to D_i$ is provable. Then $\Gamma \to C$ is also provable by applying the same inference rule I.

Finally suppose that the principal formula of I is one of $\neg A^N$. Then, I must be an application of ($\supset\to$). Recall here that a formula $\neg B$ is the abbreviation of $B \supset \bot$. The upper sequents must be of the form $\{\neg A^N\}^{m_1}, \Gamma_1 \to A^N$ and $\bot, \{\neg A^N\}^{m_2}, \Gamma_2 \to C$ such that $m_1 + m_2 = m - 1$ and Γ_1, Γ_2 is equal to Γ. Now consider the proof R of the left upper sequent

$\{\neg A^N\}^{m_1}, \Gamma_1 \to A^N$. As we did in the proof of Lemma 3.1, we trace back branches in R, which consists of sequents having A^N in the conclusion, to the places where these A^N are introduced. It is easy to see that each A^N is introduced either as an initial sequent of the form $\bot, \Delta \to A^N$, or by an application of $(\to *)$. Suppose that in at least one place, A^N is introduced by an application J of $(\to *)$, whose lower sequent is of the form $\{\neg A^N\}^k, \Sigma \to A^N$. Clearly, $k \leq m_1$. We assume here that A is of the form $D_1 * \cdots * D_w$ such that none of D_j are $*$-formulas. (Only for the simplicity's sake, we assume in the following that $D_1, \ldots, D_w$ are mutually distinct.) Then, I must have $N \cdot w$ upper sequents, each of which is of the form $\{\neg A^N\}^{t_i}, \Xi_i \to D_{n_i}$, where $1 \leq n_i \leq w$, $k = t_1 + \cdots + t_{N \cdot w}$ and the multiset $\Xi_1, \ldots, \Xi_{N \cdot w}$ is equal to Σ. For each j such that $1 \leq j \leq w$, there exist exactly N sequents with the conclusion D_j among these sequents. We enumerate them as ${S^j}_1, \ldots, {S^j}_N$. Next, for each h such that $1 \leq h \leq N$, take ${S^1}_h, \ldots, {S^w}_h$ for upper sequents and apply $(\to *)$ to them. Then, we can get a sequent of the form $\{\neg A^N\}^{u_h}, \Pi_h \to A$ for $1 \leq h \leq N$ such that $k = u_1 + \cdots + u_N$ and the multiset $\Pi_1, \ldots, \Pi_N$ is equal to Σ. Now, $l(\Sigma) \leq l(\Gamma_1) \leq l(\Gamma, C) \leq l(A) < N$. If $l(\Pi_h) > 0$ for any h such that $1 \leq h \leq N$ then $l(\Sigma) \geq N$, which is a contradiction. Thus, Π_h must be empty for some h. Let it be f. Then, $\{\neg A^N\}^{u_f} \to A$ is provable. By our assumption that A is not provable, u_f must be positive. Then, since $u_f \leq m_1 \leq m - 1 < m$ and $l(A) \leq l(A)$, $\to A$ must be provable by the hypothesis of induction. This is a contradiction.

Thus, we have shown that in any place A^N is introduced as an initial sequent of the form $\bot, \Delta \to A^N$. We will modify the proof R of $\{\neg A^N\}^{m_1}, \Gamma_1 \to A^N$ as follows. We replace every sequent $\Lambda \to A^N$ in a branch which we have traced in R, including initial sequents of the form $\bot, \Delta \to A^N$ mentioned above, by the sequent $\Lambda, \Gamma_2 \to C$. Then we will get a proof R* whose end sequent is $\{\neg A^N\}^{m_1}, \Gamma \to C$. Note that $m_1 \leq m - 1 < m$. Hence, by the hypothesis of induction, $\Gamma \to C$ is provable. This completes the proof.

5. Embeddings

In this section we show that the class $\mathcal{R}_S$ of simple residuated lattices enjoys the finite embeddability property, which we are now going to define (cf. e.g., [4], for more details).

Let $\mathcal{K}$ be a class of algebras and $\mathbf{P}$ be a partial subalgebra of an algebra $\mathbf{B}$ from $\mathcal{K}$. We say that $\mathbf{B}$ has the *finite embeddability property* in $\mathcal{K}$ iff any finite partial subalgebra of $\mathbf{B}$ can be embedded into a finite algebra $\mathbf{C}$ from $\mathcal{K}$. The class $\mathcal{K}$ has the *finite embeddability property* iff every algebra $\mathbf{B}$ from $\mathcal{K}$ has the finite embeddability property in $\mathcal{K}$.

