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A semantical study of orthologics

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Abstract

In this thesis, we investigate a type of non-classical logics from a semantical point of view. We deal with non-classical propositional logics around orthologics, and the minimum predicate extensions of some of them.

Every propositional logic we consider here is introduced so as to be characterized by a variety of algebras, and so there exists a variety of algebras that corresponds to each propositional logic. Therefore, some properties of a variety of algebras reflects on some properties of its corresponding propositional logic.

One of the topics we focus on in this thesis is the admissibility of completion of a class of algebras. There are mainly two ways of embedding an algebra into a complete one, that is, the Dedekind-MacNeille completion technique and the method via dual space construction. The first can be used when we consider an algebraic semantics of the minimum predicate extension of propositional logics and we establish the completeness theorem with respect to that semantics, because the embedding map preserves all existing joins and meets in this technique. This argument goes through for a few members of our group of logics. Indeed, we can show that their corresponding varieties of algebras admit completion by Dedekind-MacNeille completion technique, which also enables us to discuss the minimum predicate extensions of these logics. On the other hand, the second method is employed when we construct a relational semantics of a propositional logic. In fact, for some of our logics, we will build their relational semantics by using this completion method, and we will show the completeness with respect to these semantics.

Furthermore, by modifying the above completion techniques a little bit, we can prove that some members of our varieties have an algebraic property, which implies that the propositional logics that correspond to them and their minimum predicate extensions have the Craig's interpolation property.

The other topic in this thesis is the construction of a semantics of orthomodular logics. Since we can not take suitable dual spaces for orthomodular lattices, there does not exist a set-theoretic representation theorem for them which is convenient for semantics of orthomodular logics. Here we use a different representation theorem for orthomodular lattices, which is not set-theoretic, to construct Kripke-style semantics for orthomodular logics. This semantics consists of a non-empty set with some operations, instead of some relations. We show that any orthomodular logic is complete with respect to a semantics of this kind, and moreover, we discuss the infinitary extension of orthomodular logics by using this semantics.

Contents

1 Introduction

In this thesis, we investigate a type of non-classical logics which are weaker than the classical logic, from an algebraic point of view. Here *weaker* means that some of the logical laws in the classical logic are missing in logics we consider. One of the most important laws our logics do not have is the *distributive law*. The lack of the distributive law leads to some significant difficulties in the usual methods of analyzing algebraic logics. In particular, the distributive law is needed for taking *dual spaces* of original algebras, which are quite useful and tractable in analyzing the intuitionistic logic and modal logics. Because of these problems, this area of non-classical logics remains underdeveloped.

There are a few kinds of non-classical logics which do not have the distributive law. Among them, there are orthologics, orthomodular logics and their neighbors. These we will deal with here. The origin of the study of this area of logics is the mathematical formulation of quantum mechanics and the discovery of algebraic structures, called *orthomodular lattices*. We will begin this introduction with a story about the relation between quantum mechanics and orthomodular lattices according to [43].

1.1 Quantum mechanics and orthomodular lattices

The mathematical foundation of quantum mechanics was laid by J. von Neumann in 1932. In his book "*Die mathematische Grundlagen der Quantenmechanik*" [47], von Neumann formulated physical concepts by means of the theory of Hilbert space.

A *Hilbert space* H on the complex field $\mathbb{C}$ is a vector space over $\mathbb{C}$ which is equipped with an *inner product* $\langle\ ,\ \rangle$, and it is *complete* with respect to the *norm* $\|\cdot\|$ that is defined from the inner product. Here, *complete* means that every Cauchy sequence in H converges in H.

In his book, von Neumann postulated that the basic notions in quantum mechanics are *states* and *observables* . A state of a physical system corresponds to the complete knowledge of the physical system, from which we can make predictions about its development in the future. In his formulation, a state of a quantum mechanical system is mathematically described by a unit vector ψ in a Hilbert space H. This is inspired by Max Born's *probabilistic interpretation* of wave functions, that is, "the wave function ψ in Schrödinger's wave mechanics can be interpreted as a probability amplitude, i.e., $|\psi|^2$ as the probability density of a particle in the configuration space". The vector $\psi \in H$ with $\|\psi\| = 1$ and the vector $e^{i\theta}\psi$ represent the same state, and the knowledge about a state enables us only to determine expectations of observables.

On the other hand, observables correspond to measurable physical quantities, whose values are expressed by real numbers. In von Neumann's formulation, a *self-adjoint operator* is attached to every observable. In a Hilbert space H, the *adjoint operator* T^* of a linear operator T is defined as the linear operator satisfying $\langle \psi, T\varphi \rangle = \langle T^*\psi, \varphi \rangle$. An operator T is *self-adjoint* if $T = T^*$. All eigenvalues of a self-adjoint operator T are real and two eigenvectors ψ, φ of T whose eigenvalues are different are orthogonal to each other, that is $\langle \psi, \varphi \rangle = 0$. As a special self-adjoint operator, there is a kind of *projection operators*. P is a projection operator if it satisfies $P = P^* = P^2$. To every projection operator P, there corresponds a unique closed subspace of H, namely, $\{x \in H \mid Px = x\}$. Conversely, for any closed subspace C of H, we can consider a unique projection operator which is attached to C. Here a subspace C of H is *closed* if every Cauchy sequence in C converges in C.

The following *spectral resolution theorem* shows the relation between observables and self-adjoint operators. The *spectrum* of an operator T is the set of all complex numbers λ such that the operator $T - \lambda I$ does not have a bounded inverse operator. (I is the identity operator.) The spectrum of a self-adjoint operator is a subset of real numbers $\mathbb{R}$, in particular, the spectrum of a projection operator is a subset of $\{0, 1\}$.

Theorem 1.1 (See [43])
For every self-adjoint operator T, there exists a unique *spectral resolution* E on the spectrum Ω of T such that

$$T = \int_\Omega \omega E(d\omega)$$

where the map $B \mapsto E(B)$ satisfies the following:

(a) $E(B_1 \cap B_2) = E(B_1)E(B_2)$

(b) $E(\Omega - B) = E(B)' = I - E(B)$

(c) $E(\bigcup_{i\in\mathbb{N}} B_i) = \sum_{i\in\mathbb{N}} E(B_i)$

Here, if B is a Borel subset of Ω then $E(B)$ gives a projection operator, and $\{B_i\}_{i\in\mathbb{N}}$ is a sequence of mutually disjoint subsets of Ω. □

Consider a case where a quantum mechanical system is in a state ψ and where we will make a measurement of an observable whose corresponding self-adjoint operator is T. Each spectral value $\omega \in \Omega$ of T is interpreted as a possible value which may be obtained by the measurement of that observable. Then, the *probability* that the measurement has as outcome a value lying in a Borel set $B \subseteq \Omega$ is given by

$$\mu_\psi(B) = \langle \psi, E(B)\psi \rangle$$

where E is the spectral resolution of T. Furthermore, the *expectation value* of the observable T can be given by

$$\mu_\psi(T) = \int_\Omega \omega \mu_\psi(d\omega)$$

The spectral resolution theorem of observables allows us to replace observables by projection operators. Von Neumann suggested considering projection operators as representing propositions. If a proposition is true (which is determined by a measurement), then we assign the numerical value 1 to it, and if the proposition is false, we assign the numerical value 0 to it. This is the starting point of the development of the *quantum logic*.

In 1936, von Neumann and G. Birkhoff published the first article on the logic of quantum mechanics ([3]). Their main idea was that with each physical system, an orthocomplemented partially ordered set L is associated, where members of L correspond to propositions concerning the system which can be verified by experiments. The order in L corresponds to the operation of implication, and the orthocomplementation corresponds to negation. The operations meet $\cap$ and join $\cup$ in L correspond to conjunction (*and*) and disjunction (*or*), respectively. For each classical mechanical system, L is a Boolean algebra, whereas for each quantum mechanical system, L does not always fulfill the *distributive law*;

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c).$$

In their article, von Neumann and G. Birkhoff proposed L to be considered as a *modular lattice*, in which the following *modular law* holds;

$$a \leq b \text{ implies } (a \cup c) \cap b = a \cup (c \cap b).$$

However, it is proved that the lattice $\mathcal{C}(H)$ of all closed subspace of a Hilbert space H (which is isomorphic to the lattice of all projection operators of H) is modular if and only if H is finite dimensional. Therefore their postulate is not true for an infinite-dimensional Hilbert space.

Much of von Neumann's subsequent works on *continuous geometries* and *rings of operators* was motivated by his desire of constructing logical calculi satisfying the modular law. It is K.Husimi who discovered in 1937 ([21]) the condition that is satisfied in the lattice $\mathcal{C}(H)$ for any Hilbert space H, i.e.,

$$a \leq b \text{ implies } a = b \cap (b' \cup a).$$

This condition is now called *orthomodular law*.

In 1955, the orthomodular law was rediscovered independently by L.Loomis ([28])and S.Maeda ([31])in connection with their efforts to extend von Neumann's dimension theory for rings of operators. Structures studied by Loomis and Maeda are now called *orthomodular lattices*. The name *orthomodular lattice* was introduced by Kaplansky ([27]) in order to distinguish it from an orthocomplemented lattice which satisfies the modular law. The latter is now called a *modular ortholattice*. After 60s, intensive studies of orthomodular lattices have been made mainly in connection with mathematical analysis.

Before closing this section, we summarize the relation between the algebras we consider and structures of all closed subspaces $\mathcal{C}(H)$ of a Hilbert space H.

Definition 1.2

(1) $\mathfrak{A} = \langle A, \cap, \cup, ', 0, 1\rangle$ is an *ortholattice* if it satisfies the following conditions:

(i) $\langle A, \cap, \cup, 0, 1\rangle$ is a bounded lattice with the least element 0 and the greatest element 1.

(ii) $(\cdot)'$ is a unary operation (*ortho-complement*) on A which satisfies the following: For any $x, y \in A$,

(a) $x'' = x$.

(b) $x \leq y$ implies $y' \leq x'$.

(c) $x \cap x' = 0$

(2) An ortholattice $\mathfrak{A}$ is an *orthomodular lattice* if the following *orthomodular law* holds for any $x, y \in A$

$$x \leq y \quad \text{implies} \quad x = y \cap (y' \cup x).$$

(3) An ortholattice $\mathfrak{A}$ is a *modular ortholattice* if the following *modular law* holds for any $x, y, z \in A$.

$$x \leq y \quad \text{implies} \quad (x \cup z) \cap y = x \cup (z \cap y).$$ ∎

It can be shown that an ortholattice $\mathfrak{A}$ is a *Boolean algebra* if and only if it satisfies the distributive law.

We can construct a complete ortholattice of an inner product space V over a field K in the following standard way. First, a binary relation $\perp$ on V (called an *orthogonality relation*) is defined as: $a \perp b$ if and only if $\langle a, b\rangle = 0$, where $\langle a, b\rangle$ denotes the inner product of a and b. Second, for any subspace S of V, define $S^\perp := \{b \in V \mid a \perp b \text{ for all } a \in S\}$, and let $\mathcal{R}(V) := \{S \subseteq V \mid (S^\perp)^\perp = S\}$. Then it can be shown that the structure $\langle \mathcal{R}(V), \bigcap, \bigcup, (\cdot)^\perp, \{0\}, V\rangle$ is a complete ortholattice, where $\bigcap$ is the intersection and $\bigcup$ is defined by $\bigcup S_\lambda = (\bigcap S_\lambda{}^\perp)^\perp$.

Note that for a Hilbert space H over the complex field $\mathbb{C}$, $\mathcal{R}(H)$ and $\mathcal{C}(H)$ (all closed subspaces of H) coincide. In this case, the following theorem holds.

Theorem 1.3 For any Hilbert space H over $\mathbb{C}$, $\langle \mathcal{C}(H), \bigcap, \bigcup, (\cdot)^\perp, \{0\}, H\rangle$ is a complete orthomodular lattice. □

In 1966, I. Amemiya and H. Araki ([1]) proved the following theorem which characterizes the orthomodular law for Hilbert spaces.

Theorem 1.4 Let V be an inner product space over $\mathbb{C}$. Then $\mathcal{R}(V)$ is orthomodular if and only if V is complete, that is, V is a Hilbert space. □

As already mentioned above, the following theorem on the modular law holds.

Theorem 1.5 A Hilbert space H over $\mathbb{C}$ is finite dimensional if and only if $\mathcal{C}(H)$ is modular. □

As seen above, the orthomodular law and the modular law play very important roles in the theory of Hilbert spaces. The former characterizes Hilbert spaces in the set of inner product spaces and the latter represents finiteness of the dimension of the space.

So far, we have presentaed some analytical background of our research. However, our aim in this thesis is to discuss the universal-algebraic aspects of orthologics and orthomodular logics, so we will not go back to analysis or Hilbert space theory any more. Nor will we go into any topics about quantum logic or quantum physics as this would lead us even farther afield.

The fundamental work on orthologic is the construction of relational semantics for this logic by R.I.Goldblatt, which is introduced in the next section.

1.2 Orthologic and its relational semantics

Goldblatt's paper in 1974 ([15]) deals with orthologics and orthomodular logics in a similar way as in modal logics. In that paper, he defined orthologics as binary logics, constructed a relational semantics for orthologics, proved its completeness and finite model property, and extended his semantics for orthomodular logics. We will follow his approach in investigating our logics, and hence, in this section, we will introduce his method briefly, according to the paper [15].

1.2.1 Syntax of orthologics

The language consists of (i) a denumerable collection $\{p_i \mid i < \omega\}$ of propositional variables, (ii) the connectives $\neg$ and $\wedge$ of negation and conjunction, (iii) parentheses (and). The set Φ of well-formed formulas is constructed from these symbols in the usual way. The disjunction connective $\vee$ can be introduced as an abbreviation $\alpha \vee \beta := \neg(\neg\alpha \wedge \neg\beta)$. Note that there is no implication connective in the language, because it is not possible to introduce any *suitable* implication connectives in the system of orthologics in general ([25], [36]).

Usually, a logic is defined as a set of formulas which contains some axiom schemes and is closed under some inference rules. But in this case, an orthologic is defined as a set of ordered pairs of formulas, because of the lack of implication symbol. Goldblatt called a logic which is defined as a set of pairs of formulas, a *binary logic* . For formulas α, β, we denote $\alpha \vdash_{\mathbf{L}} \beta$ to mean that the pair $\langle \alpha, \beta \rangle$ is a member of the logic $\mathbf{L}$. The formal system of orthologics is defined in the following way.

Definition 1.6 (Orthologic) An *orthologic* $\mathbf{L}$ on the set Φ of formulas is a subset of the product $\Phi \times \Phi$ which includes the following axiom schemes and is closed under the following inference rules:

Axiom schemes:

(Ax1)
$$\alpha \vdash_{\mathbf{L}} \alpha$$

(Ax2)
$$\neg\neg\alpha \vdash_{\mathbf{L}} \alpha$$

(Ax3)
$$\alpha \vdash_{\mathbf{L}} \neg\neg\alpha$$

(Ax4)
$$\alpha \wedge \beta \vdash_{\mathbf{L}} \alpha$$

(Ax5)
$$\alpha \wedge \beta \vdash_{\mathbf{L}} \beta$$

(Ax6)
$$\alpha \wedge \neg\alpha \vdash_{\mathbf{L}} \beta$$

Inference Rules:

(R1)
$$\frac{\alpha \vdash_{\mathbf{L}} \beta \quad \beta \vdash_{\mathbf{L}} \gamma}{\alpha \vdash_{\mathbf{L}} \gamma}$$

(R2)
$$\frac{\alpha \vdash_{\mathbf{L}} \beta \quad \alpha \vdash_{\mathbf{L}} \gamma}{\alpha \vdash_{\mathbf{L}} \beta \wedge \gamma}$$

(R3)
$$\frac{\alpha \vdash_{\mathbf{L}} \beta}{\neg\beta \vdash_{\mathbf{L}} \neg\alpha}$$

The intersection of all orthologics on Φ, that is, the *smallest orthologic*, is denoted by $\mathbf{O}$. ∎

Of course, this formal system is formulated so as to satisfy the following algebraic characterization theorem. A *valuation* v from Φ to an ortholattice $\mathfrak{A}$ is a function $v : \Phi \to \mathfrak{A}$, which satisfies the conditions:

(1) $v(\alpha \wedge \beta) = v(\alpha) \cap v(\beta)$. (2) $v(\neg\alpha) = (v(\alpha))'$.

Theorem 1.7 (Characterization for O)
For any formulas α and β, the following two conditions are equivalent:

(1) $\langle \alpha, \beta \rangle \in \mathbf{O}$.

(2) $v(\alpha) \leq v(\beta)$ holds for any ortholattice $\mathfrak{A}$, and for any valuation v from Φ to $\mathfrak{A}$. □

Proof is very standard. (1) implies (2) is proved by showing that (2) holds for (Ax1),...,(Ax6) and is preserved by (R1), (R2), and (R3). Conversely, (2) implies (1) is proved by showing that the *Lindenbaum algebra* for **O** is an ortholattice. The Lindenbaum Algebra for an orthologic **L** is the quotient algebra $\Phi/\equiv_{\mathbf{L}}$, where $\alpha \equiv_{\mathbf{L}} \beta$ if and only if $\langle \alpha, \beta \rangle \in \mathbf{L}$ and $\langle \beta, \alpha \rangle \in \mathbf{L}$.

A few more syntactical notions and notations are introduced here. Let **L** be an orthologic. For a formula α and a non-empty (possibly infinite) subset Γ of formulas, α is **L**-*derivable from* Γ ($\Gamma \vdash_{\mathbf{L}} \alpha$) if there exist $\beta_1, \beta_2, \cdots \beta_n \in \Gamma$ such that $\beta_1 \wedge \cdots \wedge \beta_n \vdash_{\mathbf{L}} \alpha$. For a non-empty set Γ of formulas, Γ is **L**-*full* if it fulfills the following conditions.

(1) For some $\alpha \in \Phi$, $\Gamma \vdash_{\mathbf{L}} \alpha$ does not hold.

(2) If $\alpha \in \Gamma$ and $\alpha \vdash_{\mathbf{L}} \beta$, then $\beta \in \Gamma$.

(3) If $\alpha, \beta \in \Gamma$, then $\alpha \wedge \beta \in \Gamma$.

The notion of **L**-full sets corresponds to that of *proper filters* of ortholattices in an algebraic sense. The notion will be used when the canonical model for an orthologic **L** is constructed.

1.2.2 Relational semantics of orthologics

In general, there are mainly two ways of constructing semantics for logics. One is the way of algebraic semantics, an example of which we have seen above for the case of the smallest orthologic **O**, and the other is the way of relational semantics, or Kripke-style semantics, such as a non-empty set with some relations. The method of relational semantics has proved to be a great success particularly in modal logics, because it is easy to visualize and manipulate, and it is very simple to characterize models in this semantics for almost all important modal logics.

Establishing the completeness theorem for a logic with respect to a relational semantics is equivalent to proving a representation theorem of the algebra which corresponds to that logic, by way of taking its dual space. In the case of modal logics, Jónsson-Tarski's representation theorem for Boolean algebras with operators ([24]) is the foundation of relational semantics. Although an ortholattice is not a Boolean algebra, there exists a representation theorem via dual space, which is the basic result for constructing relational semantics for orthologics. This representation theorem was obtained by R.I.Goldblatt ([16]) in 1975.

An *orthogonality space* $F = \langle X, \perp \rangle$ consists of a non-empty set X and a binary relation $\perp$ on X which is irreflexive and symmetric. For a subset $Y \subseteq X$, define, $Y^* := \{a \in X \mid a \perp b \text{ for any } b \in Y\}$, and Y is $\perp$-*regular* if $Y = Y^{**}$ holds. The following theorem holds for orthogonality spaces and ortholattices.

Theorem 1.8

(1) The class $\mathcal{R}(F)$ of all $\perp$-*regular* subsets of F is a complete ortholattice where the order is set inclusion, the lattice meet is set intersection, and the orthocomplement is the operation of taking $(\cdot)^*$.

(2) let $\mathfrak{A} = \langle A, \cap, \cup, (\cdot)', 0, 1\rangle$ be an ortholattice. Define $X_{\mathfrak{A}}$ and $\perp_{\mathfrak{A}}$ as follows: $X_{\mathfrak{A}}$ is the collection of all *proper filters* of $\mathfrak{A}$, and for $x, y \in X_{\mathfrak{A}}$, $x \perp_{\mathfrak{A}} y$ if and only if there exists an element $a \in A$ such that $a \in x$ and $a' \in y$. Then the pair $F_{\mathfrak{A}} = \langle X_{\mathfrak{A}}, \perp_{\mathfrak{A}}\rangle$ is an orthogonality space.

(3) Every ortholattice $\mathfrak{A}$ can be embedded into a complete ortholattice $\mathcal{R}(F_{\mathfrak{A}})$, where the embedding $\eta : A \to \mathcal{R}(F_{\mathfrak{A}})$ is defined as: $\eta(a) := \{x \in X_{\mathfrak{A}} \mid a \in x\}$. □

Orthogonality spaces appeared already in the paper by Foulis and Randall ([14]) and the above result (1) was well known long before Foulis and Randall, which is stated in Birkhoff's book ([4]). Goldblatt has showed a way to construct an orthogonality space of an ortholattice. Then he used the orthogonality space to establish a representation of any ortholattice. To obtain his full representation theorem, we need to restrict $\mathcal{R}(F_{\mathfrak{A}})$ by introducing a topology so as to make the map η isomorphic. But even the above theorem is enough for proving the completeness theorem of the logic **O** with respect to its relational semantics, that is, the following frames and models for orthologics.

Definition 1.9

(1) $\mathcal{F} = \langle X, \perp\rangle$ is an *orthoframe* if X is a non-empty set and $\perp$ is an irreflexive symmetric binary relation.

(2) $\mathfrak{M} = \langle X, \perp, V\rangle$ is an *orthomodel* on the frame $\langle X, \perp\rangle$ if V is a function assigning to each propositional variable p_i a $\perp$-*regular* subset $V(p_i)$ of X.
The truth of a formula α at a point $x \in X$ in $\mathfrak{M}$ is defined recursively as follows: The symbol $\mathfrak{M} \models_x \alpha$ is read as " a formula α is true at x in a model $\mathfrak{M}$".

(a) $\mathfrak{M} \models_x p_i$ if and only if $x \in V(p_i)$.

(b) $\mathfrak{M} \models_x \alpha \wedge \beta$ if and only if $\mathfrak{M} \models_x \alpha$ and $\mathfrak{M} \models_x \beta$.

(c) $\mathfrak{M} \models_x \neg\alpha$ if and only if for any $y \in X$, $\mathfrak{M} \models_y \alpha$ implies $x \perp y$.

∎

Let Γ be a non-empty set of formulas and α a formula. Γ *implies* α in an orthomodel $\mathfrak{M}$ ($\mathfrak{M} : \Gamma \models \alpha$ in symbol) if for any $x \in X$, either there exists a formula β such that not $\mathfrak{M} \models_x \beta$, or else $\mathfrak{M} \models_x \alpha$. For an orthoframe $\mathcal{F}$,

Γ *$\mathcal{F}$-implies* α ($\mathcal{F} : \Gamma \models \alpha$ in symbol) if $\mathfrak{M} : \Gamma \models \alpha$ for all orthomodel $\mathfrak{M}$ on $\mathcal{F}$. For a class $\mathfrak{C}$ of orthoframes, Γ *$\mathfrak{C}$-implies* α ($\mathfrak{C} : \Gamma \models \alpha$ in symbol) if $\mathcal{F} : \Gamma \models \alpha$ for all frames $\mathcal{F}$ in $\mathfrak{C}$.

Let θ be the class of all orthoframes. Under these setting of orthomodels and the notion of validity, the following holds.

Theorem 1.10 (Soundness for O) If $\Gamma \vdash_{\mathbf{O}} \alpha$, then $\theta : \Gamma \models \alpha$. □

This soundness result is essentially equivalent to (1) of Theorem 1.8. To prove the completeness, the *canonical model* for an orthologic **L** has to be constructed. The *canonical orthomodel* for **L** is defined in the following way.

Definition 1.11 Let **L** be an orthologic. Then the *canonical model* for **L** is the structure $\mathfrak{M}_{\mathbf{L}} = \langle X_{\mathbf{L}}, \perp_{\mathbf{L}}, V_{\mathbf{L}} \rangle$, where

$$X_{\mathbf{L}} = \{x \subseteq \Phi \mid x \text{ is } \mathbf{L}\text{-full}\}.$$

$$x \perp_{\mathbf{L}} y \quad \text{if and only if} \quad \text{there exists } \alpha \text{ such that } \neg\alpha \in x \text{ and } \alpha \in y.$$

$$V_{\mathbf{L}}(p_i) = \{x \in X_{\mathbf{L}} \mid p_i \in x\}.$$

∎

Clearly this construction is a translation of the dual space construction of an ortholattice in (2) of Theorem 1.8 into logical terms. It is easily seen that $\mathfrak{M}_{\mathbf{L}}$ is indeed an orthomodel, and so, as in the theorem, the following holds.

Lemma 1.12 Let **L** be an orthologic, Γ a non-empty set of formulas, α a formula.

(1) For all $x \in X_{\mathbf{L}}$, $\mathfrak{M}_{\mathbf{L}} \models_x \alpha$ if and only if $\alpha \in x$.

(2) $\Gamma \vdash_{\mathbf{L}} \alpha$ if and only if $\mathfrak{M}_{\mathbf{L}} : \Gamma \models \alpha$. □

With the aid of the lemma above, the completeness theorem for the orthologic **O** is proved.

Theorem 1.13 (Completeness for O) If $\theta : \Gamma \models \alpha$, then $\Gamma \vdash_{\mathbf{O}} \alpha$. □

Furthermore, Goldblatt showed the following facts about the smallest orthologic **O** by analyzing his orthomodel.

(1) Any theorem of the orthologic **O** (i.e., any pair of formulas in **O**) can be translated into a theorem of the Brouwerian modal logic **B** (its modality □ corresponds to a reflexive, symmetric binary relation on Kripke frames), which is formulated as a binary logic.

(2) Filtration method works for orthomodels in establishing that **O** has the finite model property. Therefore **O** is decidable, since it is finitely axiomatazable.

(3) A semantics for the smallest calculus of orthomodular logic can be obtained by a refinement of the semantics for orthologics.

We will discuss his approach to the smallest orthomodular logic ((3) above) and its limits, in the remaining part of this section.

1.2.3 Extension to orthomodular logic and its limit

Definition 1.14 (Orthomodular logic) An *orthomodular logic* **L** is an orthologic which also includes the following axiom scheme.

(Ax7) $\alpha \wedge (\neg\alpha \vee (\alpha \wedge \beta)) \vdash_{\mathbf{L}} \beta$

The *smallest orthomodular logic*, or the *quantum logic* is denoted by **Q** . ▮

The axiom scheme (Ax7) represents the orthomodular law in a system of binary logic, and so, the algebraic characterization of **Q** by the class of orthomodular lattices holds as in the case of the orthologic **O** in Theorem 1.7.

To construct frames and models for orthomodular logics, the notion of $\perp$-regularity in orthofames has to be refined as follows: Let $\langle X, \perp\rangle$ be an orthoframe. For subset Y, Z of X with $Y \subseteq Z$, Y is *$\perp$-regular in Z*, if $Y^{\star\star} = Y$, where $(\cdot)^{\star}$ is a unary operation that depends on Z, defined by: $Y^{\star} := \{z \in Z \mid z \perp y \text{ for all } y \in Y\}$.

Definition 1.15 $\mathcal{F} = \langle X, \perp, \xi\rangle$ is a *quantum frame* , if $\langle X, \perp\rangle$ is an orthoframe and ξ is a non-empty collection of $\perp$-regular subsets of X which satisfies the following:

(1) ξ is closed under set intersection and the operation $(\cdot)^{*}$ defined by: $Y^{*} := \{z \in X \mid x \perp y \text{ for all } y \in Y\}$.

(2) For $Y, Z \in \xi$ with $Y \subseteq Z$, Y is $\perp$-regular in Z.

$\mathfrak{M} = \langle X, \perp, \xi, V\rangle$ is a *quantum model*, if $\langle X, \perp, \xi\rangle$ is a quantum frame and V is a function assigning to each p_i a member of ξ. The truth conditions are the same as in Definition 1.9. ▮

The restriction of the range of the valuation V to ξ makes the axiom scheme (Ax7) true in quantum models. Let Ω be the class of all quantum frames, Γ a non-empty subset of formulas, and α a formula. Due to our refinement, the following soundness theorem holds.

Theorem 1.16 **(Soundness for Q)** If $\Gamma \vdash_{\mathbf{Q}} \alpha$, then $\Omega : \Gamma \models \alpha$. □

Let **L** be an orthomodular logic. The canonical model for **L**, this time, is defined as follows:

Definition 1.17 The *canonical quantum frame for* **L** is the structure $\mathcal{G}_{\mathbf{L}} = \langle X_{\mathbf{L}}, \perp_{\mathbf{L}}, \xi_{\mathbf{L}} \rangle$, where $X_{\mathbf{L}}$ and $\perp_{\mathbf{L}}$ are the same as in Definition 1.11, and $\xi_{\mathbf{L}} = \{|\alpha|^{\mathbf{L}} \mid \alpha \in \Phi\}$. Here $|\alpha|^{\mathbf{L}} = \{x \in X_{\mathbf{L}} \mid \alpha \in x\}$. The *canonical quantum model for* **L** is $\mathfrak{N}_{\mathbf{L}} = \langle X_{\mathbf{L}}, \perp_{\mathbf{L}}, \xi_{\mathbf{L}}, V_{\mathbf{L}} \rangle$, where $V_{\mathbf{L}}(p_i) = |p_i|$. ▮

As in the case of orthologics, the lemma similar to Lemma 1.12 holds for orthomodular logics, from which the completeness theorem for the logic **Q** follows.

Lemma 1.18 Let **L** be an orthomodular logic, Γ a non-empty set of formulas, α a formula.

(1) For all $x \in X_{\mathbf{L}}$, $\mathfrak{N}_{\mathbf{L}} \models_x \alpha$ if and only if $\alpha \in x$.

(2) $\Gamma \vdash_{\mathbf{L}} \alpha$ if and only if $\mathcal{G}_{\mathbf{L}} : \Gamma \models \alpha$. □

Note that Lemma 1.12 (2) holds for the canonical *model* for an orthologic, whereas in the lemma above, (2) holds for the canonical *frame* for an orthomodular logic.

Theorem 1.19 **(Completeness for Q)** If $\Omega : \Gamma \models \alpha$, then $\Gamma \vdash_{\mathbf{Q}} \alpha$. □

The above approach of Goldblatt is now widely known as a *method of general frames* (*models*), which works well in modal logics quite generally. What we should find is a class of orthoframes that characterizes the orthomodular logic **Q**. A few years later, however, he reached a negative answer to this problem. In [17], he showed the following theorem by demonstrating that an inner product space is an elementary substructure of its Hilbert space completion, with respect to orthogonality relation.

Theorem 1.20 There is no first-order condition on orthogonality relations that determines a subclass of the class of orthoframes that characterizes the orthomodular logic **Q**. □

Almost all important modal logics have their frames defined by first-order conditions on the relation R. Usually, to show that a logic is characterized by a certain class of frames it is enough to prove that the class includes the frame of the canonical model. If the class can be defined by first-order language, then the question boils down to showing that the canonical frame satisfies a certain first-order condition or a set of such conditions. This is a standard approach for modal logics.

But the above theorem implies that the standard approach breaks down in the case of orthomodular logics. Indeed, there remains several fundamental questions still open about orthomodular logics. In order to tame orthomodular logics, we need some new idea in semantical analysis.

1.3 Organization of this thesis

This dissertation consists of 7 chapters and an appendix. In each chapter, we discuss the following subjects.

Chapter 1 was devoted to describing the background information of this research, especially the relation between the theory of quantum mechanics and the birth of orthomodular logics, and we explained one of the fundamental work on orthologics and orthomodular logics by Goldblatt.

In Chapter 2, we introduce propositional logics which are discussed in this thesis, and varieties of algebras corresponding to our logics. In this chapter, we also prepare several algebraic tools which are used in the following chapters, such as filters, complete algebras, some algebraic laws, and so forth, and we also show some algebraic propositions there, for future use.

One of the main topics, that is, completion of algebras is discussed in Chapter 3. There are two ways of completion of algebras in general. One is the Dedekind-MacNeille completion technique and the other is the method via dual spaces of algebras. Among our varieties of algebras, three accept both techniques, the variety of semi-ortholattices, the variety of ortholattices, and the variety of Boolean algebras. The application of these completion results to logic spreads over several directions. In this chapter, first we construct relational semantics for the smallest semi-orthologics.

In Chapter 4, as the second application of completion technique, we discuss the minimum predicate extensions of the smallest orthologic and the smallest semi-orthologic. In this case, we have to use the Dedekind-MacNeille completion technique, because all even infinite existing meets and joins must be preserved by the embedding map in constructing semantics for the predicate extensions.

The third application of completion technique is discussed in Chapter 5. We will see, by a simple observation, that for any of our varieties of algebras, the super amalgamation property of the variety is a sufficient condition for its corresponding logic to have the Craig's interpolation property. We can prove that the variety of ortholattices and the variety of semi-ortholattices have the super amalgamation property by applying the Dedekind-MacNeille completion technique in a slightly modified way. Thus we can conclude that both the smallest orthologic and the smallest semi-orthologic have the Craig's interpolation property. Moreover, with the help of results in the previous chapter, we can show that their minimum predicate extensions also have the interpolation property.

Chapter 6 is devoted to devising a Kripke-style semantics of orthomodular logics. This type of semantics is based on the representation theorem by Foulis, which employs a certain type of semigroups. We can prove the general completeness of any orthomodular logic with respect to this type of semantics. We discuss the infinitary extension of orthomodular logics using this semantics.

We summarize the thesis in Chapter 7, and discuss a few ways to tackle the completion problem of orthomodular lattices.

Finally, in Appendix we discuss syntactical research on the smallest orthologic. A Gentzen type sequent calculus for the smallest orthologic by S. Tamura is introduced there. This is a cut-free system, so we can obtain the Craig's interpolation property of the smallest orthologic by applying Maehara's method to this calculus.

2 Basic notions

In this section, we prepare some algebraic concepts and logics for our investigation. Each logic we consider here is characterized by a certain class of algebras, that can be defined by a set of *identities*. In other words, each logic is formalized by a set of axiom schemes and inference rules which corresponds to its defining set of identities.

2.1 Classes of algebras we discuss

Every algebra we treat here has the signature $\langle A, \cap, \cup, (\cdot)', 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$. A class of algebras which can be defined by a set of identities is called a *variety* . All our classes of algebras are varieties. The first is the largest class of all we consider. This class is called *the variety of semi-ortholattices* and denoted by $\mathcal{OL}^{(-)}$. To show this class forms a variety explicitly, we give its definition by the set of identities.

Definition 2.1 (Semi-ortholattice) A *semi-ortholattice* is an algebraic structure $\mathfrak{A} = \langle A, \cap, \cup, {}', 0.1\rangle$, which satisfies the following identities:

$$x \cap y = y \cap x \qquad x \cup y = y \cup x$$

$$x \cap (y \cap z) = (x \cap y) \cap z \qquad x \cup (y \cup z) = (x \cup y) \cup z$$

$$x = x \cap (x \cup y) \qquad x = x \cup (x \cap y)$$

$$x \cup 1 = 1$$

$$x \cap x' = 0$$

$$(x')' \cap x = x$$

$$x' \cap (x \cup y)' = (x \cup y)'$$

∎

In other words, a semi-ortholattice is a bounded lattice with a unary operation $(\cdot)'$ which satisfies the following: for any $x, y \in A$,

(a) $x \leq x''$.

(b) $x \cap x' = 0$.

(c) $x \leq y$ implies $y' \leq x'$.

In comparison with Definition 1.2, it is easily seen that the class $\mathcal{OL}$ of all ortholattices can be defined by the identities for $\mathcal{OL}^{(-)}$ together with the *double negation law* .

$$x = x'' \qquad \text{(double negation law)}$$

As is stated in Definition 1.2 and below that, the class $\mathcal{OML}$ of all orthomodular lattices, the class $\mathcal{MOL}$ of all modular ortholattices, and the class $\mathcal{BA}$ of Boolean algebras are defined by adding the orthomodular law, the modular law, and the distributive law respectively, to the identities for $\mathcal{OL}$, which can be rewritten down in forms of identity in the following way.

For $\mathcal{OML}$ $\quad x \cap \{(x \cap y) \cup x'\} = x \cap y \quad$ (orthomodular law)

For $\mathcal{MOL}$ $\quad x \cap \{(x \cap y) \cup z\} = (x \cap y) \cup (x \cap z) \quad$ (modular law)

For $\mathcal{BA}$ $\quad x \cap (y \cup z) = (x \cap y) \cup (x \cap z) \quad$ (distributive law)

It is easily seen that the orthomodular law follows from the modular law, and that the modular law follows from the distributive law. Similarly, the class $\mathcal{OML}^{(-)}$ of *semi-orthomodular lattices* , the class $\mathcal{MOL}^{(-)}$ of *semi-modular ortholattice* , and the class $\mathcal{BA}^{(-)}$ of *semi-Boolean algebras* are obtained by adding the above each identity respectively to the identities for $\mathcal{OL}^{(-)}$.

Obviously, all the classes we mentioned above are varieties which are *subvarieties* of $\mathcal{OL}^{(-)}$, and the relation among these varieties are:

$$\begin{array}{ccccccc} \mathcal{BA} & \subseteq & \mathcal{MOL} & \subseteq & \mathcal{OML} & \subseteq & \mathcal{OL} \\ \cap\!\!| & & \cap\!\!| & & \cap\!\!| & & \cap\!\!| \\ \mathcal{BA}^{(-)} & \subseteq & \mathcal{MOL}^{(-)} & \subseteq & \mathcal{OML}^{(-)} & \subseteq & \mathcal{OL}^{(-)} \end{array}$$

There are, of course, many subvarieties of $\mathcal{OL}^{(-)}$, other than those given here. The symbol $\mathcal{V}$ is used for a meta-variable for a variety of algebras of the signature $\langle A, \cap, \cup, {}', 0.1\rangle$, whereas the symbol $\mathcal{C}$ is used for a meta-variable for an arbitrary class of algebras of the same signature in general, and an algebra which is a member of $\mathcal{C}$ is sometimes called $\mathcal{C}$-*algebra*.

The varieties of algebras we have defined so far are used for interpreting formulas of *propositional* logics, whereas, in interpreting sentences of a *predicate logic* algebraically, we usually employ *complete* algebras of a suitable kind.

Let $\mathfrak{P} = \langle P, \leq\rangle$ be a partially ordered set. For a subset $S \subseteq P$, $a \in P$ is the *greatest lower bound* of S if a satisfies the following:

(1) $a \leq x$ for all $x \in S$.

(2) For any $u \in P$, $u \leq x$ for all $x \in S$, then $u \leq a$.

The greatest lower bound of a given subset is uniquely determined if it exists. The *least upper bound* of a given subset S is defined as the dual of the greatest lower bound of S.

Definition 2.2 Let $\mathfrak{A} = \langle A, \cap, \cup, ', 0.1\rangle$ be an algebra in a class $\mathcal{C}$. $\mathfrak{A}$ is *complete* $\mathcal{C}$-algebra if for any $S \subseteq A$, there exists the greatest lower bound of S ($\bigcap S$ in symbol) in A. ▮

It is easily shown that, if $\mathfrak{A}$ is a complete algebra, then there also exists the *least upper bound* of S ($\bigcup S$ in symbol) for any subset S of A. Complete $\mathcal{V}$-algebras may be employed when the minimum *predicate extension* of the smallest propositional logic which corresponds to $\mathcal{V}$ is considered. But this type of semantics for the predicate extension is successful only when the Lindenbaum algebra of this predicate logic can be embedded into a complete $\mathcal{V}$-algebra. The problem of *completion* of our algebras will be discussed in the next chapter.

2.2 Some properties of our algebras

Before introducing logics, we will give in this section a several properties of our algebras.

Proposition 2.3 Let $\mathfrak{A} = \langle A, \cap, \cup, ', 0.1\rangle$ be a semi-ortholattice. Then the following holds. For $x, y \in A$,

(1) $0' = 1$ and $1' = 0$.

(2) $x''' = x'$.

(3) $x' \cup y' \leq (x \cap y)'$.

(4) $(x \cup y)' \leq x' \cap y'$. □

The proof will be obtained by only simple calculations. Note that De Morgan's law ($x' \cup y' = (x \cap y)'$ and $(x \cup y)' = x' \cap y'$) does not hold in semi-ortholattices in general, but only the inequalities (3) and (4) hold. On the other hand, De Morgan's law holds in any ortholattice because of the double negation law. Therefore, in any ortholattice, the join $x \cup y$ of x and y can be regarded to an abbreviation as $(x' \cap y')'$, and so, the variety of ortholattices can be defined by equations which includes only $0, 1, \cap$, and $(\cdot)'$.

Consider the distributive law ($x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$). Then it is easily seen that the dual form of this distributive law ($x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$) can be derived from the original one and visa versa. For the modular law, which can be also expressed as:$y \leq x$ implies $x \cap (y \cup z) = y \cup (x \cap z)$, its dual form is just the same as the original one. On the other hand, it is not the case for the orthomodular law.

The orthomodular law can be expressed equivalently in the following form:

(1) $x \cap \{(x \cap y) \cup x'\} = x \cap y$.

(2) $x \geq y$ implies $y = x \cap (x' \cup y)$.

(3) $x \geq y$ and $x' \cup y = 1$ imply $x = y$.

The equivalence of these three forms of orthomodular law can be shown trivially. Among these forms of orthomodular laws and their dual forms, the following relation holds.

Proposition 2.4 Let $\mathfrak{A} = \langle A, \cap, \cup, ', 0.1\rangle$ be a semi-orthomodular lattice. Then the following are equivalent. For $x, y \in A$,

(1) $x \cup x' = 1$.

(2) $x'' = x$.

(3) $x \leq y$ implies $y = x \cup (x' \cap y)$.

Proof : (1) $\Rightarrow$ (2): Since we have $x \leq x''$, $x = x'' \cap (x''' \cup x) = x''$ holds.
(2) $\Rightarrow$ (3): Trivial.
(3) $\Rightarrow$ (1): Since $x \leq 1$, and we have (3), $1 = x \cup (x' \cap 1) = x \cup x'$ holds. □

We point out one more fact. If we have the double negation law and the orthomodular law together, the distributive law can be expressed in a simpler form, that is, the following equivalence holds in orthomodular lattices.

Proposition 2.5 Let $\mathfrak{A} = \langle A, \cap, \cup, ', 0.1\rangle$ be an orthomodular lattice. Then the following are equivalent. For $x, y, z \in A$,

(1) $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$.

(2) $x \cap (x' \cup y) = x \cap y$.

Proof : (1)$\Rightarrow$(2): Trivial. (Take x' for z.)
(2)$\Rightarrow$(1): First, note that $(x \cap y) \cup (x \cap z) \leq x \cap (y \cup z)$ is clear. So it is enough to show that $x \cap (y \cup z)$ is the least upper bound of $x \cap y$ and $x \cap z$. Take any $u \in A$ such that $x \cap y \leq u$ and $x \cap z \leq u$. Then, since we have $u \cap x \cap (y \cup z) \leq x \cap (y \cup z)$, by the orthomodular law we have only to show that $(x \cap (y \cup z)) \cap \{u \cap x \cap (y \cup z)\}' = 0$. By the fact that $x \cap y = x \cap y \cap u \leq x \cap (y \cup z) \cap u$, we have $\{x \cap (y \cup z) \cap u\}' \leq (x \cap y)' = x' \cup y'$. Similarly we have $\{x \cap (y \cup z) \cap u\}' \leq x' \cup z'$. Then, by (2), we have $\{x \cap (y \cup z) \cap u\}' \cap (x \cap (y \cup z)) \leq (x' \cup y') \cap x \cap (y \cup z) = x \cap y' \cap (y \cup z)$ and $\{x \cap (y \cup z) \cap u\}' \cap (x \cap (y \cup z)) \leq x \cap z' \cap (y \cup z)$. Therefore we conclude that $\{x \cap (y \cup z) \cap u\}' \cap (x \cap (y \cup z)) \leq x \cap (y \cup z) \cap (y' \cap z') = x \cap (y \cup z) \cap (y \cup z)' = 0$. □

We call here the law of (2) the *commutative law* . It will be clear in Chapter 6 why this is called so. The next proposition will be also used in Chapter 6.