Thus, our task in this section is to produce for any $\mathbf{M} \in \mathcal{R}_S$, and any finite partial $\mathbf{P} \subseteq \mathbf{M}$, a finite $\mathbf{A} \in \mathcal{R}_S$, into which $\mathbf{P}$ can be embedded.

Let $\mathbf{M}$ and $\mathbf{P}$ be as above. Our construction of a finite algebra into which $\mathbf{P}$ will be embeddable is to be carried out in two stages.

Firstly, observe that since $\mathbf{M}$ is simple, for any $x \in M$ if $x < 1$ there exists a positive integer m such that $x^m = 0$. We will show that we can always embed $\mathbf{P}$ into an $\mathbf{M}$ for which the m above is uniform, i.e., there exists a positive integer m that $x^m = 0$ for any $x < 1$. This is equivalent to saying that $\mathbf{M}$ can always be so chosen that it satisfies EM_m, for some m.

For any residuated lattice $\mathbf{M}$, let $\mathbf{M}^-$ stand for its $\{\cap, \cup, \cdot\}$-reduct. For an element $a \in M \setminus \{1\}$, $\mathbf{M}^-_a$ denotes the algebra $\langle (a] \cup \{1\}; \cap, \cup, \cdot, 0, 1\rangle$, where $(a]$ denotes the set $\{x : x \leq a\}$. Clearly, $\mathbf{M}^-_a$ is a subalgebra of $\mathbf{M}^-$.

Lemma 5.1. *For any $a, b, c \in M$, the set $U = \{x \leq a : x \cdot b \leq c\}$ has the greatest element equal to $a \cap (b \to c)$.*

Proof. We have both $a \cap (b \to c) \leq a$ and $(a \cap (b \to c)) \cdot b \leq c$, clearly. Now suppose that $z \leq a$ and $z \cdot b \leq c$ hold. Then, $z \leq a \cap (b \to c)$. Thus we have our lemma.

Let $\mathbf{M}$ be any simple residuated lattice, and $a \in M \setminus \{1\}$. We will define a binary operation $\to^*$ on $\mathbf{M}_a^-$, by putting:

$$x \to^* y = \begin{cases} a \cap (x \to y), & \text{if } x \not\leq y, \\ 1, & \text{otherwise.} \end{cases}$$

By Lemma 5.1, the operation $\to^*$ is well-defined. Let us call the resulting algebra $\mathbf{M}_a^+$. In what follows, we will refer to the universe of either $\mathbf{M}_a^-$ or $\mathbf{M}_a^+$, by M_a. This is unambiguous, as they share the same universe.

Lemma 5.2. *$\mathbf{M}_a^+$ is a simple residuated lattice. Moreover,* EM_k *is valid in $\mathbf{M}_a^+$ for some k.*

Proof. To show that $\mathbf{M}_a^+$ is a residuated lattice, we only have to check whether, for any $x, y \in M_a$, the element $x \to^* y$ is the largest among all $z \in M_a$ with $z \cdot x \leq y$. This was already assured by Lemma 5.1.

To show that $\mathbf{M}_a^+$ is simple, take any element $x \in M_a$ such that $x < 1$. Then, $x \leq a$. As $\mathbf{M}$ is simple, we have that for some $k < \omega$, $a^k = 0$ in $\mathbf{M}$. Thus, $x^k \leq a^k = 0$. Since the fusion in $\mathbf{M}_a^+$ coincides with the fusion in $\mathbf{M}$, $x^k = 0$ also in $\mathbf{M}_a^+$. Hence $\mathbf{M}_a^+$ is simple and EM_k is valid in $\mathbf{M}_a^+$.

Now, let $\mathbf{P}$ be the original partial algebra, and $\mathbf{M}$ the algebra into which $\mathbf{P}$ is embedded. We have:

Lemma 5.3. *There exists an algebra $\mathbf{N} \in \mathcal{R}_S$ such that $\mathbf{N} \supseteq \mathbf{P}$ and $\mathbf{N}$ satisfies* EM_k*, for some positive integer k.*

Proof. Define P_0 to be $P \setminus \{1\}$. As P_0 is finite, the join $\bigcup P_0$ is well-defined in $\mathbf{M}$, although not necessarily in $\mathbf{P}$. Since $\mathbf{M}$ is simple (hence subdirectly irreducible), $\bigcup P_0 < 1$ in $\mathbf{M}$, for otherwise, by Theorem 4.2 in [8], for some $p \in P_0$, $p = 1$, which is a contradiction.

Thus, the element $c = \bigcup P_0$ is different from 1. Now, consider $\mathbf{M}_c^+$ defined as above. By Lemma 5.2, EM_k is valid in $\mathbf{M}_c^+$ for some k. Observe also that, for any $a, b \in P$, and $\star \in \{\cup, \cap, \cdot, \to\}$, if $a \star b$ is defined in P, then $a \star b$ in $\mathbf{P}$ is equal to $a \star b$ in $\mathbf{M}_c^+$. This is trivial for all operations except $\to$. However, if $a \to b$ is defined, then $a \to b$ belongs to P, and therefore it is either equal to 1 or smaller than c in $\mathbf{M}$. Thus, it remains unaltered in

$\mathbf{M}_c^+$. Therefore, $\mathbf{M}_c^+$ may be taken as the algebra $\mathbf{N}$ whose existence was required by the lemma.

Before entering the second stage of our construction, we will define an auxiliary notion. Let us call the algebra $\mathbf{W} = \langle W; \cup, \cdot, 0, 1\rangle$ a *bounded, commutative, semilattice-ordered monoid* (or, for short, a *bocsoid,* although we do not particularly like this acronym) iff:

1. $\langle W; \cup, 0, 1\rangle$ is a join-semilattice with the greatest element 1 and the smallest element 0,
2. $\langle W; \cdot, 1\rangle$ is a commutative monoid satisfying: $x \cdot (y \cup z) = (x \cdot y) \cup (x \cdot z)$.

Lemma 5.4. *Any finite bocsoid is a reduct of a residuated lattice.*

Proof. Let $\mathbf{W}$ be a finite bocsoid. Define the following two operations on its universe W:

- $x \cap y = \bigcup\{z \mid z \leq x \ \&\ z \leq y\}$,
- $x \to y = \bigcup\{w \mid w \cdot x \leq y\}$.

Under these definitions $\langle W; \cup, \cap, \cdot, \to, 0, 1\rangle$ becomes a residuated lattice, as it is straightforward to check. Note that if $\cdot$ is idempotent, then we get the well-known fact that every finite distributive lattice is a Heyting algebra.

Let us call a bocsoid $\mathbf{W}$ *k-potent* iff there is a positive integer k such that $\mathbf{W}$ satisfies $x^{k+1} = x^k$.

Lemma 5.5. *All k-potent bocsoids are locally finite.*

Proof. Let $\mathbf{W}$ be a k-potent bocsoid, and Z be a finite subset of W. By distributivity of $\cdot$ over $\cup$ each element generated by Z is of the form $\bigcup_{i=0}^{j-1} z_i$, for some positive integer j, where each z_i is either 1, or 0, or a fusion of some elements from Z.

As fusion is associative and commutative, we can dispense with parentheses and write each z_i as $a_0^{n_0} \dots a_{l-1}^{n_{l-1}}$, with $a_0, \dots, a_{l-1} \in Z$. Since $x^{k+1} = x^k$ holds, for a certain fixed k, the exponents in this expression are bounded from above by k, i.e., if $n_j > k$, then $a_0^{n_0} \dots a_j^{n_j} \dots a_{l-1}^{n_{l-1}} =$

$a_0^{n_0} \ldots a_j^k \ldots a_{l-1}^{n_{l-1}}$. It follows, that an $X \subseteq W$, with $X = \{a_0, \ldots, a_{l-1}\}$, can generate, by means of fusion alone, at most $(k+1)^l + 1$ distinct elements. Thus, as there are only finitely many subsets of Z, the closure of Z under fusion is finite. As semilattices are locally finite, the closure of the latter under join is finite as well. This proves the lemma.