Proposition 2.6 Let $\mathfrak{A}$ be an orthomodular lattice which satisfies the commutative law. Then for $x, y, z \in A$, the following holds:

(1) $x \cap (x' \cup y) = y \cap (y' \cup x)$

(2) $(((x \cup y') \cap y) \cup z') \cap z = (((x \cup z') \cap z) \cup y') \cap y$ □

The proof will be given by simple calculations.

2.3 Other algebraic concepts

Several notions for subsets of an algebra are useful in semantical analysis, especially in constructing a dual space of that algebra. Here we introduce the notion of *filters* and show a simple fact on them. Let $\mathfrak{A} = \langle A, \cap, \cup, ', 0.1 \rangle$ be a semi-ortholattice.

Definition 2.7 A subset $F \subseteq A$ is a *filter* of $\mathfrak{A}$ if it satisfies the following three conditions: For $x, y \in A$,

(1) $1 \in F$.

(2) $x \in F$ and $x \leq y$ imply $y \in F$.

(3) $x, y \in F$ implies $x \cap y \in F$.

A filter F of $\mathfrak{A}$ is *proper* if $0 \notin F$. A proper filter F of $\mathfrak{A}$ is *prime* if it satisfies the following:

(4) $x \cup y \in F$ implies either $x \in F$ or else $y \in F$.

∎

In general, a subset $S \subseteq A$, which satisfies the condition (2) above is said to be *upward-closed* . A *downward-closed* subset is defined dually. As seen easily from above, the notion of filter can be defined on structures which has an order relation and a meet operation ($\cap$). For any non-empty subset $S \subseteq A$, the *filter generated by* S (in symbol $[S)$) is defined by:

$$[S) := \{y \in A \mid \exists x_1, x_2, \ldots, x_n \in S \text{ such that } x_1 \cap \cdots \cap x_n \leq y\}$$

It is easily proved that $[S)$ is the smallest filter which contains S.

The notion of proper filters is used for completion of ortholattices, as is shown in Chapter 1, and that of prime filters is used for Boolean algebras.

2.4 Logics

In this section, we introduce formal systems of propositional logics, each of which has algebras in the previous section as its algebraic semantics. We adopt a framework of *binary logics* by Goldblatt, which appeared in Chapter 1. First, we define the system for a binary logic which corresponds to the variety $\mathcal{OL}^{(-)}$, and then, we extend this system by introducing a several axiom schemes.

In propositional case, our language consists of the following primitive symbols: a collection of propositional variables $p, q, r, p_0, p_1, \ldots$ etc., a propositional constant $\bot$ (falsity), a unary connective $\neg$ (negation), binary connectives $\wedge$ (conjunction) and $\vee$ (disjunction), and a pair of parentheses (,).

The set Φ of all formulas of this language is defined by the following three formation rules:

(1) $\bot$ is a formula, and each propositional variable p_i is a formula.

(2) If α is a formula, then so is $(\neg\alpha)$.

(3) If α and β are formulas, then so are $(\alpha \wedge \beta)$ and $(\alpha \vee \beta)$.

As already seen in Chapter 1, a binary logic **L** is defined as a set of pairs of formulas in the following way. For formulas α, β, we denote $\alpha \vdash_{\mathbf{L}} \beta$ to mean that the pair $\langle \alpha, \beta \rangle$ is a member of the logic **L**.

Definition 2.8 (Semi-orthologic) A *semi-orthologic* **L** on the set Φ of formulas is a subset of the product $\Phi \times \Phi$ which includes the following axioms and is closed under the following inference rules:

Axiom schemes:

(Ax1)
$$\alpha \vdash_{\mathbf{L}} \alpha$$

(Ax2)
$$\alpha \vdash_{\mathbf{L}} \neg\neg\alpha$$

(Ax3)
$$\alpha \wedge \beta \vdash_{\mathbf{L}} \alpha$$

(Ax4)
$$\alpha \wedge \beta \vdash_{\mathbf{L}} \beta$$

(Ax5)
$$\alpha \vdash_{\mathbf{L}} \alpha \vee \beta$$

(Ax6)
$$\beta \vdash_{\mathbf{L}} \alpha \vee \beta$$

(Ax7)
$$\alpha \wedge \neg\alpha \vdash_{\mathbf{L}} \beta$$

(Ax8)
$$\perp \vdash_{\mathbf{L}} \alpha$$

Inference Rules:

(R1)
$$\frac{\alpha \vdash_{\mathbf{L}} \beta \quad \beta \vdash_{\mathbf{L}} \gamma}{\alpha \vdash_{\mathbf{L}} \gamma}$$

(R2)
$$\frac{\alpha \vdash_{\mathbf{L}} \beta \quad \alpha \vdash_{\mathbf{L}} \gamma}{\alpha \vdash_{\mathbf{L}} \beta \wedge \gamma}$$

(R3)
$$\frac{\alpha \vdash_{\mathbf{L}} \gamma \quad \beta \vdash_{\mathbf{L}} \gamma}{\alpha \vee \beta \vdash_{\mathbf{L}} \gamma}$$

(R4)
$$\frac{\alpha \vdash_{\mathbf{L}} \beta}{\neg\beta \vdash_{\mathbf{L}} \neg\alpha}$$

A semi-orthologic is sometimes called simply *a logic*. The intersection of all semi-orthologics on Φ, that is, the *smallest semi-orthologic* is denoted by $\mathbf{OL}^{(-)}$. ∎

A propositional extension of $\mathbf{OL}^{(-)}$ is accomplished by adding some axiom schemes to the logic $\mathbf{OL}^{(-)}$. Let $\mathbf{L}$ be a logic, and (**Axs**) an axiom scheme. Then the smallest logic which contains both $\mathbf{L}$ and (**Axs**) is denoted by $\mathbf{L} \oplus (\mathbf{Axs})$. Examples of such axiom schemes are the following:

(**Dbn**): $\neg\neg\alpha \vdash \alpha$

(**Oml**): $\alpha \wedge (\neg\alpha \vee (\alpha \wedge \beta)) \vdash \beta$

(**Mod**): $\alpha \wedge ((\alpha \wedge \beta) \vee \gamma) \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$

(**Dis**): $\alpha \wedge (\beta \vee \gamma) \vdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$

After these preparations, we can formulate *the smallest orthologic* (**OL**), *the smallest orthomodular logic* (**OML**), *the smallest modular orthologic* (**MOL**), and *the classical logic* (**CL**) as follows:

$$\mathbf{OL} = \mathbf{OL}^{(-)} \oplus (\mathbf{Dbn}) \qquad \mathbf{MOL} = \mathbf{OL} \oplus (\mathbf{Mod})$$

$$\mathbf{OML} = \mathbf{OL} \oplus (\mathbf{Oml}) \qquad \mathbf{CL} = \mathbf{OL} \oplus (\mathbf{Dis})$$

Similarly, the family of the smallest logics which do not have the double negation law, that is, *the smallest semi-orthologic*, ($\mathbf{OL}^{(-)}$) *the smallest semi-orthomodular logic*, ($\mathbf{OML}^{(-)}$) *the smallest semi-modular orthologic*, ($\mathbf{MOL}^{(-)}$) and *the smallest semi-classical logic* ($\mathbf{CL}^{(-)}$) are formulated as follows:

$$\mathbf{OL}^{(-)} \qquad \mathbf{MOL}^{(-)} = \mathbf{OL}^{(-)} \oplus (\mathbf{Mod})$$

$$\mathbf{OML}^{(-)} = \mathbf{OL}^{(-)} \oplus (\mathbf{Oml}) \qquad \mathbf{CL}^{(-)} = \mathbf{OL}^{(-)} \oplus (\mathbf{Dis})$$

Figure 1 below shows the relation among these 8 logics with the inconsistent logic. Each circle indicates one of these logics. If a logic includes another logic properly, then the former is located in the upper part, whereas the latter is in the lower, and both are tied with a line.

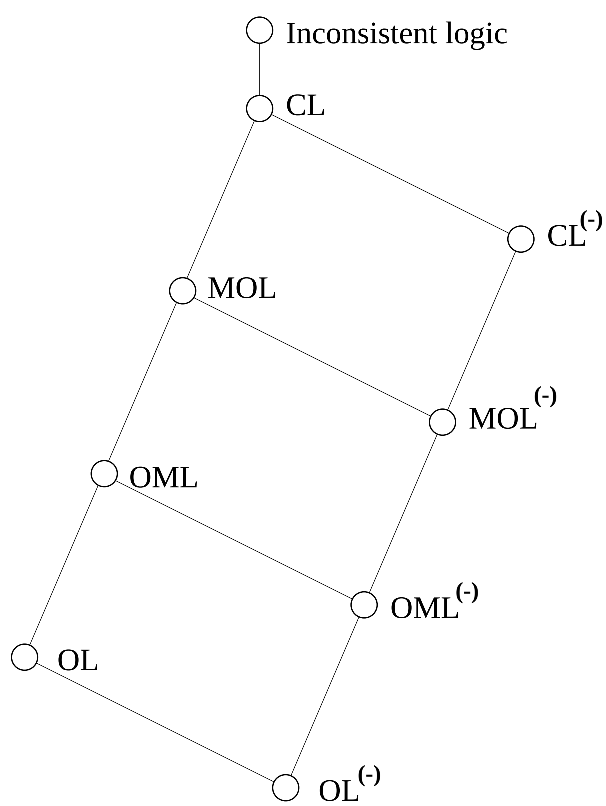

Figure 1: The relation among logics

Of course, there exist tremendously many logics and those 8 logics are only a several examples. Each logic which contains the **OML**, for example, is called an *orthomodular* logic. Such a naming is used for logics which is above other smallest logics.

Now, we give some algebraic characterization theorems for logics. The first one is for the semi-orthologic $\mathbf{OL}^{(-)}$. Let $\mathfrak{A}$ be a semi-ortholattice. A *valuation* v is a function from Φ to $\mathfrak{A}$ that satisfies the following conditions:

(1) $v(\bot) = 0$.

(2) $v(\neg\alpha) = (v(\alpha))'$.

(3) $v(\alpha \wedge \beta) = v(\alpha) \cap v(\beta)$.

(4) $v(\alpha \vee \beta) = v(\alpha) \cup v(\beta)$.

Then, the following characterization theorem holds for $\mathbf{OL}^{(-)}$.

Theorem 2.9 (Characterization for $\mathbf{OL}^{(-)}$)
For any formulas, α and β, the following two conditions are equivalent:

(1) $\langle \alpha, \beta \rangle \in \mathbf{OL}^{(-)}$.

(2) $v(\alpha) \leq v(\beta)$ holds for any $\mathfrak{A} \in \mathcal{OL}^{(-)}$, and for any valuation v from Φ to $\mathfrak{A}$. □

Similar characterization theorem holds for each of 8 logics which are given above, that is, the logics **OL**, **OML**, **MOL**, **CL**, $\mathbf{OML}^{(-)}$, $\mathbf{MOL}^{(-)}$, and $\mathbf{CL}^{(-)}$ are characterized by the varieties $\mathcal{OL}$, $\mathcal{OML}$, $\mathcal{MOL}$, $\mathcal{BA}$, $\mathcal{OML}^{(-)}$, $\mathcal{MOL}^{(-)}$, and $\mathcal{BA}^{(-)}$, respectively. Moreover, a given subvariety $\mathcal{V}$ of $\mathcal{OL}^{(-)}$, we construct by adding certain set of axiom schemes the smallest logic $\mathbf{L}(\mathcal{V})$ for which the following, similar characterization theorem as Theorem 2.4 holds.

Theorem 2.10 (Characterization for $\mathbf{L}(\mathcal{V})$)
For any formulas, α and β, the following two conditions are equivalent:

(1) $\langle \alpha, \beta \rangle \in \mathbf{L}(\mathcal{V})$.

(2) $v(\alpha) \leq v(\beta)$ holds for any $\mathfrak{A} \in \mathcal{V}$, and for any valuation v from Φ to $\mathfrak{A}$. □

Proofs of all such algebraic characterization theorem are quite similar for the proof of Theorem 1.7.

2.5 Note

The notions of ortholattices, orthomodular lattices, modular ortholattices, and Boolean algebras are familiar in general, whereas the notions of their counterpart which are missing the double negation law are the author's original. So the terminology *semi- ... lattice* in this thesis is ad hoc and not so popular.

Let $\mathfrak{A} := \langle A, \cap, \cup, (\cdot)', 0, 1\rangle$ be the reduct of a Heyting algebra $\langle A, \cap, \cup, \supset, 0, 1\rangle$, where the operation $(\cdot)'$ is defined by: $x \cap y = 0 \Leftrightarrow x \leq (y \supset 0) = y'$. Then it is easily proved that $\mathfrak{A}$ is a semi-Boolean algebra, and moreover, it can be shown that $(x \cup y)' = x' \cap y'$ holds for any $x, y \in A$. This implies that the class of such reducts of Heyting algebras is a proper subclass of the variety $\mathbf{BA}^{(-)}$. In other words, the orthocomplement $(\cdot)'$ in a semi-ortholattice is surely weaker than the *Heyting complement* .

3 Completion of algebras

For a class of algebras $\mathcal{C}$, the *completion problem* of $\mathcal{C}$-algebras is asking whether any algebra $\mathfrak{A} \in \mathcal{C}$ can be embedded into a *complete* $\mathcal{C}$-algebra. In this chapter, we will show that the varieties $\mathcal{OL}$, $\mathcal{OL}^{(-)}$ and $\mathcal{BA}$ admit completion by employing *Dedekind-MacNeille completion technique* ([29]) and a completion technique using their dual spaces. These facts imply some properties of their corresponding algebraic logics. First, the completion technique by means of dual spaces gives us a tool for constructing relational semantics of propositional logics. Secondly, the completion of a variety of algebras enables us to establish algebraic completeness theorem of its predicate logic which is the minimum first-order extension of the propositional logic corresponding to that variety. Thirdly, the *Craig interpolation property* of that propositional logic and its first-order extension. The first one is discussed in this chapter, and the second and the last one will be discussed in Chapter 4 and 5.

3.1 Completion techniques

In this section, we explain two techniques of completion using the example of the variety $\mathcal{OL}^{(-)}$ of semi-ortholattices, because both of them works for this variety, and in the next section, we show that varieties $\mathcal{OL}$, $\mathcal{BA}$, and $\mathcal{BA}^{(-)}$ admit completion by making use of those techniques. The definition of admissibility of completion of a class of algebras is the following:

Definition 3.1 Let $\mathcal{C}$ be a class of algebras. $\mathcal{C}$ *admits completion* if it satisfies that for any $\mathfrak{A} \in \mathcal{C}$, there exists a *complete* $\mathcal{C}$-algebra $\mathfrak{C}$ and an *embedding* $\eta : \mathfrak{A} \to \mathfrak{C}$, where the *embedding* η is a mapping satisfying the following:

(1) $\eta(0) = 0$ and $\eta(1) = 1$.

(2) $\eta(x') = (\eta(x))'$

(3) $\eta(x \cap y) = \eta(x) \cap \eta(y)$.

(4) $\eta(x \cup y) = \eta(x) \cup \eta(y)$.

∎

Dedekind-MacNeille completion technique is one of the most standard methods of completion. This is originally introduced to make a partially ordered set complete, especially building reals from rationals. We give below the main theorem of Dedekind-MacNeille completion technique without proof, which we will employ in this chapter. A more detailed description is given in [11]. Let $\langle P, \leq \rangle$ be a poset. We will use the following notations.

Notation

For $a \in P$,
$\uparrow a := \{ b \in P \mid a \leq b \}$
$\downarrow a := \{ b \in P \mid b \leq a \}$

For $A \subseteq P$,
$A^u := \{ b \in P \mid a \leq b \text{ for } \forall a \in A\}$
$A^l := \{ b \in P \mid b \leq a \text{ for } \forall a \in A\}$

Theorem 3.2 Let $\boldsymbol{DM}(P) := \{A \subseteq P \mid A^{ul} = A\}$. Then, the following holds.

(1) $\langle \boldsymbol{DM}(P), \subseteq \rangle$ is a complete lattice, where $\bigcap A_i = \Cap A_i$, and $\bigcup A_i = (\Cup A_i)^{ul}$ for $\{A_i \mid i \in I\} \subseteq \boldsymbol{DM}(P)$. (The symbols $\Cap$ and $\Cup$ are the operations of set-theoretic intersection and union respectively.)

(2) The map $\eta : P \to \boldsymbol{DM}(P)$ which is defined by $\eta(x) = \downarrow x$ is an order-isomorphism of P into $\boldsymbol{DM}(P)$.

(3) For a subset $\{x_\lambda\}_{\lambda \in \Lambda} \subseteq P$, if $\bigcap_\lambda x_\lambda$ exists in P, then $\eta(\bigcap_\lambda x_\lambda) = \Cap_\lambda \eta(x_\lambda)$. Similarly, if $\bigcup_\lambda x_\lambda$ exists in P, then $\eta(\bigcup_\lambda x_\lambda) = (\Cup_\lambda \eta(x_\lambda))^{ul}$. □

The property that arbitrary existing meets and joins are preserved is important when the predicate extension of a propositional logic which corresponds to a variety is considered.

Now we consider the variety $\mathcal{OL}^{(-)}$. Take any semi-ortholattice $\mathfrak{A} = \langle A, \cap, \cup, (\cdot)', 0, 1\rangle \in \mathcal{OL}^{(-)}$. For this $\mathfrak{A}$, we will construct a complete semi-ortholattice into which $\mathfrak{A}$ is embeddable.

The first construction is by means of the Dedekind-MacNeille completion. Let $\mathfrak{B} = \langle \boldsymbol{DM}(A), \subseteq \rangle$. Then $\mathfrak{B}$ is a complete lattice with the operations specified in the above theorem. So it remains to show how to introduce an orthocomplement for $\mathfrak{B}$, and an embedding. Define a unary operation $(\cdot)^\perp$ on $\boldsymbol{DM}(A)$ by:

$$S^\perp := \{a \in A \mid a' \in S^u\}.$$

Then this operation has the following properties.

Proposition 3.3 For subsets $S, T \in DM(A)$ and an element $x \in A$,

(1) $(S^\perp)^{ul} = S^\perp$.

(2) $S \Cap S^\perp = \{0\}$.

(3) $S \subseteq T$ implies $T^\perp \subseteq S^\perp$.

(4) $S \subseteq S^{\perp\perp}$.

(5) $(\downarrow x)^\perp = \downarrow x'$.

Proof : (1): It is obvious that $S^\perp \subseteq (S^\perp)^{ul}$. For the converse, take $a \in (S^\perp)^{ul}$. For any $b \in S^\perp$ and any $c \in S$, $b' \in S^u$ and $c \leq b'$ hold, and so we have $b \leq b'' \leq c$. This implies that $c' \in (S^\perp)^u$, and $a \leq c'$. Thus $c \leq c'' \leq a'$ holds and $a' \in S^u$, which means that $a \in S^\perp$.

(2): Clearly $S \Cap S^{\perp} \supseteq \{0\}$. For $a \in S \Cap S^{\perp}$, $a \in S$ and $a' \in S^u$ hold, so we have $a \leq a'$. This implies that $a \leq a \cap a' = 0$, and so $a = 0$.
(3): Suppose $S \subseteq T$ and take $a \in T^{\perp}$. Then $a' \in T^u \subseteq S^u$. Therefore $a \in S$.
(4): Take $a \in S$. For any $b \in S^{\perp}$, $b' \in S^u$, so we have $a \leq b'$ and then, $b \leq b'' \leq a'$. This implies that $a' \in (S^{\perp})^u$, which means that $a \in S^{\perp\perp}$.
(5): Take $a \in (\downarrow x)^{\perp}$. Then $a' \in (\downarrow x)^u = \uparrow x$, and so we have $x \leq a'$ and $a \leq a'' \leq x'$. This means that $a \in \downarrow x'$. Conversely, take $a \in \downarrow x'$. This implies that $a \leq x'$ and so, $x \leq x'' \leq a'$. Therefore $a' \in \uparrow x = (\downarrow x)^u$, which means that $a \in (\downarrow x)^{\perp}$. □

This proposition shows that the structure $\mathfrak{B} = \langle DM(A), \Cap, \bigcup, (\cdot)^{\perp}, \{0\}, A\rangle$ forms a complete semi-ortholattice. With the help of (5) of the above theorem, it is easily seen that the map $\eta : P \to \boldsymbol{DM}(P)$, which is defined by $\eta(x) = \downarrow x$, becomes a required embedding in Definition 3.1. Therefore we conclude that a semi-ortholattice can be embedded into a complete semi-ortholattice by an embedding which preserves any existing meets and joins.

Another construction is through the dual space of a semi-ortholattice. The basic idea is the same as in the case of an ortholattice ([16]). A slight refinement is needed. In this case, the dual space is defined as follows:
A *semi orthogonality space* is a structure $F = \langle X, \leq, \cap, 1, \perp\rangle$, which satisfies the following conditions.

(1) $\langle X, \leq, \cap, 1\rangle$ is a meet-semilattice with the greatest element 1.

(2) $\perp$ is a binary operation on X which has the following properties: For $x, y, z \in X$,

  (i) $x \perp 1$ holds for all $x \in X$.

  (ii) $x \perp x$ if and only if $x = 1$.

  (iii) $x \perp y$ implies $y \perp x$.

  (iv) $x \perp y$ and $y \leq z$ imply $x \perp z$.

  (v) $x \perp y$ and $x \perp z$ imply $x \perp (y \cap z)$.

For a subset $Y \subseteq X$, define $Y^* := \{x \in X \mid x \perp y \text{ for all } y \in Y\}$.
For a semi orthogonality space F and a semi-ortholattice $\mathfrak{A} = \langle A, \cap, \cup, (\cdot)', 1, 0\rangle$, the following lemma holds.

Lemma 3.4

(1) The class $\mathbb{F}(F)$ of all filters of F forms a complete semi-ortholattice under the following operations: For $\{S_\lambda\}_{\lambda \in \Lambda} \subseteq \mathbb{F}(F)$ and $S \in \mathbb{F}(F)$,

  (a) $\bigcap S_\lambda = \Cap_\lambda S_\lambda$.