Theorem 5.6. *The class $\mathcal{R}_S$ of simple residuated lattices has the finite embeddability property.*

Proof. We have to show that every finite partial algebra $\mathbf{P}$ embeddable in $\mathcal{R}_S$ can be embedded into a finite algebra from $\mathcal{R}_S$. ¿From Lemma 5.3 we know that there is a simple residuated lattice $\mathcal{N}$ satisfying $x^{k+1} = x^k$, for some positive integer k, and such that $\mathbf{N} \supseteq \mathbf{P}$. In fact we can take $\mathbf{N} = \mathbf{M}_\mathbf{c}^+$, where the latter has been defined in the proof of Lemma 5.3. Let also P_0 and c be the same as in that proof.

Let $\mathbf{N}^-$ be the $\{\cup, \cdot, 0, 1\}$-reduct of $\mathbf{N}$. Thus, $\mathbf{N}^-$ is a bocsoid. Let $\mathbf{W}$ be the sub-bocsoid of $\mathbf{N}^-$ generated by P. By the properties of $\mathbf{N}$, the bocsoid $\mathbf{N}^-$ is k-potent, indeed, it satisfies:

$$x^k = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases} \tag{simp}$$

because, by Lemma 5.3, so does $\mathbf{N}$. As P is finite, it follows, by Lemma 5.5, that so is $\mathbf{W}$. Now, let $\mathbf{S}$ be the residuated lattice resulting from endowing $\mathbf{W}$ with the operations defined in the proof of Lemma 5.4. Clearly, $\mathbf{S}$ is a finite residuated lattice; since $\mathbf{W}$ satisfies (simp), $\mathbf{S}$ is simple.

All that remains is to show that $\mathbf{P}$ is indeed a partial subalgebra of $\mathbf{S}$. There are two operations, for which something could go wrong, the ones that have been defined anew: $\cap$ and $\rightarrow$. However, if $x, y \in P$ and $x \cap y$, or $x \rightarrow y$, is defined in P, then, it is not difficult to verify from the definitions that the original operations coincide with the new ones. This finishes the proof.

Let $\mathcal{E}_k$ stand for the variety of k-potent residuated lattices, defined relative to $\mathcal{R}$ by the identity: $x^k = x^{k+1}$. Thus, $\mathcal{E}_k$ corresponds to the logic $\mathbf{FL_{ew}}[\mathrm{E}_k]$. Analogously, let $\mathcal{EM}_k$ stand for the variety of k-potent residuated lattices, defined relative to $\mathcal{R}$ by the identity: $x \vee \neg x^k = 1$; this corresponds to $\mathbf{FL_{ew}}[\mathrm{EM}_k]$.

Theorem 5.7. *Both $\mathcal{E}_k$ and $\mathcal{EM}_k$ have the finite embeddability property.*

Proof. Let $\mathbf{P}$ be a partial subalgebra of an algebra $\mathbf{N}$ from $\mathcal{E}_k$ ($\mathcal{EM}_k$). Let $\mathbf{N}^-$ and $\mathbf{W}$ be as in the proof of Theorem 5.6. Then, $\mathbf{W}$ is a finite bocsoid satisfying $x^k = x^{k+1}$ ($x \vee \neg x^k = 1$), and the rest of the proof applies without any changes.

6. Conclusions and other remarks

Let us start with proving what we have announced in the title, namely:

Theorem 6.1. *The variety $\mathcal{R}$ of residuated lattices is generated by its finite simple members.*

Proof. It suffices to show that for any non-theorem A of $\mathbf{FL_{ew}}$ there is a finite simple residuated lattice $\mathbf{S}$ falsifying A. By Theorem 2.1, there exist a simple residuated lattice $\mathbf{M}$, and a valuation v on M, with $v(A) < 1$. Let Σ be the set of all subformulas of A. Take $P = v(\Sigma) \cup \{0, 1\}$, and define a partial algebra $\mathbf{P}$ putting, for $\star \in \{\cap, \cup, \rightarrow, \cdot\}$, $v(B) \star v(C) = v(B \star C)$, if $B \star C$ belongs to Σ, and leaving it undefined otherwise[1]. Thus defined, $\mathbf{P}$ is a finite partial algebra from $\mathcal{R}_S$, and $\mathbf{P} \subseteq \mathbf{M}$. The construction from Section 5 yields a finite simple residuated lattice $\mathbf{S}$, into which $\mathbf{P}$ can also be embedded. For a variable p, let us put $w(p) = v(p)$, if p occurs in A, and be arbitrary otherwise. Extending w to a valuation, and, as usual, retaining the same symbol for it, we obtain, $w(A) < 1$ in $\mathbf{S}$, which ends the proof.