(b) $\bigcup S_\lambda = \Sigma_\lambda S_\lambda := [\,\Cup_\lambda S_\lambda) = \{a \in X \mid \exists b_1, b_2, \ldots, b_n \in \Cup_\lambda S_\lambda,$ such that $b_1 \cap b_2 \cap \cdots \cap b_n \leq a\}.$

(c) $S' = S^*$.

(2) For a semi-ortholattice $\mathfrak{A}$, the class of all filters of $\mathfrak{A}$ is denoted by $X_\mathfrak{A}$, and for $x, y \in X_\mathfrak{A}$, a binary relation $\perp_\mathfrak{A}$ is defined by: $x \perp_\mathfrak{A} y$ if and only if there exists an element $a \in A$, such that $a \in x$ and $a' \in y$. Then, the structure $F_\mathfrak{A} = \langle X_\mathfrak{A}, \subseteq, \Cap, A, \perp_\mathfrak{A} \rangle$ is a semi orthogonal space.

Proof : (1): Obviously $\mathbb{F}(F)$ is the greatest element and $\{1\}$ is the least element in $\mathbb{F}(F)$. It is easily check that if $\{S_\lambda\}_{\lambda \in \Lambda}$ is a family of filters in F, then $\Cap_\lambda S_\lambda$ is the filter of the greatest lower bound, whereas $\Sigma_\lambda S_\lambda$ is the filter of the least upper bound. Similarly, for a subset $S \subseteq A$, S^* is also a filter by the properties of the relation $\perp$. Thus, we only have to check the conditions of the operation $(\cdot)^*$ for a semi-ortholattice. The definition of $(\cdot)^*$ infers that $S \subseteq S^{**}$ and that $S \subseteq T$ imply $T^* \subseteq S^*$. For the last one, $S \Cap S^* \supseteq \{1\}$ is obvious. Take $a \in S \Cap S^*$, then $a \perp a$ holds, and so $a = 1$. Therefore $S \Cap S^* = \{1\}$.
(2): It is clear that $\langle X_\mathfrak{A}, \subseteq, \Cap, A, \rangle$ is a meet-semilattice with the greatest element A. Therefore we have to check the conditions for $\perp_\mathfrak{A}$.
(i): For any filter x in A, $x \perp_\mathfrak{A} A$ holds obviously.
(ii): $A \perp_\mathfrak{A} A$ is clear. For the converse, suppose $x \perp_\mathfrak{A} x$ for a filter $x \subseteq A$. Then there exists $a \in A$ such that $a, a' \in x$, which implies that $0 = a \cap a' \in x$. Thus $x = A$.
(iii): Suppose $x \perp_\mathfrak{A} y$ for filters x, y in A. Then there exists $a \in A$ such that $a \in x$ and $a' \in y$. Since $a \leq a''$, we have $a'' \in x$. Therefore $y \perp_\mathfrak{A} x$ holds.
(iv): Suppose that $x \perp_\mathfrak{A} y$ and $y \subseteq z$ for filters x, y, z in A. Then there exists $a \in A$ such that $a \in x$ and $a' \in y \subseteq x$. Therefore $x \perp_\mathfrak{A} z$ holds.
(v): Suppose $x \perp_\mathfrak{A} y$ and $x \perp_\mathfrak{A} z$ for filters x, y, z in A. Then there exist $a, b \in A$ such that $a \in x$, $a' \in y$, and $b \in x$, $b' \in z$. Then we have $a \cap b \in x$ and $a' \cup b' \in y \Cap z$. Since $y \Cap z$ is a filter and $a' \cup b' \leq (a \cap b)'$, $(a \cap b)' \in y \Cap z$. Therefore $x \perp_\mathfrak{A} (y \Cap z)$ holds. □

The join $S_1 \cup S_2$ of $S_1, S_2 \in \mathbb{F}(F)$ is sometimes denoted by $S_1 + S_2$, which is the smallest filter containing both S_1 and S_2. By this lemma, we can construct a complete semi-ortholattice $\mathbb{F}(F_\mathfrak{A})$ of a given semi-ortholattice $\mathfrak{A}$. Then, as shown below, an embedding $\eta : \mathfrak{A} \to \mathbb{F}(F_\mathfrak{A})$ is turned out to be:

$$\eta(a) := \{x \in X_\mathfrak{A} \mid a \in x\}.$$

Clearly η is one to one. $\eta(1) = X_\mathfrak{A}$ and $\eta(0) = \{A\}$ are immediate. Next, we will show that $\eta(a') = (\eta(a))^*$ for an element $a \in A$. Take $x \in \eta(a')$, the we have $a' \in x$. For any $y \in \eta(a)$, $a \in y$ holds. Thus we have $x \perp_\mathfrak{A} y$, which implies that $x \in (\eta(a))^*$. For the converse, take $x \in (\eta(a))^*$. As a filter in $\eta(a)$, we can take $\uparrow a$, and $x \perp_\mathfrak{A} \uparrow a$. Then there exists $b \in A$ such that $b \in x$ and $b' \in \uparrow a$, which means $a \leq b'$. Thus, $b \leq b'' \leq a'$ holds, and so we have $x \in \eta(a')$. For the meet operation, it is easy to show that $\eta(a \cap b) = \eta(a) \bigcap \eta(b)$ for $a, b \in A$. For the join operation, we will show that $\eta(a \cup b) = \eta(a) \bigcup \eta(b) = \Sigma\{\eta(a), \eta(b)\} := \eta(a) + \eta(b)$ for elements $a, b \in A$. Take $x \in \eta(a \cup b)$, then $a \cup b \in x$ holds. Then $\uparrow a \in \eta(a)$ and $\uparrow b \in \eta(b)$, we have $\uparrow a \Cap \uparrow b \subseteq x$. Thus $x \in \eta(a) + \eta(b)$. Conversely, take $x \in \eta(a) + \eta(b)$. Then there exist $y \in \eta(a)$ and $z \in \eta(b)$, such that $y \Cap z \subseteq x$. Since we have $a \cup b \in y \Cap z$, $a \cup b \in x$, which implies that $x \in \eta(a \cup b)$. Therefore, we have proved that this η is an embedding. Note that, in this case, the embedding η does not always preserve all existing meets and joins.

Thus, we have completed the proof of the following theorem by different two methods.

Theorem 3.5 (Completion of semi-ortholattices) The variety $\mathcal{OL}^{(-)}$ admits completion. □

3.2 Completion of $\mathcal{OL}$ and $\mathcal{BA}$

3.2.1 The variety $\mathcal{OL}$

It is already mentioned that $\mathcal{OL}$ admits completion in Chapter 1 (Theorem 1.8), in which the technique by means of its dual space is introduced. The Dedekind-MacNeille completion technique also works for $\mathcal{OL}$. Here we will show this fact.

Consider an ortholattice $\mathfrak{A} = \langle A, \cap, \cup, (\cdot)', 0, 1\rangle$. By the same argument as for the variety $\mathcal{OL}^{(-)}$, the structure $\mathfrak{B} = \langle DM(A), \Cap, \bigcup, (\cdot)^{\perp}, \{0\}, A\rangle$ forms a complete semi-ortholattice. So we have only to show that the double negation law holds for $\mathfrak{B}$ in this time. For any $S \in DM(A)$, take $a \in S^{\perp\perp}$. Then $a' \in (S^{\perp})^u$. Take any $b \in S^u$, then, since $b = b''$, $b' \in S^{\perp}$. This implies that $b' \leq a'$, and so $a = a'' \leq b'' = b$. Thus, $a \in S^{ul} = S$. Therefore $S = S^{\perp\perp}$. Our proof is completed.

Theorem 3.6 (Completion of ortholattices) The variety $\mathcal{OL}$ admits completion, in which arbitrary existing meets and joins are preserved. □

3.2.2 The variety $\mathcal{BA}$

The method of completion for Boolean algebras by means of their dual spaces is well known as (a part of) *Stone Representation theorem.* Here we will point out that the Dedekind-MacNeille completion technique also works for the variety of Boolean algebras.

Take a Boolean algebra $\mathfrak{A} = \langle A, \cap, \cup, (\cdot)', 0, 1\rangle$. Then of course, $\mathfrak{A}$ is an ortholattice, it is already shown that $\mathfrak{B} = \langle DM(A), \Cap, \bigcup, (\cdot)^{\perp}, \{0\}, A\rangle$ is a complete ortholattice and that the map η is our desired embedding. So we have only to prove that $\mathfrak{B}$ satisfies the distributive law. For this purpose, it is enough to show that $S \Cap T = S \Cap (S^{\perp} \bigcup T)$ holds for $S, T \in DM(A)$ due to Proposition 2.5 in Chapter 2. $S \Cap T \subseteq S \Cap (S^{\perp} \bigcup T)$ is trivial. For the converse, take $a \in S \Cap (S^{\perp} \bigcup T) = S \Cap (S \Cap T^{\perp})^{\perp}$. Then, this implies that $a \in S$ and $a \in (S \Cap T^{\perp})^{\perp}$, which means $a' \in (S \Cap T^{\perp})^{u}$. Now take any $b \in T^{u}$, then $b' \in T^{\perp}$ since $b = b''$. Because S and $T^{\perp}$ are downward-closed, $a \cap b' \in S \Cap T^{\perp}$. Then we have $a \cap b' \leq a'$, which implies $a = a'' \leq (a \cap b')' = a' \cup b$. Therefore, we have $a = a \cap a \leq a \cap (a' \cup b) = a \cap b \leq b$, which means that $a \in T^{ul} = T$. Thus $a \in S \Cap T$ and $S \Cap (S^{\perp} \bigcup T) \subseteq S \Cap T$. Our proof is completed. Finally, the following theorem holds for the variety $\mathcal{BA}$.

Theorem 3.7 (Completion of Boolean algebras) The variety $\mathcal{BA}$ admits completion, in which arbitrary existing meets and joins are preserved.

□

3.3 Semantics of $\mathbf{OL}^{(-)}$

One of the applications of completion of a class of algebras is to construct relational semantics of the logic which corresponds to the class. In this section we will explain how to build a relational semantics for the logic $\mathbf{OL}^{(-)}$ and show completeness theorem with respect to that semantics.

The construction of semantics for $\mathbf{OL}^{(-)}$ is based on Lemma 3.4. (1) of this lemma corresponds to the soundness, whereas (2) corresponds to the completeness. The situation is quite similar as in the case of the orthologic **OL** in Chapter 1. First we define relational semantics for the logic $\mathbf{OL}^{(-)}$ and prove its soundness.

Definition 3.8

(1) $\mathcal{F} = \langle X, \leq, \cap, 1, \perp \rangle$ is a *semi-orthoframe* if it is a semi orthogonality space.

(2) $\mathfrak{M} = \langle X, \leq, \cap, 1, \perp, V \rangle$ is a *semi-orthomodel* on the frame $\langle X, \leq, \cap, 1, \perp \rangle$ if V is a function assigning to each propositional variable p_i a filter $V(p_i)$ of X.

The truth of a formula α at a point $x \in X$ in $\mathfrak{M}$ is defined recursively as follows: The symbol $\mathfrak{M} \models_x \alpha$ is read as " a formula α is true at x in a model $\mathfrak{M}$".

(a) $\mathfrak{M} \models_x \perp$ if and only if $x = 1$.

(b) $\mathfrak{M} \models_x p_i$ if and only if $x \in V(p_i)$.

(c) $\mathfrak{M} \models_x \alpha \wedge \beta$ if and only if $\mathfrak{M} \models_x \alpha$ and $\mathfrak{M} \models_x \beta$.

(d) $\mathfrak{M} \models_x \alpha \vee \beta$ if and only if there exist $y, z \in X$ such that $\mathfrak{M} \models_y \alpha$ and $\mathfrak{M} \models_z \beta$ and $y \cap z \leq x$.

(e) $\mathfrak{M} \models_x \neg\alpha$ if and only if for any $y \in X$, $\mathfrak{M} \models_y \alpha$ implies $x \perp y$.

∎

For a formula α, we denote $\|\alpha\|^{\mathfrak{M}} := \{x \in G \mid \mathfrak{M} \models_x \alpha\}$. Then we can restate the above truth conditions in the following form:

(a) $\|\perp\|^{\mathfrak{M}} = \{1\}$.

(b) $\|p_i\|^{\mathfrak{M}} = V(p_i)$.

(c) $\|\alpha \wedge \beta\|^{\mathfrak{M}} = \|\alpha\|^{\mathfrak{M}} \cap \|\beta\|^{\mathfrak{M}}$.

(d) $\|\alpha \vee \beta\|^{\mathfrak{M}} = \|\alpha\|^{\mathfrak{M}} + \|\beta\|^{\mathfrak{M}}$.

(e) $\|\neg\alpha\|^{\mathfrak{M}} = (\|\alpha\|^{\mathfrak{M}})^*$.

We need the following lemma to show the soundness.

Lemma 3.9 Let $\mathfrak{M}$ be a semi-orthomodel and α a formula. Then $\|\alpha\|^{\mathfrak{M}}$ is a filter of X.
Proof : We will prove this by induction on the construction of α. The cases $\alpha := \perp,\ p_i,\ \beta \wedge \gamma$ are trivial. For the case $\alpha := \beta \vee \gamma$, it is also easy to show by rewriting the condition like (d) above. For the case $\alpha := \neg\beta$, it is straightforward by the properties of the relation $\perp$. □

Let Γ be a non-empty set of formulas and α a formula. Γ *implies* α in an semi-orthomodel $\mathfrak{M}$ ($\mathfrak{M} : \Gamma \models \alpha$)if for any $x \in X$, either there exists a formula β such that not $\mathfrak{M} \models_x \beta$, or else $\mathfrak{M} \models_x \alpha$. For an semi-orthoframe $\mathcal{F}$, Γ $\mathcal{F}$-*implies* α ($\mathcal{F} : \Gamma \models \alpha$) if $\mathfrak{M} : \Gamma \models \alpha$ for all semi-orthomodel $\mathfrak{M}$ on $\mathcal{F}$. For a class $\mathfrak{C}$ of semi-orthoframes, Γ $\mathfrak{C}$-*implies* α ($\mathfrak{C} : \Gamma \models \alpha$) if $\mathcal{F} : \Gamma \models \alpha$ for all frames $\mathcal{F}$ in $\mathfrak{C}$.
Let ϑ be the class of all semi-orthoframes. Under these settings of semi-orthomodels and the notion of validity, the following holds.

Theorem 3.10 (Soundness for $\mathbf{OL}^{(-)}$) If $\Gamma \vdash_{\mathbf{OL}^{(-)}} \alpha$, then $\vartheta : \Gamma \models \alpha$.
Proof : Almost trivial. We will show only a several cases of axiom schemes and inference rules. Take any semi-orthomodel $\mathfrak{M}$ and any point x in $\mathfrak{M}$.

Axiom schemes

(Ax2): $\alpha \vdash \neg\neg\alpha$
Suppose that $\mathfrak{M} \models_x \alpha$, and take any y such that $\mathfrak{M} \models_y \neg\alpha$. Then we have $x \perp y$, which implies that $\mathfrak{M} \models_x \neg\neg\alpha$.

(Ax5): $\alpha \vdash \alpha \vee \beta$
Suppose that $\mathfrak{M} \models_x \alpha$. Then, since we have $\mathfrak{M} \models_1 \beta$ and $x \cap 1 \leq x$, $\mathfrak{M} \models_x \alpha \vee \beta$ holds.

(Ax7): $\alpha \wedge \neg\alpha \vdash \beta$

Suppose $\mathfrak{M} \models_x \alpha \wedge \neg\alpha$. Then this means that $\mathfrak{M} \models_x \alpha$ and $\mathfrak{M} \models_x \neg\alpha$, which implies $x \perp x$. Then we have $x = 1$. Therefore $\mathfrak{M} \models_x \beta$.

Inference Rules

(R3): $\dfrac{\alpha \vdash \gamma \quad \beta \vdash \gamma}{\alpha \vee \beta \vdash \gamma}$

Suppose $\mathfrak{M} : \alpha \models \gamma$, $\mathfrak{M} : \beta \models \gamma$ and $\mathfrak{M} \models_x \alpha \vee \beta$. Then there exist $y, z \in X$ such that $\mathfrak{M} \models_y \alpha$, $\mathfrak{M} \models_z \beta$ and $y \cap z \leq x$. The first implies that $\mathfrak{M} \models_y \gamma$ and the second implies that $\mathfrak{M} \models_z \gamma$ by our supposition, and so, together with the third, we have that $\mathfrak{M} \models_x \gamma$.

(R4): $\dfrac{\alpha \vdash \beta}{\neg\beta \vdash \neg\alpha}$

Suppose $\mathfrak{M} : \alpha \models \beta$ and $\mathfrak{M} \models_x \neg\beta$. Take any $y \in X$ such that $\mathfrak{M} \models_y \alpha$. Then, this implies that $\mathfrak{M} \models_y \beta$ and hence, $x \perp y$. Therefore $\mathfrak{M} \models_x \neg\alpha$.

□

To prove the completeness, the *canonical model* for a semi-orthologic is needed. Lemma 3.4 (2) is the basic result for constructing its canonical model. Let $\mathbf{L}$ be a semi-orthologic. A subset y of formulas is $\mathbf{L}$-*theory* if it satisfies the following:

(1) $\neg\perp \in y$.

(2) $\alpha, \beta \in y$ implies $\alpha \wedge \beta \in y$.

(3) $\alpha \in y$ and $\alpha \vdash_{\mathbf{L}} \beta$ imply $\beta \in y$.

As is easily seen, any $\mathbf{L}$-theory is a filter in Φ. Then, the canonical model for the logic $\mathbf{L}$ is defined in the following way.

Definition 3.11 The *canonical model* for $\mathbf{L}$ is the structure $\mathfrak{M}_{\mathbf{L}} = \langle X_{\mathbf{L}}, \subseteq, \Cap, \Phi, \perp_{\mathbf{L}}, V_{\mathbf{L}} \rangle$, where

$X_{\mathbf{L}} = \{x \subseteq \Phi \mid x \text{ is an } \mathbf{L}\text{-theory}\}$.

$x \perp_{\mathbf{L}} y$ if and only if there exists α such that $\neg\alpha \in x$ and $\alpha \in y$.

$V_{\mathbf{L}}(p_i) = \{x \in X_{\mathbf{L}} \mid p_i \in x\}$.

∎

Here we want to make sure the following facts on $\mathbf{L}$-theories.

Proposition 3.12

(1) For a formula α, $y := \{\gamma \in \Phi \mid \alpha \vdash_{\mathbf{L}} \gamma\}$ is an $\mathbf{L}$-theory.

(2) Let α be a formula, and z an $\mathbf{L}$-theory such that $\neg\alpha \notin z$. Let $y := \{\gamma \in \Phi \mid \alpha \vdash_{\mathbf{L}} \gamma\}$. Then, for any formula σ, either $\neg\sigma \notin z$ or $\sigma \notin y$.

Proof : (1): It is obvious by some simple calculations.
(2): Take any formula σ and suppose that $\neg\sigma \in z$ and $\sigma \in y$. The latter means that $\alpha \vdash_{\mathbf{L}} \sigma$, which implies $\neg\sigma \vdash_{\mathbf{L}} \neg\alpha$. Since we suppose that $\neg\sigma \in z$, we deduce that $\neg\alpha \in z$, which leads a contradiction. □

Now we will proceed to prove the completeness theorem. First, we have to check that the canonical model for a semi-orthologic $\mathbf{L}$ is indeed a semi-orthomodel.

Lemma 3.13 $\mathfrak{M}_{\mathbf{L}}$ is a semi-orthomodel.
Proof : Clearly $\langle X_{\mathbf{L}}, \subseteq, \Cap, \Phi\rangle$ is a meet-semilattice with the greatest element Φ. So we have to check the conditions for $\perp_{\mathbf{L}}$ for a semi orthogonality space and that $V_{\mathbf{L}}(p_i)$ is a filter of $X_{\mathbf{L}}$. For $x, y, z \in X_{\mathbf{L}}$,
(i): $x \perp_{\mathbf{L}} \Phi$ is obvious.
(ii):$\Phi \perp_{\mathbf{L}} \Phi$ is obvious. Suppose $x \perp_{\mathbf{L}} x$, then there exists a formula α such that $\alpha \in x$ and $\neg\alpha \in x$. This imply that $\alpha \wedge \neg\alpha \in x$, and so $\beta \in x$ for any formula β. Thus we have $x = \Phi$.
(iii): Suppose $x \perp_{\mathbf{L}} y$. Then there exists a formula α such that $\alpha \in x$ and $\neg\alpha \in y$. By (Ax2), the former implies that $\neg\neg\alpha \in x$, and so we have $y \perp_{\mathbf{L}} x$.
(iv): It is trivial that $x \perp_{\mathbf{L}} y$ and $y \subseteq z$ imply $x \perp_{\mathbf{L}} z$.
(v): Suppose $x \perp_{\mathbf{L}} y$ and $x \perp_{\mathbf{L}} z$. Then there exist formulas α, β such that $\alpha \in x$, $\neg\alpha \in y$ and $\beta \in x$, $\neg\beta \in z$. These imply $\alpha \wedge \beta \in x$ and $\alpha \vee \beta \in y \Cap z$, and the latter implies $\neg(\alpha \wedge \beta) \in x \Cap y$. Thus we have $x \perp_{\mathbf{L}} (y \Cap z)$.
On conditions for $V_{\mathbf{L}}(p_i)$, clearly $\Phi \in V_{\mathbf{L}}(p_i)$. Suppose $x \in V_{\mathbf{L}}(p_i)$ and $x \subseteq y$. Then $p_i \in x \subseteq y$, and so, $p_i \in y$. Thus $y \in V_{\mathbf{L}}(p_i)$. Finally, suppose $x, y \in V_{\mathbf{L}}(p_i)$. Then $p_i \in x$, $p_i \in y$, which imply that $p_i \in x \Cap y$. Thus $x \Cap y \in V_{\mathbf{L}}(p_i)$. Therefore, $V_{\mathbf{L}}(p_i)$ is a filter of $X_{\mathbf{L}}$ and $\mathfrak{M}_{\mathbf{L}}$ is a semi-orthomodel. □

Next is the key theorem for the completeness result.

Theorem 3.14 (Fundamental theorem for semi-orthologic) For a semi-orthologic $\mathbf{L}$, let $x \in X_{\mathbf{L}}$ and α a formula. Then $\mathfrak{M}_{\mathbf{L}} \models_x \alpha$ if and only if $\alpha \in x$.
Proof : We prove it by induction on the construction of α. The cases $\alpha := \perp, p_i, \beta \wedge \gamma$ are trivial. We will show the cases $\alpha := \beta \vee \gamma$ and $\alpha := \neg\beta$. Take any $x \in X_{\mathbf{L}}$
Case $\alpha := \beta \vee \gamma$: Suppose that $\mathfrak{M}_{\mathbf{L}} \models_x \beta \vee \gamma$. Then there exist $y, z \in X_{\mathbf{L}}$ such that $\mathfrak{M}_{\mathbf{L}} \models_y \beta$ and $\mathfrak{M}_{\mathbf{L}} \models_z \gamma$ and $y \Cap z \subseteq x$. Then by induction hypothesis, we have $\beta \in Y$ and $\gamma \in z$, which imply $\beta \vee \gamma \in y$ and $\beta \vee \gamma \in z$, and so, $\beta \vee \gamma \in x$. Conversely, suppose $\beta \vee \gamma \in x$. Put $y := \{\delta \in \Phi \mid \beta \vdash_{\mathbf{L}} \delta\}$ and $x := \{\delta \in \Phi \mid \gamma \vdash_{\mathbf{L}} \delta\}$. Then obviously $\beta \in y$ and $\gamma \in z$, which means $\mathfrak{M}_{\mathbf{L}} \models_y \beta$ and $\mathfrak{M}_{\mathbf{L}} \models_z \gamma$. On the other hand, If $\delta \in y \Cap z$, then we deduce

$\beta \vee \gamma \vdash_{\mathbf{L}} \delta$, and so $\delta \in x$, that is, $y \Cap z \subseteq x$. Therefore $\mathfrak{M}_{\mathbf{L}} \models_x \beta \vee \gamma$.
Case $\alpha := \neg\beta$: Suppose $\neg\beta \in x$. Take $y \in X_{\mathbf{L}}$ such that $\mathfrak{M}_{\mathbf{L}} \models_y \beta$. Then by induction hypothesis, $\beta \in y$, and so we have $x \perp_{\mathbf{L}} y$. Therefore $\mathfrak{M}_{\mathbf{L}} \models_x \neg\beta$. Conversely, suppose $\neg\beta \notin x$. Put $y := \{\gamma \in \Phi \mid \beta \vdash_{\mathbf{L}} \gamma\}$, then of course, $\beta \in y$. By Proposition 3.12 (2), for any formula σ, either $\neg\sigma \notin x$ or $\sigma \notin y$ holds. Then, for this y, $\mathfrak{M}_{\mathbf{L}} \models_y \beta$ and $x \not\perp_{\mathbf{L}} y$. Thus we have $\mathfrak{M}_{\mathbf{L}} \not\models_x \beta$. □

Corollary 3.15 Let $\mathbf{L}$ be a semi-orthologic, Γ a non-empty set of formulas, and α a formula. Then $\Gamma \vdash_{\mathbf{L}} \alpha$ if and only if $\mathfrak{M}_{\mathbf{L}} : \Gamma \models \alpha$.
Proof : Suppose $\Gamma \vdash_{\mathbf{L}} \alpha$. Then there exist formulas $\beta_1, \beta_2, \dots \beta_n$ such that $\beta_1 \wedge \cdots \wedge \beta_n \vdash_{\mathbf{L}} \alpha$. Take any x in $\mathfrak{M}_{\mathbf{L}}$ and suppose that $\mathfrak{M}_{\mathbf{L}} \models_x \gamma$ for any $\gamma \in \Gamma$. In particular, $\mathfrak{M}_{\mathbf{L}} \models_x \beta_i$ for $i = 1, \dots n$. Then by the previous theorem, we have $\beta_1, \beta_2, \dots \beta_n \in x$ and so, $\beta_1 \wedge \cdots \wedge \beta_n \in x$. because of our supposition, $\alpha \in x$, which means that $\mathfrak{M}_{\mathbf{L}} \models_x \alpha$. Thus we have shown that $\mathfrak{M}_{\mathbf{L}} : \Gamma \models \alpha$. Conversely, suppose that $\Gamma \not\vdash_{\mathbf{L}} \alpha$. Put $y := \{\gamma \in \Phi \mid \Gamma \vdash_{\mathbf{L}} \gamma\}$. Then $\alpha \notin y$. Thus, for any $\beta \in \Gamma$, $\beta \in y$ but $\alpha \notin y$, which means that $\mathfrak{M}_{\mathbf{L}} \models_y \beta$ for any $\beta \in \Gamma$, but $\mathfrak{M}_{\mathbf{L}} \not\models_y \alpha$. Therefore $\mathfrak{M}_{\mathbf{L}} : \Gamma \not\models \alpha$. □

We have come to show the completeness theorem for the semi-orthologic $\mathbf{OL}^{(-)}$ at last.

Theorem 3.16 (Completeness for $\mathbf{OL}^{(-)}$) Let ϑ be the class of all semi-orthomodel. If $\vartheta : \Gamma \models \alpha$, then $\Gamma \vdash_{\mathbf{OL}^{(-)}} \alpha$.
Proof : Consider $\mathfrak{M}_{\mathbf{OL}^{(-)}}$. Then this is indeed a semi-orthomodel. Therefore the theorem follows immediately from the previous corollary. □

3.4 Note

The fact that the variety of ortholattices admits Dedekind-MacNeille completion is well known result, and this can be strengthen by replacing ortholattices by orthoposets, which are bounded posets with the orthocomplement ([4], [6]). On the other hand, the completion by means of the dual space of ortholattices is used for the representation theorem of ortholattices, where a topology is introduced in order to restrict the range of the embedding η and make the map onto ([16]). The technique of restricting the range of the embedding by introducing a topology is a common way of representation of this kind. The most famous example is the representation of Boolean algebras by M.H. Stone [45].

For the variety of modular ortholattices, any kind of completion fails ([20]). The proof of this result is not so easy. The basic fact, which was shown by I. Kaplansky [27], is that every complete modular ortholattice is a *continuous geometry*.

The completion problem of the variety of orthomodular lattices is still open, though there are several partial results on that problem. ([7], [19], [44]).

There is an orthomodular lattice which do not admit Dedekind-MacNeille completion ([18]). The author found that any orthomodular lattice can be embedded into a complete semi-orthomodular lattice by introducing the dual space like a semi-orthogonality space. The completion problem of the variety of orthomodular lattices will be discussed in Chapter 7.

4 Predicate extensions of the orthologic OL and the semi-orthologic $\mathbf{OL}^{(-)}$

As seen in the previous chapter, the varieties $\mathcal{OL}^{(-)}$, $\mathcal{OL}$ and $\mathcal{BA}$ admit the Dedekind-MacNeille completion. One of the important features of this technique is that the embedding map preserves all existing meets and joins in the original structure, which brings us the possibility of extending these propositional logics that correspond to the varieties to predicate logics. In this chapter, we discuss the predicate extensions of the logics $\mathbf{OL}^{(-)}$ and $\mathbf{OL}$, especially, we explain the former predicate calculus precisely. The minimum predicate extension $\mathbf{P}(\mathbf{OL}^{(-)})$ of the smallest semi-orthologic $\mathbf{OL}^{(-)}$ is obtained syntactically by adding the axioms and rules for quantification to the propositional system. We adopt the style of predicate logic with the *equality symbol.* Several syntactical tools are needed to deal with quantification, as usual. The algebraic semantics of the minimum predicate semi-orthologic is based on the *complete semi-ortholattices.*

4.1 Syntax of the predicate semi-orthologic with equality

The first order language $\mathcal{L}$ consists of the following symbols:

- a denumerable set $\mathbf{V}$ of variable symbols: $x, y, z, x_1, x_2, ...$
- a denumerable set of predicate symbols: $P, Q, R, ...$
- the equality symbol: $\doteq$
- connectives: $\wedge, \vee, \neg, \bot, \forall, \exists$
- parentheses: (,)

Since $\mathcal{L}$ has no function symbols, the set Term($\mathcal{L}$) of all *terms* of $\mathcal{L}$ is just the set $\mathbf{V}$. The set Form($\mathcal{L}$) of all formulas on the language $\mathcal{L}$ is defined by the following formation rules:

(1) $\bot$ is a formula.

(2) For any terms $t_1, t_2, \ldots, t_n, t, s$, and any predicate symbol P of *arity* n, $P(t_1, t_2, \ldots, t_n)$ and $(t \doteq s)$ are formulas.

(3) If α and β are formulas and x a variable, then so are $(\neg\alpha)$, $(\forall x\alpha)$, $(\exists x\alpha)$, $(\alpha \wedge \beta)$, and $(\alpha \vee \beta)$.

For a formula α, FV(α) denotes the set of all free variables which occur in α. A formula α, in which free variables x, y, z occur, for example, is sometimes denoted by $\alpha(x, y, z)$. For a non-empty set $\mathbf{M}$, let $\mathcal{L}_{\mathbf{M}}$ be the extended language obtained from $\mathcal{L}$ by adding all names of elements in $\mathbf{M}$. In this case, Form($\mathcal{L}_{\mathbf{M}}$) is obtained by the same formation rules above, where Term($\mathcal{L}_{\mathbf{M}}$) consists of all variables and all names of elements in $\mathbf{M}$. Sent$_{\mathbf{M}}$ denotes the set of all sentences on $\mathcal{L}_{\mathbf{M}}$, where a *sentence* means a formula which has no free variables.

Before introducing the notion of *substitution*, we mention a simple technicality here: Since all variables can be enumerated, and the number of variables which occur in a formula is finite, it is possible to choose a fresh variable for a formula uniquely, i.e., to take the smallest-numbered variable of all the fresh ones for the formula.

Substitution operation $[x/t]$, in a formula α, of a term t for a variable x is defined as follows:

If x does not occur free in α, then $\alpha[x/t]$ is α itself.

If x occurs free in α, then

- if t is a name a of an element in $\mathbf{M}$, then $\alpha[x/a]$ is a formula which is obtained by substituting a for each free occurrence of x in α.
- if t is a variable y, and x does not occur in a scope of $\forall y$ in α, then $\alpha[x/y]$ is obtained by substituting y for each free occurrence of x in α.
- if t is a variable y, and x occurs in a scope of $\forall y$ in α, then $\alpha[x/y]$ is obtained, first replacing each bound occurrence of y by a fresh variable for α, and then substituting y for each free occurrence of x.

Definition 4.1 (Minimum predicate semi-orthologic with equality) The *minimum predicate orthologic* (with equality) $\mathbf{P}(\mathbf{OL}^{(-)})$ on $\mathcal{L}$ is the smallest subset of the product Form($\mathcal{L}$) $\times$ Form($\mathcal{L}$) (we write $\alpha \vdash \beta$ to mean the pair $\langle \alpha, \beta \rangle \in \mathbf{P}(\mathbf{OL}^{(-)})$) which includes the following axioms and is closed under the following inference rules:

Axioms:

(Ax1),...(Ax8) are the same as in Definition 2.8.

(Ax9)

$$\alpha \vdash x \doteq x$$

(Ax10)

$$x \doteq y \vdash y \doteq x$$

(Ax11)

$$P \wedge (x \doteq y) \vdash P[x/y]$$

(Ax12)

$$\forall x \alpha(x) \vdash \alpha(x)[x/y]$$

(Ax13)

$$\alpha(x)[x/y] \vdash \exists x \alpha(x)$$

Inference Rules:

(R1), (R2), (R3) and (R4) are the same as in Definition 2.8.

(R5)
$$\frac{\alpha \vdash \beta(x)}{\alpha \vdash \forall x \beta(x)}$$
(R6)
$$\frac{\beta(x) \vdash \alpha}{\exists x \beta(x) \vdash \alpha}$$

where x is not free in α in both rules.

In the above axioms and rules, α, β denote any formulas, P any atomic formula, and $\alpha(x), \beta(x)$ any formulas such that x occurs free in α and β. ∎

4.2 Semantics for $\mathbf{P}(\mathbf{OL}^{(-)})$ and completeness

As a semantical model for the minimum predicate semi-orthologic $\mathbf{P}(\mathbf{OL}^{(-)})$, here is introduced a structure, consisting of a complete semi-ortholattice and an assignment of formulas to elements in this lattice.

Definition 4.2 (Model for $\mathbf{P}(\mathbf{OL}^{(-)})$) A *model* (for $\mathcal{L}$) is a structure $\mathfrak{M} = \langle \mathfrak{L}, \mathbf{M}, [[\cdot]]_{\mathfrak{M}} \rangle$, where

- $\mathfrak{L} = \langle \mathbf{L}, \bigcap, \bigcup, ', 0, 1 \rangle$ is a complete semi-ortholattice.
- $\mathbf{M}$ is a non-empty set. Let $\overline{\mathbf{M}}$ be the set of names of all elements in $\mathbf{M}$.
- $[[\cdot]]_{\mathfrak{M}}$ is a map from atomic sentences of $\mathcal{L}_{\mathbf{M}}$ to elements in $\mathfrak{L}$ which satisfies the following conditions: for any $a, b, a_1, ..., a_n, b_1, ..., b_n \in \overline{\mathbf{M}}$, and for any atomic formula $P(x_1, ..., x_n)$,

(1) $[[a \doteq a]]_{\mathfrak{M}} = 1$

(2) $[[a \doteq b]]_{\mathfrak{M}} = [[b \doteq a]]_{\mathfrak{M}}$

(3) $[[P(a_1, ..., a_n)]]_{\mathfrak{M}} \cap [[a_1 \doteq b_1]]_{\mathfrak{M}} \cap \cdots \cap [[a_n \doteq b_n]]_{\mathfrak{M}} \leq [[P(b_1, ..., b_n)]]_{\mathfrak{M}}$

(4) $[[\perp]]_{\mathfrak{M}} = 0$

The domain of $[[\cdot]]$ is extended to $\mathrm{Sent}_{\mathbf{M}}$ in the following inductive way:

$[[\alpha \wedge \beta]]_{\mathfrak{M}} = [[\alpha]]_{\mathfrak{M}} \cap [[\beta]]_{\mathfrak{M}}$ $\qquad$ $[[\forall x \alpha]]_{\mathfrak{M}} = \bigcap_{a \in \overline{\mathbf{M}}} [[\alpha[x/a]]]_{\mathfrak{M}}$

$[[\alpha \vee \beta]]_{\mathfrak{M}} = [[\alpha]]_{\mathfrak{M}} \cup [[\beta]]_{\mathfrak{M}}$ $\qquad$ $[[\exists x \alpha]]_{\mathfrak{M}} = \bigcup_{a \in \overline{\mathbf{M}}} [[\alpha[x/a]]]_{\mathfrak{M}}$

$[[\neg \alpha]]_{\mathfrak{M}} = ([[\alpha]]_{\mathfrak{M}})'$

∎

Definition 4.3 (Validity) Let $\mathfrak{M}$ be a model. For $\alpha, \beta \in \text{Form}(\mathcal{L})$, suppose $\text{FV}(\alpha) \uplus \text{FV}(\beta) = \{x_1, x_2, ..., x_n\}$.

(1) α *implies* β in $\mathfrak{M}$ ($\alpha \models_{\mathfrak{M}} \beta$) if for any $a_1, ..., a_2 \in \mathbf{M}$, $[[\alpha[x_1/a_1] \cdots [x_n/a_n]]]_{\mathfrak{M}} \leq [[\beta[x_1/a_1] \cdots [x_n/a_n]]]_{\mathfrak{M}}$ holds.

(2) α *implies* β ($\alpha \models \beta$) if $\alpha \models_{\mathfrak{M}} \beta$ for any model $\mathfrak{M}$.

∎

Theorem 4.4 (The soundness theorem for $\mathbf{P}(\mathbf{OL}^{(-)})$) For any $\alpha, \beta \in \text{Form}(\mathcal{L})$, if $\langle \alpha, \beta \rangle \in \mathbf{P}(\mathbf{OL}^{(-)})$, then $\alpha \models \beta$.

Proof : We will proceed by usual induction on the height of derivation. Let $\mathfrak{M} = \langle \mathfrak{L}, \mathbf{M}, [[\cdot]]_{\mathfrak{M}} \rangle$ be an arbitrary model.

1. For axioms: We will give the whole proofs only for the cases of (Ax3), (Ax11) and (Ax12). Other axioms can be treated similarly.

(Ax3): $\alpha \wedge \beta \vdash \alpha$.

Let $\text{FV}(\alpha) \cup \text{FV}(\beta) = \{x_1, ..., x_n\}$. For any $a_1, ..., a_n \in \overline{\mathbf{M}}$,

$$\begin{aligned} & [[(\alpha \wedge \beta)[x_1/a_1] \cdots [x_n/a_n]]]_{\mathfrak{M}} \\ = \; & [[\alpha[x_1/a_1] \cdots [x_n/a_n]]]_{\mathfrak{M}} \cap [[\beta[x_1/a_1] \cdots [x_n/a_n]]]_{\mathfrak{M}} \\ \leq \; & [[\alpha[x_1/a_1] \cdots [x_n/a_n]]]_{\mathfrak{M}} \end{aligned}$$

Thus $\alpha \wedge \beta \models_{\mathfrak{M}} \beta$ holds.

(Ax11): $P \wedge (x \doteq y) \vdash P[x/y]$. ($P$ is an atomic formula.)

If x does not occur free in P, this axiom is a special case of (Ax4), then we have just shown above. Suppose $\text{FV}(P) = \{x, z_1, ..., z_n\}$. For any $a, b, c_1, ..., c_n \in \overline{\mathbf{M}}$, the substitution results of both handsides are,

$$\begin{aligned} & (P \wedge (x \doteq y))[x/a][y/b][z_1/c_1] \cdots [z_n/c_n] \\ \equiv \; & P[x/a][z_1/c_1] \cdots [z_n/c_n] \wedge (a \doteq b) \\ & (P[x/y])[x/a][y/b][z_1/c_1] \cdots [z_n/c_n] \\ \equiv \; & P[x/b][z_1/c_1] \cdots [z_n/c_n] \end{aligned}$$

Because of the condition (3) on $[[\cdot]]_{\mathfrak{M}}$, we have,

$$\begin{aligned} & [[P[x/a][z_1/c_1] \cdots [z_n/c_n]]]_{\mathfrak{M}} \cap [[a \doteq b]]_{\mathfrak{M}} \\ \leq \; & [[P[x/b][z_1/c_1] \cdots [z_n/c_n]]]_{\mathfrak{M}} \end{aligned}$$

Therefore $P \wedge (x \doteq y) \models_{\mathfrak{M}} P[x/y]$ holds.

(Ax12): $\forall x \alpha(x) \vdash \alpha(x)[x/y]$.

By the assumption, x occurs free in α. Let $\text{FV}(\alpha) = \{x, z_1, ..., z_n\}$.

Then $\mathrm{FV}(\forall x\alpha) = \{z_1, ..., z_n\}$ and $\mathrm{FV}(\alpha[x/y]) = \{y, z_1, ..., z_n\}$. For any $a, b, c_1, ..., c_n \in \overline{\mathbf{M}}$, the substitution results of both handsides are,

$$\begin{array}{rl} & (\forall x\alpha)[x/a][y/b][z_1/c_1]\cdots[z_n/c_n] \\ \equiv & (\forall x\alpha)[z_1/c_1]\cdots[z_n/c_n] \\ & (\alpha[x/y])[x/a][y/b][z_1/c_1]\cdots[z_n/c_n] \\ \equiv & \alpha[x/b][z_1/c_1]\cdots[z_n/c_n] \end{array}$$

Therefore,

$$\begin{array}{rl} & [\![(\forall x\alpha)[z_1/c_1]\cdots[z_n/c_n]]\!]_{\mathfrak{M}} \\ = & \bigcap_{d\in\overline{\mathbf{M}}} [\![\alpha[x/d][z_1/c_1]\cdots[z_n/c_n]]\!]_{\mathfrak{M}} \\ \leq & [\![\alpha[x/b][z_1/c_1]\cdots[z_n/c_n]]\!]_{\mathfrak{M}} \end{array}$$

Thus $\forall x\alpha(x) \models_{\mathfrak{M}} \alpha(x)[x/y]$ holds.

2. For inference rules: We give a proof only for the case of (R5). A similar argument works for other cases.

(R5):

$$\frac{\alpha \vdash \beta}{\alpha \vdash \forall x\beta} \qquad x \text{ is not free in } \alpha\text{, but } x \text{ occurs free in } \beta.$$

Let $\mathrm{FV}(\alpha) = \{y_1, ..., y_n\} \not\ni x$ and $\mathrm{FV}(\beta) = \{x, z_1, ..., z_m\}$. Then $\mathrm{FV}(\forall x\beta) = \{z_1, ..., z_m\}$. For any $a, b_1, ..., b_n, c_1, ..., c_m \in \overline{\mathbf{M}}$,

$$\begin{array}{rcl} [\![\alpha[x/a][y_1/b_1]\cdots[z_m/c_m]]\!]_{\mathfrak{M}} & = & [\![\alpha[y_1/b_1]\cdots[y_n/b_n]]\!]_{\mathfrak{M}} \\ [\![\forall x\beta[x/a][y_1/b_1]\cdots[z_m/c_m]]\!]_{\mathfrak{M}} & = & \bigcap_{d\in\overline{\mathbf{M}}} [\![\beta[x/d][z_1/c_1]\cdots[z_m/c_m]]\!]_{\mathfrak{M}} \end{array}$$

By induction hypothesis, we have

$$[\![\alpha[y_1/b_1]\cdots[y_n/b_n]]\!]_{\mathfrak{M}} \leq [\![\beta[x/a][z_1/c_1]\cdots[z_m/c_m]]\!]_{\mathfrak{M}}$$

Thus,

$$[\![\alpha[y_1/b_1]\cdots[y_n/b_n]]\!]_{\mathfrak{M}} \leq \bigcap_{d\in\overline{\mathbf{M}}} [\![\beta[x/d][z_1/c_1]\cdots[z_m/c_m]]\!]_{\mathfrak{M}}$$

Therefore $\alpha \models_{\mathfrak{M}} \forall x\beta$ holds.

□

Theorem 4.5 (The completeness theorem for $\mathbf{P}(\mathbf{OL}^{(-)})$) For any $\alpha, \beta \in \mathrm{Form}(\mathcal{L})$, if $\alpha \models \beta$, then $\langle\alpha, \beta\rangle \in \mathbf{P}(\mathbf{OL}^{(-)})$.