As mentioned already in Corollary 2.5, from Theorem 2.1 we can derive that $\mathbf{FL_{ew}}$ is determined by the class of all simple residuated lattices. It is shown in [8] that the logic $\mathbf{FL_{ew}}[\mathrm{EM}_k]$, which is obtained from $\mathbf{FL_{ew}}$

[1] Strictly speaking, we use $\star$ somewhat ambiguously here, as it stands for an algebraic operation in one context and for a logical connective in the other. However, as this ambiguity causes no harm (logical formulae are elements of the absolutely free algebra of the appropriate type anyway) we rely on the reader to sort it out.

by adding the axiom EM_k: $p \vee \neg p^k$, is determined by the class of all simple residuated lattices satisfying $x^k = x^{k+1}$. The sequence of logics $\{\mathbf{FL_{ew}}[\mathrm{EM}_k]\}_k$ is a monotone decreasing sequence of logics, whose first member is classical logic. It is interesting to see to which logic this sequence will converge, more precisely, what is the intersection of them. As a corollary of Theorem 6.1, we have:

Theorem 6.2. *The intersection of the logics* $\mathbf{FL_{ew}}[\mathrm{EM}_k]$ *for* $k < \omega$ *is equal to* $\mathbf{FL_{ew}}$.

Proof. We only have to show that any non-theorem can be falsified on an algebra that satisfies:

$$x^k = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases}$$

for some positive integer k. The algebra $\mathbf{S}$ from the previous proof is such.

The argument employed in Section 5 bears certain resemblance to the filtration method best known in the context of semantic structures for modal logics. Note that similar techniques were used in [7] and [3] to show finite model property of some intuitionistic modal logics.

It should also be obvious that the technique can be employed to show that some subvarieties of $\mathcal{R}$ are generated by their finite members. For instance it follows from Theorem 5.7 that the varieties associated with logics $\mathbf{FL_{ew}}[\mathrm{EM}_k]$, for any positive k, are generated by their finite simple members, and the varieties associated with logics $\mathbf{FL_{ew}}[\mathrm{E}_k]$, for any positive k, are generated by their finite (not necessarily simple) members.

We finish off with two questions. The first concerns possible analogues of our result for other varieties of residuated lattices. Although, as the situation within the variety of Heyting algebras suggests, any thorough characterisation of subvarieties of $\mathcal{R}$ generated by their finite members is rather elusive, we would like to ask:

Question 6.3. *Which (well-known) subvarieties of $\mathcal{R}$ are generated by their finite/simple/finite and simple members?*

We do not even know whether the variety of classical residuated lattices is generated by its finite simple members, although this seems a plausible conjecture, as it is generated both by its finite members and by its simple members. Other natural candidates to consider are: linear residuated lattices, distributive residuated lattices.

Among residuated structures a prominent place occupy *partially ordered commutative residuated integral monoids,* commonly referred to as *pocrims* (cf. [2] for more about pocrims). The class $\mathcal{P}_k$ of k-potent (i.e., satisfying $x^{k+1} = x^k$; some authors prefer to call these $k{+}1$-potent) pocrims is a variety, for each positive integer k. After [1], we ask:

Question 6.4. *Does $\mathcal{P}_k$ have the finite embeddability property for $k > 1$?*

For $k = 1$ the answer is affirmative and easy. For it can easily be checked that the technique from [9] embeds any idempotent pocrim into an idempotent residuated lattice. This actually only restates the well-known fact that idempotent pocrims are precisely Brouwerian semilattices. Thus, if $\mathbf{P}$ is a finite partial subalgebra of an algebra $\mathbf{A} \in \mathcal{P}_1$, then it is also a partial subalgebra of an algebra $\mathbf{B}$ from $\mathcal{E}_1$. Thus, by Theorem 5.7, $\mathbf{P}$ is finitely embeddable in $\mathcal{E}_1$; hence, by taking the appropriate reduct, also in $\mathcal{P}_1$. This simple reasoning, however, fails for $k > 1$, for whereas it remains true that any k-potent pocrim can be embedded into a residuated lattice, the latter may in general fail to be k-potent.

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