Proof : We will show this by applying the Lindenbaum construction of the canonical model for our logic $\mathbf{P}(\mathbf{OL}^{(-)})$.

Define a relation $\approx$ on Form($\mathcal{L}$) as follows: for $\alpha, \beta \in$ Form($\mathcal{L}$), $\alpha \approx \beta$ if and only if $\langle \alpha, \beta \rangle \in \mathbf{P}(\mathbf{OL}^{(-)})$ and $\langle \beta, \alpha \rangle \in \mathbf{P}(\mathbf{OL}^{(-)})$. Then, since the relation $\approx$ is indeed a congruence relation, the equivalence class $\|\alpha\| := \{\beta \in \text{Form}(\mathcal{L}) \mid \alpha \approx \beta\}$ and the operations $\cap$ as: $\|\alpha\| \cap \|\beta\| := \|\alpha \wedge \beta\|$, and $(\cdot)^{\perp}$ as: $\|\alpha\|^{\perp} := \|\neg\alpha\|$ are well defined on the quotient set $\|\mathcal{L}\| := \{\|\alpha\| \mid \alpha \in \text{Form}(\mathcal{L})\}$. Further, an order on $\|\mathcal{L}\|$ can be defined as: $\|\alpha\| \leq \|\beta\|$ if and only if $\langle \alpha, \beta \rangle \in \mathbf{P}(\mathbf{OL}^{(-)})$.

In this construction, the structure $\mathfrak{L}_0 = \langle \|\mathcal{L}\|, \cap, (\cup), ^{\perp}, \|\neg\bot\|, \|\bot\| \rangle$ turns out to be a semi-ortholattice, in which $\|\forall x \alpha(x)\|$ is the greatest lower bound of the set $\{\alpha(x)[x/y] \mid y \in \mathbf{V}\}$. It is clear that $\|\forall x \alpha(x)\|$ is a lower bound, for we have the axiom $\forall x \alpha(x) \vdash \alpha(x)[x/y]$. To show it is the greatest lower bound, suppose $\beta \vdash \alpha(x)[x/y]$ for all $y \in \mathbf{V}$. If $x \notin \text{FV}(\beta)$, then take x for y. As $\alpha(x)[x/x] \equiv \alpha(x)$, we can deduce,

$$\frac{\beta \vdash \alpha(x)}{\beta \vdash \forall x \alpha(x)}$$

Therefore we have $\beta \vdash \forall x \alpha(x)$. If $x \in \text{FV}(\beta)$, then take, for y, z which does not occur free in α and β, and does not occur bound in α. For $\alpha(x)[x/z][z/x] \equiv \alpha(x)$, we can deduce,

$$\frac{\dfrac{\beta \vdash \alpha(x)[x/z]}{\beta \vdash \forall z(\alpha(x)[x/z])} \quad \dfrac{\forall z(\alpha(x)[x/z]) \vdash \alpha(x)}{\forall z(\alpha(x)[x/z]) \vdash \forall x \alpha(x)}}{\beta \vdash \forall x \alpha(x)}$$

Thus we have $\beta \vdash \forall x \alpha(x)$.

By Theorem 3.5, $\mathfrak{L}_0$ can be embedded into a complete semi-ortholattice $\mathfrak{L}_1$, in which all the operations in $\mathfrak{L}_0$, including infinite meet and infinite join, are preserved. Consider a model $\mathfrak{N} = \langle \mathfrak{L}_1, \mathbf{V}, \|\cdot\| \rangle$. Note that every formula is a sentence in this model $\mathfrak{N}$ and that we may identify the set of all names of variables $\overline{\mathbf{V}}$ with the set of variables $\mathbf{V}$. We claim that this is indeed a model, i.e., it will be shown by induction that we can put $[[\alpha]]_{\mathfrak{N}} := \|\alpha\|$ for any formula α.

- If α is atomic, we can easily check that $\|\cdot\|$ satisfies the conditions (1),(2), (3) and (4) in Definition 4.2.

- If α is not atomic, we can show the following by simple calculation:

$$[[\alpha \wedge \beta]]_{\mathfrak{N}} = [[\alpha]]_{\mathfrak{N}} \cap [[\beta]]_{\mathfrak{N}} = \|\alpha\| \cap \|\beta\| = \|\alpha \wedge \beta\|$$

$$[[\neg\alpha]]_{\mathfrak{N}} = [[\alpha]]_{\mathfrak{N}}{}^{\perp} = (\|\alpha\|)^{\perp} = \|\neg\alpha\|$$

$$[[\forall x \alpha]]_{\mathfrak{N}} = \bigcap_{y \in \mathbf{V}} [[\alpha[x/y]]]_{\mathfrak{N}} = \bigcap_{y \in \mathbf{V}} \|\alpha[x/y]\| = \|\forall x \alpha\|$$

Therefore, $\|\cdot\|$ satisfies the conditions on $[\![\cdot]\!]_{\mathfrak{N}}$, and so $\mathfrak{N}$ is a (canonical) model for the logic $\mathbf{P}(\mathbf{OL}^{(-)})$.

Now we are in a position to prove the completeness result. Suppose $\alpha \models \beta$. Then for any model $\mathfrak{M}, \alpha \models_{\mathfrak{M}} \beta$. Take $\mathfrak{N}$ for $\mathfrak{M}$, then we have $\|\alpha\| \leq \|\beta\|$, which means $\langle \alpha, \beta \rangle \in \mathbf{P}(\mathbf{OL}^{(-)})$. □

4.3 The minimum predicate extension of OL

Since we have Theorem 3.6, we can construct an algebraic semantics for the minimum predicate extension $\mathbf{P}(\mathbf{OL})$ of the smallest orthologic $\mathbf{OL}$ in the same manner. Of course in this case, some of the axiom schemes and some of the inference rules are derivable due to the double negation law. As a model of this logic, we may take a complete ortholattice for $\mathfrak{L}$ in Definition 4.2. Then we can show the soundness and the completeness of the predicate logic $\mathbf{P}(\mathbf{OL})$ quite the same as Theorem 4.4 and 4.5.

4.4 Note

The formulation of the predicate calculus in this chapter and the construction of its models are based on a work by J.L.Bell [2]. It is only for simplicity and not essential that we adopt the formal system with the equality symbol and without function symbols.

5 Amalgamation and interpolation

Craig's interpolation property is a syntactical property of propositional and predicate logics. But there are sometimes semantic criteria to decide whether a given logic has this property or not. The most famous example is the amalgamation property of a class of Heyting algebras. By using this criterion, it can be proved that there are only 7 logics that have the Craig's interpolation property among the intermediate propositional logics, that is, logics between the classical logic and the intuitionistic logic ([34]).

For our propositional logics, however, the situation is a bit different, because our language do not have an implication symbol. Therefore Craig's interpolation property must be formulated by the deducibility symbol ($\vdash_{\mathbf{L}}$) instead of the implication symbol, and we have obtained, so far, only a sufficient algebraic condition for a logic to have the interpolation property. That is the *super amalgamation property* of a variety of algebras. To show that a variety of algebras have this property, the techniques for completion of varieties of algebras in Chapter 3 can be modifies and applied.

In this chapter, we will prove that the varieties $\mathcal{OL}$ and $\mathcal{OL}^{(-)}$ have the super amalgamation property, and hence, each logic which correspond to them respectively, has the Craig's interpolation property.

Then we will also show the Craig's interpolation property of the minimum predicate extension of the semi-orthologic $\mathbf{P}(\mathbf{OL}^{(-)})$ and that of the orthologic $\mathbf{P}(\mathbf{OL})$ by using quite similar argument.

5.1 Interpolation property and super amalgamation property

First, we introduce the notion of interpolation property of our propositional logics.

Definition 5.1 (Interpolation property) Let $\mathbf{L}$ be a semi-orthologic. $\mathbf{L}$ has the *interpolation property* (IP for short) if for any formulas α and β, $\langle \alpha, \beta \rangle \in \mathbf{L}$ implies that there exists a formula γ which satisfies the following conditions:

(1) $\langle \alpha, \gamma \rangle \in \mathbf{L}$, and $\langle \gamma, \beta \rangle \in \mathbf{L}$.

(2) Only these propositional variables which are common to α and β, or the constant $\perp$, occur in the formula γ.

∎

Any formula γ satisfying the above conditions (1) and (2) is sometimes called a *interpolant* of α and β. Next, we define a property of a class of algebras which will be shown to be a sufficient condition for a logic to have the interpolation property.

Definition 5.2 (Super amalgamation property) ([10])
A class of algebras $\mathcal{C}$ has the *amalgamation property* (AP for short), if for any $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{C}$ (let their universes be A_0, A_1, A_2 respectively), and any embeddings f_1, f_2 such that $f_1 : \mathfrak{A}_0 \to \mathfrak{A}_1$ and $f_2 : \mathfrak{A}_0 \to \mathfrak{A}_2$, there exist a $\mathfrak{A} \in \mathcal{C}$ and embeddings g_1, g_2 such that $g_1 : \mathfrak{A}_1 \to \mathfrak{A}$ and $g_2 : \mathfrak{A}_2 \to \mathfrak{A}$, and $g_1(f_1(a)) = g_2(f_2(a))$ holds for any $a \in A_0$. A class of algebras $\mathcal{C}$ has the *super amalgamation property* (SAP for short), if it has the AP and satisfies the following additional condition: For any $b \in A_i$ and $c \in A_j$ ($\{i, j\} = \{1, 2\}$), if $g_i(b) \leq g_j(c)$ in $\mathfrak{A}$, then there exists $d \in A_0$ such that $b \leq f_i(d)$ in $\mathfrak{A}_i$ and $f_j(d) \leq c$ in $\mathfrak{A}_j$ hold. ∎

In the above definition, *embeddings* mean the same as in Definition 3.1, and the algebra $\mathfrak{A}$, into which both algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are embedded, is called a *target algebra*. Remind that for any subvariety $\mathcal{V}$ of $\mathcal{OL}^{(-)}$, the smallest logic that corresponds to $\mathcal{V}$ is denoted by $\mathbf{L}(\mathcal{V})$. It can be shown that the super-amalgamation property of $\mathcal{V}$ is a sufficient condition for the logic $\mathbf{L}(\mathcal{V})$ to have the interpolation property. That is, the following lemma holds.

Lemma 5.3 If $\mathcal{V}$ has the SAP, then $\mathbf{L}(\mathcal{V})$ has the IP.
Proof : Consider formulas $\varphi = \varphi(p_1, \cdots, p_l, r_1, \cdots, r_n)$ constructed only from variables in $\{p_1, \cdots, p_l, r_1, \cdots, r_n\}$, and $\psi = \psi(q_1, \cdots, q_m, r_1, \cdots, r_n)$ constructed only from variables in $\{q_1, \cdots, q_m, r_1, \cdots, r_n\}$. Suppose there exists no formula $\chi = \chi(r_1, \cdots r_n)$ such that $\langle \varphi, \chi \rangle \in \mathbf{L}(\mathcal{V})$ and $\langle \chi, \psi \rangle \in \mathbf{L}(\mathcal{V})$. Then it is enough to show $\langle \varphi, \psi \rangle \notin \mathbf{L}(\mathcal{V})$. Let $\mathfrak{A}_0, \mathfrak{A}_1$ and $\mathfrak{A}_2$ be free $\mathcal{V}$-algebras generated by the sets $\{c_1, \cdots, c_n\}$, $\{a_1, \cdots, a_l, c_1, \cdots, c_n\}$ and $\{b_1, \cdots, b_m, c_1, \cdots, c_n\}$ respectively. Then, $\mathfrak{A}_0$ is embedded into both $\mathfrak{A}_1$ and $\mathfrak{A}_2$ by identity maps. So, by the SAP of $\mathcal{V}$, there exist an algebra $\mathfrak{A} \in \mathcal{V}$ and embeddings $g_1 : \mathfrak{A}_1 \to \mathfrak{A}$, $g_2 : \mathfrak{A}_2 \to \mathfrak{A}$ such that, $g_1(a) = g_2(a)$ for any $a \in A_0$ and that for any $b \in A_i$ and for any $c \in A_j$ ($i \neq j$), if $g_i(b) \leq g_j(c)$, then there exists $d \in A_0$ such that $b \leq d$ in $\mathfrak{A}_i$ and $d \leq c$ in $\mathfrak{A}_j$. Therefore by the assumption about the formulas φ and ψ, $g_1(\varphi(a_1, \cdots, a_l, c_1, \cdots, c_n)) \not\leq g_j(\psi(b_1, \cdots, b_m, c_1, \cdots c_n))$ in $\mathfrak{A}$. Define a valuation v into $\mathfrak{A}$ as:

$$v(p_i) = g_1(a_i) \qquad \text{for } i = 1, \cdots, l$$

$$v(q_j) = g_2(b_j) \qquad \text{for } j = 1, \cdots, m$$

$$v(r_k) = g_1(c_k) = g_2(c_k) \qquad \text{for } k = 1, \cdots, n.$$

Then, because g_1 and g_2 are homomorphisms, $v(\varphi) \not\leq v(\psi)$ in $\mathfrak{A}$. Therefore, $\langle \varphi, \psi \rangle \notin \mathbf{L}(\mathcal{V})$. □

This lemma is fundamental for the rest of this chapter. Below, we will give some examples of logics having the Craig's interpolation property, by applying this lemma. In fact, we will show that the four subvarieties of $\mathcal{OL}^{(-)}$ have the super-amalgamation property.

5.2 Super amalgamation property of some varieties

To show that a class $\mathcal{C}$ of algebras has the super amalgamation property, here we take the following way: Take arbitrary $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{C}$ (with their universes A_0, A_1, A_2, respectively), and suppose that there are embeddings $f_1 : \mathfrak{A}_0 \to \mathfrak{A}_1$ and $f_2 : \mathfrak{A}_0 \to \mathfrak{A}_2$. Without loss of generality, we can suppose $\mathfrak{A}_0$ is a subalgebra of $\mathfrak{A}_1$ and $\mathfrak{A}_2$, i.e., f_1 and f_2 may be taken to be identity functions: for any $x \in A_0$, $f_1(x) = f_2(x) = x$.

Put $B := A_1 \uplus A_2$ and define an order $\preceq$ on B as follows: for $x, y \in B$,

$$x \preceq y \Leftarrow \text{def} \Rightarrow \begin{cases} x \le y \ \text{ in } A_i & \text{if } x, y \in A_i. \\ \exists z \in A_0 \ \text{ s.t. } x \le z \text{ in } A_i,\ z \le y \text{ in } A_j & \text{if } x \in A_i \text{ and } y \in A_j, \\ & \text{where } \{i, j\} = \{1, 2\}. \end{cases}$$

It is easily checked that this $\preceq$ is an order on B. A unary operation $(\cdot)'$ on B is inherited from each $(\cdot)'$ on A_i. Finally let $\mathfrak{B} := \langle B, \preceq, ' \rangle$. This algebraic structure is not always a lattice, because it is not guaranteed that there exists the greatest lower bound$(x \cap y)$, and the least upper bound $(x \cup y)$ in B for any given $x, y \in B$. This is a kind of partially ordered set with some properties which depends on the original class $\mathcal{C}$. We call this $\mathfrak{B}$ a $\mathcal{C}$-*poset*.

If this $\mathcal{C}$-poset can be embedded into a $\mathcal{C}$-algebra, then we have that the class $\mathcal{C}$ has the super-amalgamation property.

5.2.1 The variety $\mathcal{OL}^{(-)}$

First, we consider the case of semi-ortholattices. Let $\mathfrak{B} := \langle B, \preceq, ' \rangle$ be a $\mathcal{OL}^{(-)}$-poset. It is easily seen that the following holds for this $\mathfrak{B}$. For $x, y \in B$,

(a) $x \preceq x''$.

(b) The infimum of x and x' (i.e. $x \cap x'$) exists in B and it equals 0.

(c) $x \preceq y$ implies $y' \preceq x'$.

Now, consider the Dedekind-MacNeille completion of $\mathfrak{B}$ and take $\mathfrak{C} = \langle DM(B), \Cap, \bigcup, (\cdot)^{\perp}, \{0\}, B \rangle$. Then for this algebra, quite the same proposition holds as Proposition 3.3, that is,

Proposition 5.4 For subsets $S, T \in DM(B)$ and an element $x \in A$,

(1) $(S^{\perp})^{ul} = S^{\perp}$.

(2) $S \Cap S^{\perp} = \{0\}$.

(3) $S \subseteq T$ implies $T^{\perp} \subseteq S^{\perp}$.

(4) $S \subseteq S^{\perp\perp}$.

(5) $(\downarrow x)^{\perp} = \downarrow x'$.

Therefore, this $\mathfrak{C}$ turns out to be a complete semi-ortholattice, and we take this $\mathfrak{C}$ for our target algebra $\mathfrak{A}$. The desired embeddings $\eta_i : \mathfrak{A}_i \to \mathfrak{A}$ $(i = 1, 2)$ are given by:

$$\eta_i(x) = \downarrow x \quad \text{for } x \in A_i$$

It is obvious that this map satisfies the conditions for the SAP. Thus, we have proved the following theorem for the variety $\mathcal{OL}^{(-)}$.

Theorem 5.5 The variety $\mathcal{OL}^{(-)}$ has the super-amalgamation property. □

5.2.2 The variety $\mathcal{OL}$

To prove the SAP for the variety $\mathcal{OL}$, it is enough to notice that a $\mathcal{OL}$-poset $\mathfrak{B}$ satisfies the same assumptions as $\mathcal{OL}^{(-)}$-posets together with $x'' \preceq x$, by which we can show that the Dedekind-MacNeille completion $DM(B)$ has the property $S = S^{\perp\perp}$. Of course, other part of the proof for $\mathcal{OL}^{(-)}$-posets can go through also for $\mathcal{OL}$-poset. Therefore the following theorem holds.

Theorem 5.6 The variety $\mathcal{OL}$ has the super-amalgamation property. □

5.3 Interpolation property of the P(OL$^{(-)}$) and P(OL)

As well as the propositional logics $\mathbf{OL}^{(-)}$ and **OL**, the interpolation property for the minimum predicate extension of these two logics, which are introduced in Chapter 4, can be shown by the quite similar argument.

Theorem 5.7 (The interpolation property of P(OL$^{(-)}$)) Let $\alpha, \beta \in$ Form($\mathcal{L}$). If $\langle \alpha, \beta \rangle \in \mathbf{P}(\mathbf{OL}^{(-)})$, then there exists $\gamma \in$ Form($\mathcal{L}$) which satisfies the following conditions:

(1) $\langle \alpha, \gamma \rangle \in \mathbf{P}(\mathbf{OL}^{(-)})$ and $\langle \gamma, \beta \rangle \in \mathbf{P}(\mathbf{OL}^{(-)})$.

(2) Only these predicate symbols which are common to α and β, or $\perp$, occur in the formula γ.

Proof : Consider two formulas $\alpha \equiv \alpha(P_1, ..., P_m, R_1, ..., R_\ell, \perp)$ and $\beta \equiv \beta(Q_1, ..., Q_n, R_1, ..., R_\ell, \perp)$, which are constructed from (atomic) predicate symbols $\{P_1, ..., P_m, R_1, ..., R_\ell, \perp\}$ and $\{Q_1, ..., Q_n, R_1, ..., R_\ell, \perp\}$ respectively. Note that all the symbols P_i, Q_j, R_k are distinct from each other. Although each predicate symbol has its own arity, and some finite number (possibly 0) of variables occur in it, here we omit them, to avoid an overloaded notation. Suppose that there exists no formula $\gamma \equiv \gamma(R_1, ..., R_\ell, \bot)$, such that $\langle\alpha, \gamma\rangle \in \mathbf{P}(\mathbf{OL}^{(-)})$ and $\langle\gamma, \beta\rangle \in \mathbf{P}(\mathbf{OL}^{(-)})$. We have only to prove that $\langle\alpha, \beta\rangle \notin \mathbf{P}(\mathbf{OL}^{(-)})$. Construct three subalgebras of the Lindenbaum algebra as follows:

$\mathfrak{L}_0$ which is generated from $\{R_1, ..., R_\ell, \bot\}$.
$\mathfrak{L}_1$ which is generated from $\{P_1, ..., P_m, R_1, ..., R_\ell, \bot\}$.
$\mathfrak{L}_2$ which is generated from $\{Q_1, ..., Q_n, R_1, ..., R_\ell, \bot\}$.

For each generating subset, we take all the substitution instances of the relevant variables. Then $\mathfrak{L}_0$, $\mathfrak{L}_1$, and $\mathfrak{L}_2$ are semi-ortholattices, and $\mathfrak{L}_0$ is embedded into both $\mathfrak{L}_1$ and $\mathfrak{L}_2$. By the SAP of the variety of semi-ortholattices, there exist a complete semi-ortholattice $\mathfrak{L}$ and embeddings $g_1 : \mathfrak{L}_1 \to \mathfrak{L}$ and $g_2 : \mathfrak{L}_2 \to \mathfrak{L}$, and these satisfy some required conditions. With those conditions, we have $g_1(\|\alpha\|) \not\leq g_2(\|\beta\|)$. Take a model for the predicate logic $\mathbf{P}(\mathbf{OL}^{(-)})$ $\mathfrak{M} = \langle\mathfrak{L}, \mathbf{V}, [[\cdot]]_{\mathfrak{M}}\rangle$, where $[[\cdot]]_{\mathfrak{M}}$ maps every substitution instance of each atomic sentence $P_1, ..., R_\ell$ of $\mathcal{L}_{\mathbf{V}}$ to an element in $\mathfrak{L}$ in the following way.

$[[P_i]]_{\mathfrak{M}} = g_1(\|P_i\|)$ for $i = 1, ..., m$.
$[[Q_j]]_{\mathfrak{M}} = g_2(\|Q_j\|)$ for $j = 1, ..., n$.
$[[R_k]]_{\mathfrak{M}} = g_1(\|R_k\|) = g_2(\|R_k\|)$ for $k = 1, ..., \ell$.

Then, this $\mathfrak{M}$ is indeed a model for $\mathbf{P}(\mathbf{OL}^{(-)})$ and because g_1 and g_2 are homomorphisms, $[[\alpha]]_{\mathfrak{M}} \not\leq [[\beta]]_{\mathfrak{M}}$. Thus $\langle\alpha, \beta\rangle \notin \mathbf{P}(\mathbf{OL}^{(-)})$. □

The quite the same proof also works for the predicate logic $\mathbf{P}(\mathbf{OL})$ and it can be shown that the predicate logic $\mathbf{P}(\mathbf{OL})$ also has the interpolation property of the following form.

Theorem 5.8 (The interpolation property of P(OL)) Let $\alpha, \beta \in \mathrm{Form}(\mathcal{L})$. If $\langle\alpha, \beta\rangle \in \mathbf{P}(\mathbf{OL})$, then there exists $\gamma \in \mathrm{Form}(\mathcal{L})$ which satisfies the following conditions:

(1) $\langle\alpha, \gamma\rangle \in \mathbf{P}(\mathbf{OL})$ and $\langle\gamma, \beta\rangle \in \mathbf{P}(\mathbf{OL})$.

(2) Only these predicate symbols which are common to α and β, or $\bot$, occur in the formula γ. □

5.4 Note

Algebraic equivalent conditions for a propositioanl logic to have the Craig's interpolation property have been intensively studied for some classes of nonclassical logics ([42],[30],[35]). Amalgamation property and super amalgamation property are typical as such conditions. In our case, the situation is a bit different, because Craig's interpolation property must be formulated without an implication symbol. But it may be possible that the super amalgamation property is an equivalent condition for a semi-orthologics to have the Craig's interpolation property. The question is how to prove the necessity. If it would be shown, then we could decide completely which logics between $\mathbf{OL}^{(-)}$ and **CL** have the Craig's interpolation property, like in the case of intermediate logics.

For a subvariety $\mathcal{V}$ of the variety of Heyting algebras, the Craig's interpolation property of the logic $\mathbf{L}(\mathcal{V})$ is equivalent to the amalgamation property of the variety $\mathcal{V}$, and moreover, the super amalgamation property follows from the amalgamation property in the case of a subvariety of the variety of Heyting algebras. This fact is fundamental in deciding 7 logics between the classical logic and the intuitionistic logic which have the interpolation property ([34]).

Completion technique is not always necessary for showing that a class $\mathcal{C}$ of algebras has the super amalgamation property, in other words, the target $\mathcal{C}$-algebra is not necessary to be a complete $\mathcal{C}$-algebra. However, for a variety $\mathcal{V}$ of algebras, if a completion technique works well for $\mathcal{V}$-poset as well as $\mathcal{V}$-algebra, then we can utilize the fact for proving that the minimum predicate extension of the logic $\mathbf{L}(\mathcal{V})$ also has the Craig's interpolation property, as seen in this Chapter.

The classical logic **CL** and its predicate extension have the Craig's interpolation property. But our proof method does not work well for the case of the variety $\mathcal{BA}$. It seems difficult to show that the Dedekind-MacNeille completion of a $\mathcal{BA}$-poset has the distributive law, since the proof of Theorem 3.7 depends highly on the fact that $\mathcal{BA}$ is a distributive lattice. Craig's interpolation property of **CL** and $\mathbf{P}(\mathbf{CL})$ is proved in a completely different way, for example, by a syntactical method.

6 Kripke-style semantics of orthomodular logics

In this chapter, we discuss Kripke-style semantics of orthomodular logics. For the smallest orthomodular logic **OML**, Goldblatt built a relational semantics (*quantum models*) and showed the completeness theorem for **OML** with respect to this semantics ([15]), as we have seen in Chapter 1. His method to construct *quantum models* is restricting the range of the valuation V of orthomodels in order to make the resulting models satisfy the orthomodular law.

Our approach is completely different. The construction of our semantics for orthomodular logics is based on a different type of representation theorem, in which non-empty set with two operations is employed, instead of some relations.

Here we will give a Kripke-style semantics of orthomodular logics and show the completeness theorem for any orthomodular logic which corresponds to a subvariety of $\mathcal{OML}$ with respect to this semantics. Furthermore, we will extend this semantics to one for infinitary orthomodular logics.

6.1 Semantics of orthomodular logics

6.1.1 Frames and models

Kripke-style semantics for propositional logics like the intuitionistic logic and modal logics has a close connection to a set-theoretic (Stone-like) representation theorem of algebras which correspond to these logics. There is a representation theorem also for orthomodular lattices by D.J.Foulis ([13]), although it is not set-theoretic. His representation technique is by using particular semigroups, called *Rickart * semigroups* ([12]). Here we adopt his method in order to characterize orthomodular logics in a universal form. First, we only introduce our several semantical tools, and their properties will be discussed in the following section.

Definition 6.1 (Orthomodular frame) $\mathcal{F} = \langle G, \cdot, *, \mathbf{0}\rangle$ is an *orthomodular frame*, if it satisfies the following conditions from (1) to (4).

(1) $\langle G, \cdot\rangle$ is a semigroup.

(2) The element $\mathbf{0} \in G$ satisfies that $x \cdot \mathbf{0} = \mathbf{0} \cdot x = \mathbf{0}$ for any $x \in G$. We call this $\mathbf{0}$ the *zero element*.

(3) $*$ is a unary operation which satisfies the following: for any $x, y \in G$, $(x \cdot y)^* = y^* \cdot x^*$ and $(x^*)^* = x$.

Before giving the condition (4), we introduce some notions. An element $e \in G$ is a *projection* if $e = e^* = e \cdot e$. The set of all projections in G is denoted by $P(G)$, i.e.

$$P(G) := \{e \in G \mid e = e^* = e \cdot e\}$$

For a non-empty subset M of G, the set $M^{(r)}$ of *right annihilators of* M and the set $M^{(\ell)}$ of *left annihilators of* M are defined by:

$$M^{(r)} := \{y \in G \mid x \cdot y = \mathbf{0} \text{ for any } x \in M\}$$

$$M^{(\ell)} := \{y \in G \mid y \cdot x = \mathbf{0} \text{ for any } x \in M\}$$

Then the condition (4) is given as follows:

(4) For any $x \in G$, there exists a projection $e \in P(G)$ such that the right annihilator of the singleton set $\{x\}$ can be expressed as:

$$\{x\}^{(r)} = e \cdot G \quad (:= \{e \cdot y \mid y \in G\}\ )$$

The above condition (4) will play one of the central roles in the characterization of propositional orthomodular logics and when we extend our semantics to infinitary logics, we generalize this condition somehow to fit our purpose.

We need one more technical notion. The condition (4) says that for any $x \in G$, there exists $e \in P(G)$ such that $\{x\}^{(r)} = e \cdot G$. But, we cannot assume, in general, that any $e \in P(G)$ can be represented as $\{x\}^{(r)} = e \cdot G$ for some $x \in G$. Now we say that e in $P(G)$ is *closed* if for this e, there exists an element $x \in G$ such that $e \cdot G = \{x\}^{(r)}$. The set of all closed projections is denoted by $P_c(G)$, i.e.,

$$P_c(G) := \{e \in P(G) \mid \exists x \in G, \quad e \cdot G = \{x\}^{(r)}\}$$

∎

We define models on the basis of orthomodular frames.

Definition 6.2 (Orthomodular model) $\mathfrak{M} = \langle \mathcal{F}, u\rangle = \langle G, \cdot, *, \mathbf{0}, u\rangle$ is a *orthomodular model on a frame* $\mathcal{F}$, if $\mathcal{F}$ is an orthomodular frame and u is a function which assigns to each propositional variable p_i a closed projection $u(p_i)$ of G. The notion of *truth* in an orthomodular model is defined inductively as follows: the symbol '$\mathfrak{M} \models_x \alpha$' is read as " a formula α is true at a point x in a model $\mathfrak{M}$".

(0) $\mathfrak{M} \models_x \bot$ if and only if $x = \mathbf{0}$

(1) $\mathfrak{M} \models_x p_i$ if and only if $x \in u(p_i) \cdot G$

(2) $\mathfrak{M} \models_x \alpha \wedge \beta$ if and only if $\mathfrak{M} \models_x \alpha$ and $\mathfrak{M} \models_x \beta$

(3) $\mathfrak{M} \models_x \neg\alpha$ if and only if for any $y \in G$, $\mathfrak{M} \models_y \alpha$ implies $y^* \cdot x = \mathbf{0}$

We denote $\mathfrak{M} \not\models_x \alpha$ if $\mathfrak{M} \models_x \alpha$ does not hold. ∎

For a formula α, we denote $\|\alpha\|^{\mathfrak{M}} := \{x \in G \mid \mathfrak{M} \models_x \alpha\}$. Then we can restate the above truth conditions in the following form:

(0) $\|\bot\|^{\mathfrak{M}} = \{\mathbf{0}\}$

(1) $\|p_i\|^{\mathfrak{M}} = u(p_i) \cdot G$

(2) $\|\alpha \wedge \beta\|^{\mathfrak{M}} = \|\alpha\|^{\mathfrak{M}} \cap \|\beta\|^{\mathfrak{M}}$

(3) $\|\neg\alpha\|^{\mathfrak{M}} = \{x \in G \mid y^* \cdot x = \mathbf{0}$ for any $y \in \|\alpha\|^{\mathfrak{M}}\}$

Next we define the notion of *validity* in our orthomodular models. Let Γ be a non-empty set of formulas and α a formula. For an orthomodular model $\mathfrak{M}$, we say "Γ *implies* α at x in $\mathfrak{M}$" (in symbol, $\mathfrak{M} : \Gamma \models_x \alpha$) if and only if $\mathfrak{M} \models_x \beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_n$ for some formulas $\beta_1, \beta_2, \cdots, \beta_n \in \Gamma$ implies $\mathfrak{M} \models_x \alpha$. We say "$\Gamma$ *implies* α in $\mathfrak{M}$" ($\mathfrak{M} : \Gamma \models \alpha$) if and only if $\mathfrak{M} : \Gamma \models_x \alpha$ holds at any x in the model $\mathfrak{M}$. For an orthomodular frame $\mathcal{F}$, we say "Γ *implies* α in $\mathcal{F}$" ($\mathcal{F} : \Gamma \models \alpha$)if and only if $\mathfrak{M} : \Gamma \models \alpha$ holds for any model $\mathfrak{M}$ on the frame $\mathcal{F}$. For a class $\mathcal{C}$ of orthomodular frames, we say "Γ *implies* α in $\mathcal{C}$" ($\mathcal{C} : \Gamma \models \alpha$) if and only if $\mathcal{F} : \Gamma \models \alpha$ holds for any member $\mathcal{F}$ in $\mathcal{C}$.

Now the relation between the provability of a logic and the validity in a class of frames are defined in the following way.

Definition 6.3 Let $\mathbf{L}$ be an orthomodular logic and let Φ be the set of all formulas. Let $\mathcal{C}$ be a class of orthomodular frames.

(1) $\mathcal{C}$ *characterizes* $\mathbf{L}$ if any formulas α, β, $\alpha \vdash_{\mathbf{L}} \beta$ is equivalent to $\mathcal{C} : \alpha \models \beta$.

(2) $\mathcal{C}$ *strongly characterizes* $\mathbf{L}$ if for any non-empty subset $\Gamma \subseteq \Phi$ and any formula α, $\Gamma \vdash_{\mathbf{L}} \alpha$ is equivalent to $\mathcal{C} : \Gamma \models \alpha$.

∎

In Section 6.1 and 6.2, we will show that the smallest orthomodular logic **OML** has the strong characterization theorem of type (2) above, with respect to the class of all orthomodular frames.

6.1.2 Properties of orthomodular frames

In order to prove that our semantics is sound for the minimum orthomodular logic **OML**, we have to make use of some algebraic properties of orthomodular frames. We give some results on orthomodular frames in this section, according to Maeda's book [32]. Let $\mathcal{F} = \langle G, \cdot, *, \mathbf{0} \rangle$ be a fixed orthomodular frame throughout this section. First, we show that a unary operation $(\cdot)^{\perp}$

and a partially order relation $\leq$ can be naturally introduced in any orthomodular frame.

Proposition 6.4 For $e_1, e_2 \in P(G)$, if $e_1 \cdot G = e_2 \cdot G$, then $e_1 = e_2$, where $e \cdot G := \{e \cdot x \mid x \in G\}$

Proof : Since $e_1 \in e_2 \cdot G$, there exists an element $s \in G$ such that $e_1 = e_2 \cdot s$. Therefore $e_2 \cdot e_1 = e_2 \cdot e_2 \cdot s = e_2 \cdot s = e_1$. Similarly we have that $e_1 \cdot e_2 = e_2$. Thus we have $e_1 = {e_1}^* = (e_2 \cdot e_1)^* = {e_1}^* \cdot {e_2}^* = e_1 \cdot e_2 = e_2$. □

The next corollary follows from this proposition.

Corollary 6.5 For any $x \in G$, there exists the unique projection $e \in P(G)$ such that $\{x\}^{(r)} = e \cdot G$. □

This corollary enables us to get a correspondence between x and the unique projection e for x such that $\{x\}^{(r)} = e \cdot G$ as a function from G to $P(G)$. We denote this e by $x^{\perp}$. Note that the next two equations hold obviously. For any $x \in G$, we have $x^{\perp} \in P(G)$, and

$$\{x\}^{(r)} = x^{\perp} \cdot G$$

$$x \cdot x^{\perp} = x^{\perp} \cdot x^* = \mathbf{0}$$

Proposition 6.6 For any $e, f \in P(G)$, the following three conditions are equivalent: (1) $e \cdot f = e$ (2) $f \cdot e = e$ (3) $e \cdot G \subseteq f \cdot G$

Proof : $e \cdot f = e$ if and only if $(e \cdot f)^* = e^*$ if and only if $f^* \cdot e^* = e^*$ if and only if $f \cdot e = e$. Thus (1) and (2) are equivalent. To show (2) implies (3), take $p \in e \cdot G$. Then there exists $s \in G$ such that $p = e \cdot s$. Therefore $p \in f \cdot e \cdot G \subseteq f \cdot G$. Then, (3) implies (2), because $e \in e \cdot G \subseteq f \cdot G$, and thus, there exists $t \in G$ such that $e = f \cdot t$. Therefore $f \cdot e = f \cdot f \cdot t = f \cdot t = e$. □

By Propositions 6.4 and 6.6, we can define a partially order relation $\leq$ on $P(G)$ as follows: $e, f \in P(G)$,

$$e \leq f \quad \text{if and only if} \quad e \cdot f = e$$

The elements $\mathbf{0}$ and $\mathbf{0}^{\perp}$ have a special role in an orthomodular frame. The next proposition shows this.

Proposition 6.7

(1) Both $\mathbf{0}$ and $\mathbf{0}^{\perp}$ are projections.

(2) In $P(G)$, $\mathbf{0}$ is the least element, whereas $\mathbf{0}^{\perp}$ is the greatest element with respect to the order $\leq$.

Proof : (1): By the definition, it is clear that $\mathbf{0}^{\perp}$ is a projection. Also by the definition, $\mathbf{0} \cdot \mathbf{0} = \mathbf{0}$. Thus we have only to show that $\mathbf{0}^* = \mathbf{0}$. By operating

$*$ to both sides of $\mathbf{0} \cdot \mathbf{0}^* = \mathbf{0}$, we have $\mathbf{0} = \mathbf{0}^* \cdot \mathbf{0} = \mathbf{0}^*$.
(2): By the definition, $\mathbf{0} \cdot x = x \cdot \mathbf{0} = \mathbf{0}$ holds for any $x \in G$. This implies that $\mathbf{0}$ is the least element among all projections. As for the element $\mathbf{0}^\perp$, we can prove that $x \cdot \mathbf{0}^\perp = \mathbf{0}^\perp \cdot x = x$ for any $x \in G$. Indeed, note that $\{\mathbf{0}\}^{(r)} = \mathbf{0}^\perp \cdot G = G$. So, for any $x \in G$, there exists $s \in G$ such that $x = \mathbf{0}^\perp \cdot s$. Therefore $\mathbf{0}^\perp \cdot x = \mathbf{0}^\perp \cdot \mathbf{0}^\perp \cdot s = \mathbf{0}^\perp \cdot s = x$ holds for any $x \in G$. Moreover, for any $y \in G$, $y \cdot \mathbf{0}^\perp = (y \cdot \mathbf{0}^\perp)^{**} = (\mathbf{0}^{\perp^*} \cdot y^*)^* = (y^*)^* = y$. Consequently, we have that $x \cdot \mathbf{0}^\perp = \mathbf{0}^\perp \cdot x = x$ for any $x \in G$. This implies that $\mathbf{0}^\perp$ is the greatest element among all projections. □

The above proof shows that $\mathbf{0}^\perp$ is the unit element of a given orthomodular frame. The following are properties of the unary operation $(\cdot)^\perp$.

Lemma 6.8 Let x, y be elements in G, and e, f projections. The operation $(\cdot)^\perp$ on G has the following properties.

(1) If $x \cdot e = \mathbf{0}$, then $e \leq x^\perp$. (2) $x^\perp \leq (y \cdot x)^\perp$.

(3) If $e \leq f$, then $f^\perp \leq e^\perp$. (4) $x = x \cdot x^{\perp\perp}$, $e \leq e^{\perp\perp}$.

(5) $x^\perp = x^{\perp\perp\perp}$. (6) If $e \cdot x = x \cdot e$, then $e^\perp \cdot x = x \cdot e^\perp$.

Proof : (1): Suppose $x \cdot e = \mathbf{0}$. Take $p \in e \cdot G$, then there exists $s \in G$ such that $p = e \cdot s$. Therefore $x \cdot p = x \cdot e \cdot s = \mathbf{0}$. This means that $p \in \{x\}^{(r)} = x^\perp \cdot G$. Hence $e \cdot G \subseteq x^\perp \cdot G$, and so $e \leq x^\perp$.
(2): Take $p \in x^\perp \cdot G$, then there exists $s \in G$ such that $p = x^\perp \cdot s$. Therefore $(y \cdot x) \cdot p = y \cdot x \cdot x^\perp \cdot s = \mathbf{0}$, which implies that $p \in \{y \cdot x\}^{(r)} = (y \cdot x)^\perp \cdot G$. Hence $x^\perp \cdot G \subseteq (y \cdot x)^\perp \cdot G$ and so $x^\perp \leq (y \cdot x)^\perp$.
(3): Suppose $e \leq f$, which is equivalent to $e \cdot f = e$. For $p \in f^\perp \cdot G$, there exists $s \in G$ such that $p = f^\perp \cdot s$. Then, $e \cdot p = e \cdot f \cdot f^\perp \cdot s = \mathbf{0}$. Therefore $p \in \{e\}^{(r)} = e^\perp \cdot G$. This means that $f^\perp \cdot G \subseteq e^\perp \cdot G$. Hence $f^\perp \leq e^\perp$.
(4): By definition, $\{x\}^{(r)} = x^\perp \cdot G$ and $\{x^\perp\}^{(r)} = x^{\perp\perp} \cdot G$ hold. From the fact that $x^\perp \cdot x^* = (x \cdot x^\perp)^* = \mathbf{0}^* = \mathbf{0}$, we have that $x^* \in x^{\perp\perp} \cdot G$. Therefore there exists $s \in G$ such that $x^* = x^{\perp\perp} \cdot s$. Thus $x = x^{**} = (x^{\perp\perp} \cdot s)^* = s^* \cdot x^{\perp\perp}$. Hence we have $x \cdot x^{\perp\perp} = s^* \cdot x^{\perp\perp} \cdot x^{\perp\perp} = s^* \cdot x^{\perp\perp} = x$. Take $x = e$ in particular, this means that $e \leq e^{\perp\perp}$.
(5): By (4), $x^\perp \leq x^{\perp\perp\perp}$ holds. Conversely, take $p \in \{x^{\perp\perp}\}^{(r)} = x^{\perp\perp\perp} \cdot G$. Then $x^{\perp\perp} \cdot p = \mathbf{0}$. By this, together with (4), we have $x \cdot p = x \cdot x^{\perp\perp} \cdot p = \mathbf{0}$, which means $p \in \{x\}^{(r)} = x^\perp \cdot G$. Hence $x^{\perp\perp\perp} \leq x^\perp$.
(6): Suppose $e \cdot x = x \cdot e$. Then $e \cdot (x \cdot e^\perp) = x \cdot e \cdot e^\perp$ holds. Thus $x \cdot e^\perp \in \{e\}^{(r)} = e^\perp \cdot G$. Therefore there exists $s \in G$ such that $x \cdot e^\perp = e^\perp \cdot s$. Hence we have $e^\perp \cdot x \cdot e^\perp = e^\perp \cdot e^\perp \cdot s = e^\perp \cdot s = x \cdot e^\perp$. On the other hand, by our assumption, $x^* \cdot e = e \cdot x^*$ holds. Thus $e \cdot (x^* \cdot e^\perp) = x^* \cdot e \cdot e^\perp = \mathbf{0}$. Thus $x^* \cdot e^\perp \in \{e\}^{(r)} = e^\perp \cdot G$. Therefore there exists $t \in G$ such that $x^* \cdot e^\perp = e^\perp \cdot t$. Hence we have $e^\perp \cdot x^* \cdot e^\perp = e^\perp \cdot e^\perp \cdot t = e^\perp \cdot t = x^* \cdot e^\perp$. By operating $*$ to

both sides, we have $e^\perp \cdot x \cdot e^\perp = e^\perp \cdot x$. Therefore $x \cdot e^\perp = e^\perp \cdot x$. □

For closed projections, the following characterization lemma holds.

Lemma 6.9 For any $e \in P(G)$, $e \in P_c(G)$ if and only if $e^{\perp\perp} = e$.
Proof : If $e \in P_c(G)$, then there exists $x \in G$ such that $e = x^\perp$. Therefore, by (4) of the previous lemma, we have $e^{\perp\perp} = x^{\perp\perp\perp} = x^\perp = e$. Conversely, suppose $e^{\perp\perp} = e$. Then $\{e^\perp\}^{(r)} = e^{\perp\perp} \cdot G = e \cdot G$ holds. Thus $e \in P_c(G)$. □

Corollary 6.10 Both $\mathbf{0}$ and $\mathbf{0}^\perp$ are closed projections.
Proof : By Lemma 6.8 (5) and Lemma 6.9, it is obvious that $\mathbf{0}^\perp$ is closed. As for the projection $\mathbf{0}$, it suffices to show that $\mathbf{0}^{\perp\perp} = \mathbf{0}$. To show this, we will prove that $\mathbf{0}^{\perp\perp} \cdot G = \{\mathbf{0}\}$. For any $x \in \mathbf{0}^{\perp\perp} \cdot G = \{\mathbf{0}^\perp\}^{(r)}$, $\mathbf{0}^\perp \cdot x = \mathbf{0}$ holds. But by Proposition 6.7 (2), we have $\mathbf{0}^\perp \cdot x = x$, and hence $x = \mathbf{0}$. Thus, $\mathbf{0}^{\perp\perp} \cdot G = \{\mathbf{0}\} = \mathbf{0} \cdot G$. By Proposition 6.4, $\mathbf{0}^{\perp\perp} = \mathbf{0}$. □

So far, we have discussed some operations in an orthomodular frame, i.e., the unary operation $(\cdot)^\perp$ and the 0-ary operations (constants) $\mathbf{0}$ and $\mathbf{0}^\perp$. The rest of this section is devoted to explain about a binary operation $\sqcap$. Our purpose below is to prove the following theorem.

Theorem 6.11 For a *finite* subset $H \subseteq G$, there exists an element $e \in P(G)$ such that $H^{(r)} = e \cdot G$. □

Let $H = \{x_1, x_2, \ldots, x_n\}$. By the frame condition, for each x_i, there exists $e_i \in G$ such that $\{x_i\}^{(r)} = e_i \cdot G$. Then it is easy to show that $H^{(r)} = \bigcap_{i=1}^n \{x_i\}^{(r)} = \bigcap_{i=1}^n e_i \cdot G$. In order to show the above theorem, it is enough to consider only the case that H is a two-element subset in order to prove this theorem. It is likely that the condition (4) of Definition 6.1 is properly weaker than the above theorem, but the theorem shows that they are *equivalent*, that is, if we assume only the one-element version, then we can obtain the finitely-many-element version of that kind of condition. This is the point of orthomodular frames. On the other hand, there is a real gap between the finite version and the infinite version, the latter will appear in the semantics of infinitary orthomodular logics. For the proof of Theorem 6.11, we must prepare some propositions and lemmas first.

Proposition 6.12 For any $x \in G$, there exists a projection $f \in P(G)$ such that $\{x\}^{(\ell)} = G \cdot f$ $(:= \{y \cdot f \mid y \in G\})$.
Proof : By the frame condition, for x^*, there exists $f \in P(G)$ such that $\{x^*\}^{(r)} = f \cdot G$. We show that this f satisfies the condition of this proposition. Take $p \in G \cdot f$, then there exists $s \in G$ such that $p = s \cdot f$. Since we have $x^* \cdot f = \mathbf{0}$, $f \cdot x = \mathbf{0}$ holds, and so $p \cdot x = s \cdot f \cdot x = \mathbf{0}$. Hence $p \in \{x\}^{(\ell)}$. Conversely, take $q \in \{x\}^{(\ell)}$, then $q \cdot x = \mathbf{0}$, and so $x^* \cdot q^* = \mathbf{0}$. Therefore

$q^* \in f \cdot G$, which means that there exists $t \in G$ such that $q^* = f \cdot t$. Hence $q = t^* \cdot f \in G \cdot f$. Consequently $\{x\}^{(\ell)} = G \cdot f$. □

We can show by the similar argument as for Proposition 6.4 that the element f in the above proposition is uniquely determined. It is also clear that the above f can be written also in the following form, that is: $f = (x^*)^\perp$. For a subset $M \subseteq G$, we abbreviate $M^{(\ell)(r)}$ for $(M^{(\ell)})^{(r)}$, and $M^{(r)(\ell)}$ for $(M^{(r)})^{(\ell)}$.

Proposition 6.13

(1) For any $e \in P_c(G)$, $e \cdot G = \{e\}^{(\ell)(r)}$ holds.

(2) For any $x \in G$, there exists $z \in P(G)$ such that $\{x\}^{(\ell)(r)} = \{z\}^{(r)}$ holds.

Proof : (1): By Lemma 6.9, $\{e^\perp\}^{(r)} = e \cdot G$ holds for $e \in P_c(G)$. For $p \in \{e\}^{(\ell)(r)}$, $q \cdot p = \mathbf{0}$ for any $q \in \{e\}^{(\ell)}$. We have $e^\perp \in \{e\}^{(\ell)}$ because $e^\perp \cdot e = (e \cdot e^\perp)^* = \mathbf{0}$. Then we can take $e^\perp$ for q, hence we have $e^\perp \cdot p = \mathbf{0}$. This means that $p \in \{e^\perp\}^{(r)} = e \cdot G$. For the converse, take $s \in e \cdot G$, then there exists $t \in G$ such that $s = e \cdot t$. Since for any $u \in \{e\}^{(\ell)}$, $u \cdot e = \mathbf{0}$ holds, we have $u \cdot s = u \cdot e \cdot t = \mathbf{0}$. Therefore $s \in \{e\}^{(\ell)(r)}$.
(2): By the previous proposition, for $x \in G$, there exists $f \in P(G)$ such that $\{x\}^{(\ell)} = G \cdot f$. Then, $\{x\}^{(\ell)(r)} = (G \cdot f)^{(r)} = \{f\}^{(r)}$. Thus we can take this f for z. □

The following lemma is crucial to prove Theorem 6.11.

Lemma 6.14 For any $e_1, e_2 \in P_c(G)$, there exists the element $x \in P_c(G)$ uniquely such that $e_1 \cdot G \cap e_2 \cdot G = x \cdot G$ holds.
Proof : We will show by three steps that it is enough to put $x = e_1 \cdot ({e_2}^\perp \cdot e_1)^\perp$.
(1): $e_1 \cdot ({e_2}^\perp \cdot e_1)^\perp \cdot G \quad \subseteq \quad e_1 \cdot G \cap e_2 \cdot G$
It is trivial that $e_1 \cdot ({e_2}^\perp \cdot e_1)^\perp \cdot G \subseteq e_1 \cdot G$. By Proposition 6.13 (1), $e_2 \cdot G = \{e_2\}^{(\ell)(r)}$ holds. Take $p \in e_1 \cdot ({e_2}^\perp \cdot e_1)^\perp \cdot G$, then there exists $s \in G$ such that $p = e_1 \cdot ({e_2}^\perp \cdot e_1)^\perp \cdot s$. On the other hand, we have $\{e_2\}^{(\ell)} = G \cdot {e_2}^\perp$ by Proposition 6.12. So, for any $q \in \{e_2\}^{(\ell)}$, there exists $t \in G$ such that $q = t \cdot {e_2}^\perp$. Therefore $q \cdot p = (t \cdot {e_2}^\perp) \cdot e_1 \cdot ({e_2}^\perp \cdot e_1)^\perp \cdot s = t \cdot ({e_2}^\perp \cdot e_1) \cdot ({e_2}^\perp \cdot e_1)^\perp \cdot s = \mathbf{0}$. Thus we have $p \in \{e_2\}^{(\ell)(r)} = e_2 \cdot G$, and so $e_1 \cdot ({e_2}^\perp \cdot e_1)^\perp \cdot G \subseteq e_2 \cdot G$. Hence $e_1 \cdot ({e_2}^\perp \cdot e_1)^\perp \cdot G \subseteq e_1 \cdot G \cap e_2 \cdot G$ holds.
(2): $e_1 \cdot G \cap e_2 \cdot G \quad \subseteq \quad e_1 \cdot ({e_2}^\perp \cdot e_1)^\perp \cdot G$
We have $e_2 \cdot G = \{e_2\}^{(\ell)(r)} = \{{e_2}^\perp\}^{(r)} \cdot G$ by Propositions 6.12 and 6.13. Take $p \in e_1 \cdot G \cap e_2 \cdot G$, then ${e_2}^\perp \cdot p = \mathbf{0}$ and there exists $s \in G$ such that $p = e_1 \cdot s$. Therefore ${e_2}^\perp \cdot e_1 \cdot p = {e_2}^\perp \cdot e_1 \cdot e_1 \cdot s = {e_2}^\perp \cdot e_1 \cdot s = {e_2}^\perp \cdot p = \mathbf{0}$. Thus $p \in \{{e_2}^\perp \cdot e_1\}^{(r)} = ({e_2}^\perp \cdot e_1)^\perp \cdot G$ holds. Moreover, since $e_1 \cdot p = e_1 \cdot e_1 \cdot s = e_1 \cdot s = p$, we have $p = e_1 \cdot p \in e_1 \cdot ({e_2}^\perp \cdot e_1)^\perp \cdot G$. Hence $e_1 \cdot G \cap e_2 \cdot G \subseteq e_1 \cdot ({e_2}^\perp \cdot e_1)^\perp \cdot G$.

So far, we have shown that $e_1 \cdot G \cap e_2 \cdot G = e_1 \cdot ({e_2}^\perp \cdot e_1)^\perp \cdot G$. Last, we have only to show that x is a closed projection.
(3): $e_1 \cdot ({e_2}^\perp \cdot e_1)^\perp \in P_c(G)$

For $x := e_1 \cdot ({e_2}^{\perp} \cdot e_1)^{\perp}$, we will prove that $\{x\}^{(\ell)(r)} = x \cdot G$. Take $p \in x \cdot G$, then there exists $s \in G$ such that $p = x \cdot s$. For any $q \in \{x\}^{(\ell)}$, since $q \cdot x = \mathbf{0}$, we have $q \cdot p = q \cdot x \cdot s = \mathbf{0}$, which means $p \in \{x\}^{(\ell)(r)}$. Thus $x \cdot G \subseteq \{x\}^{(\ell)(r)}$. To the converse, it is obvious that $\{e_1\}^{(\ell)} \subseteq \{x\}^{(\ell)}$, and so we have $\{x\}^{(\ell)(r)} \subseteq \{e_1\}^{(\ell)(r)}$. Also, by similar argument as in the proof of (1), $\{e_2\}^{(\ell)} \subseteq \{x\}^{(\ell)}$, and so $\{x\}^{(\ell)(r)} \subseteq \{e_2\}^{(\ell)(r)}$. Therefore $\{x\}^{(\ell)(r)} \subseteq \{e_1\}^{(\ell)(r)} \cap \{e_2\}^{(\ell)(r)}$. Hence we have that $\{x\}^{(\ell)(r)} = \{e_1\}^{(\ell)(r)} \cap \{e_2\}^{(\ell)(r)} = e_1 \cdot G \cap e_2 \cdot G = x \cdot G$. Then, by Proposition 6.13 (2), there exists $z \in G$ such that $x \cdot G = \{x\}^{(\ell)(r)} = \{z\}^{(r)}$. Thus we conclude that $x \in P_c(G)$. □

We denote $e_1 \sqcap e_2 := e_1 \cdot ({e_2}^{\perp} \cdot e_1)^{\perp}$. By the previous lemma, $e_1 \sqcap e_2$ turns out to be the greatest lower bound of $\{e_1, e_2\}$ with respect to the order $\leq$ on $P(G)$. Now we are in a position to show Theorem 6.11. Let the finite subset $H \subseteq G$ be $\{x_1, x_2, \dots x_n\}$. Then $e := x_1 \sqcap x_2 \sqcap \cdots \sqcap x_n$ satisfies the requirement of the theorem, and hence our proof is completed. Define $e_1 \sqcup e_2 := ({e_1}^{\perp} \sqcap {e_2}^{\perp})^{\perp}$ for $e_1, e_2 \in P_c(G)$. Because of Lemma 6.8 (3), it is easily seen that $e_1 \sqcup e_2$ is the least upper bound of $\{e_1, e_2\}$. For the binary operation $\sqcap$, the following lemma holds.

Lemma 6.15 Let $e, f \in P_c(G)$.

(1) If $e \cdot f = f \cdot e$, then $e \cdot f \in P_c(G)$ and $e \sqcap f = e \cdot f$.

(2) In general, $e \sqcap f = e \sqcap (f^{\perp} \cdot e)^{\perp} = e \cdot (f^{\perp} \cdot e)^{\perp}$.

Proof : (1): $(e \cdot f)^* = f^* \cdot e^* = f \cdot e \cdot e \cdot f = e \cdot f$ and $(e \cdot f) \cdot (e \cdot f) = e \cdot (e \cdot f) \cdot f = e \cdot f$. Thus $e \cdot f \in P(G)$. Obviously $e \cdot f \leq (e \cdot f)^{\perp\perp}$ by Lemma 6.8 (4). Conversely, we have $e^{\perp} \leq (e \cdot f)^{\perp}$ by Lemma 6.8 (2), then by (3), $(e \cdot f)^{\perp\perp} \leq e^{\perp\perp} = e$, which means that $e \cdot (e \cdot f)^{\perp\perp} = (e \cdot f)^{\perp\perp}$. Similar argument implies that $f \cdot (e \cdot f)^{\perp\perp} = (e \cdot f)^{\perp\perp}$. Therefore $(e \cdot f) \cdot (e \cdot f)^{\perp\perp} = e \cdot (e \cdot f)^{\perp\perp} = (e \cdot f)^{\perp\perp}$. This means that $(e \cdot f)^{\perp\perp} \leq e \cdot f$ and so, we have $(e \cdot f)^{\perp\perp} = e \cdot f$. Hence $e \cdot f \in P_c(G)$. Now we will show that $e \cdot f$ is the greatest lower bound of $\{e, f\}$. It is trivial that $e \cdot f \leq e$ and that $e \cdot f \leq f$. Take any $g \in P_c(G)$ such that $e \cdot g = g$ and that $f \cdot g = g$. Then $(e \cdot f) \cdot g = e \cdot g = g$, which means that $g \leq e \cdot f$. Thus we have that $e \sqcap f = e \cdot f$.
(2): We have $e \sqcap f = e \cdot (f^{\perp} \cdot e)^{\perp}$ by Lemma 6.14. Put $u := f^{\perp} \cdot e$, then $e^{\perp} \leq (f^{\perp} \cdot e)^{\perp} = u^{\perp}$ by Lemma 6.8 (2). So, $e^{\perp} \cdot u^{\perp} = u^{\perp} \cdot e^{\perp}$. This implies $e \cdot u^{\perp} = u^{\perp} \cdot e$ by Lemma 6.8 (6). Therefore by (1) we have $e \sqcap u^{\perp} = e \cdot u^{\perp} = e \cdot (f^{\perp} \cdot e)^{\perp} = e \sqcap f$. □

6.1.3 The soundness theorem

We have prepared some basic properties of orthomodular frames for proving the soundness theorem in the previous section. We need one more lemma on orthomodular models. We will establish it first and then show the soundness.

Lemma 6.16 Let $\mathfrak{M} = \langle \mathcal{F}, u \rangle = \langle G, \cdot, *, \mathbf{0}, u \rangle$ be an orthomodular model. Then, for any formula α, there exists the unique closed projection $e \in P_c(G)$ such that

$$\|\alpha\|^{\mathfrak{M}} := \{x \in G \mid \mathfrak{M} \models_x \alpha\} = e \cdot G$$

Proof : Uniqueness follows clearly from Proposition 6.4, once existence will be proved. So we will define $e(\alpha)$ for a formula α inductively as follows and it really fulfills the condition.

(0) $e(\bot) := \mathbf{0}$.

(1) $e(p_i) := u(p_i)$.

(2) $e(\alpha \wedge \beta) := e(\alpha) \sqcap e(\beta)$.

(3) $e(\neg\alpha) := (e(\alpha))^{\perp}$.

Taking account of the definition of truth condition and Lemma 6.14, it is trivial that $e(\alpha)$ defined above satisfies the condition of this lemma in the cases (0), (1), and (2). For the case (3), we will show that $\|\neg\alpha\|^{\mathfrak{M}} := \{x \in G \mid y^* \cdot x = \mathbf{0}$ for any $y \in \|\alpha\|^{\mathfrak{M}}\} = e(\alpha)^{\perp} \cdot G = \{e(\alpha)\}^{(r)}$. Take $p \in \|\neg\alpha\|^{\mathfrak{M}}$, then for any $q \in \|\alpha\|^{\mathfrak{M}}$, $q^* \cdot p = \mathbf{0}$ holds. By induction hypothesis, $e(\alpha) \in e(\alpha) \cdot G = \|\alpha\|^{\mathfrak{M}}$, and so $e(\alpha) \cdot p = e(\alpha)^* \cdot p = \mathbf{0}$. Thus we have $p \in \{e(\alpha)\}^{(r)} = e(\alpha)^{\perp} \cdot G$. Conversely, take $p \in \{e(\alpha)\}^{(r)}$, $e(\alpha) \cdot p = \mathbf{0}$ holds. For any $q \in \|\alpha\|^{\mathfrak{M}}$, by induction hypothesis, there exists $s \in G$ such that $q = e(\alpha) \cdot s$. Then $q^* \cdot p = s^* \cdot e(\alpha) \cdot p = \mathbf{0}$. Thus $p \in \|\neg\alpha\|^{\mathfrak{M}}$. We can conclude that $\|\neg\alpha\|^{\mathfrak{M}} = e(\alpha)^{\perp} \cdot G$.

□

Theorem 6.17 Let Γ be a non-empty set of formulas and α a formula and $\mathcal{C}$ be the class of all orthomodular frames. Then $\Gamma \vdash_{\mathbf{OML}} \alpha$ implies $\mathcal{C} : \Gamma \models \alpha$.
Proof : Take an arbitrary model $\mathfrak{M} \in \mathcal{C}$. By Lemma 6.16 there exists a map $e : \Phi \to P_c(G)$ such that for any formula α, $\|\alpha\|^{\mathfrak{M}} = e(\alpha) \cdot G$. Therefore we have $\mathfrak{M} : \alpha \models \beta$ if and only if $e(\alpha) \cdot G \subseteq e(\beta) \cdot G$ if and only if $e(\alpha) \leq e(\beta)$ for any formulas α and β. Thus, we have only to prove that for all the axiom schemes and inference rules of Q, $\alpha \vdash_Q \beta$ implies $e(\alpha) \leq e(\beta)$. This is almost trivial. We mention here only the case of (Ax8):$\alpha \wedge (\neg\alpha \vee (\alpha \wedge \beta)) \vdash \beta$. For this case, it is enough to show that $e \sqcap (e^{\perp} \sqcup (e \sqcap f)) \leq f$ holds for any $e, f \in P_c(G)$. Since $e \sqcap f \leq e$, we have $e \cdot (e \sqcap f) = (e \sqcap f) \cdot e$, then by Lemma 6.8 (6), $e \cdot (e \sqcap f)^{\perp} = (e \sqcap f)^{\perp} \cdot e$ holds. By Lemma 6.15 (1), we have $f \geq e \sqcap f = e \sqcap (e \sqcap f) = e \sqcap ((e \sqcap f)^{\perp} \cdot e)^{\perp} = e \sqcap ((e \sqcap f)^{\perp} \sqcap e)^{\perp} = e \sqcap ((e \sqcap f) \sqcup e^{\perp})$.
□

6.2 Canonical model and completeness theorem

To show that an orthomodular logic is complete with respect to our semantics, we will introduce how to construct the *canonical model* of a logic **L**. Our construction is carried out via *Lindenbaum construction.* Let **L** be an orthomodular logic which has been fixed through this chapter, and Φ the set of all formulas of our propositional language. A congruence relation $\equiv_{\mathbf{L}}$ on Φ is defined as:

$$\alpha \equiv_{\mathbf{L}} \beta \quad \text{if and only if} \quad \alpha \vdash_{\mathbf{L}} \beta \ \text{ and } \ \beta \vdash_{\mathbf{L}} \alpha$$

It is easily seen that $\equiv_{\mathbf{L}}$ is indeed a congruence relation. The equivalence class of each formula α is denoted by $[\alpha]$, that is, $[\alpha] := \{\beta \in \Phi \mid \alpha \equiv_{\mathbf{L}} \beta\}$, and the quotient set $\{[\alpha] \mid \alpha \in \Phi\}$ is denoted by $[\Phi]$. Since $\equiv_{\mathbf{L}}$ is a congruence relation, an order $\leq_{\mathbf{L}}$ and operations $\wedge$ and $\neg$ on Φ can be defined as follows:

$$[\alpha] \leq_{\mathbf{L}} [\beta] \quad \text{if and only if} \quad \alpha \vdash_{\mathbf{L}} \beta$$

$$[\alpha] \wedge [\beta] := [\alpha \wedge \beta] \quad \text{and} \quad \neg[\alpha] := [\neg\alpha]$$

Note that we use the same symbols also for operation on $[\Phi]$.

6.2.1 Canonical model construction

Definition 6.18 (Canonical frame and canonical model) The *canonical frame* $\mathcal{F}_{\mathbf{L}}$ for an orthomodular logic **L** is a structure of quadruple $\langle G(\mathbf{L}), \circ, *, \theta\rangle$ where:

(1) $G(\mathbf{L})$ is the set of *monotone* maps φ from $[\Phi]$ to $[\Phi]$ having the *residual map* $\varphi^\sharp$ for each $\varphi \in G(\mathbf{L})$. Here, monotonicity means that for any formulas α, β, $[\alpha] \leq_{\mathbf{L}} [\beta]$ implies $\varphi([\alpha]) \leq_{\mathbf{L}} \varphi([\beta])$, and the residual map $\varphi^\sharp$ for φ means a monotone map that satisfies $\varphi \circ \varphi^\sharp([\chi]) \leq_{\mathbf{L}} [\chi]$ and $\varphi^\sharp \circ \varphi([\chi]) \geq_{\mathbf{L}} [\chi]$ for any $[\chi] \in [\Phi]$, where $\circ$ is the composition operator for maps.

(2) $*$ is a unary operation on $G(\mathbf{L})$ defined as: $\varphi^*([\chi]) := \neg(\varphi^\sharp(\neg[\chi]))$ for any $\varphi \in G(\mathbf{L})$, where $[\chi]$ is a variable for an element in $[\Phi]$ as an input of map φ.

(3) θ is the *zero map*, that is $\theta([\chi]) = [\bot]$ holds for all $[\chi] \in [\Phi]$.

Clearly $\theta \in G(\mathbf{L})$. The *canonical model* for **L** is a structure $\mathfrak{M}_{\mathbf{L}} = \langle \mathcal{F}_{\mathbf{L}}, u_{\mathbf{L}}\rangle$, where $\mathcal{F}_{\mathbf{L}}$ is the canonical frame defined above and $u_{\mathbf{L}}$ is a map from the set of all propositional variables to $G(\mathbf{L})$ defined by: $u_{\mathbf{L}}(p_i)([\chi]) := ([\chi] \vee \neg[p_i]) \wedge [p_i]$ for each $[\chi] \in [\Phi]$ and for each propositional variable p_i. ∎

We will prove in the following that $\mathfrak{M}_{\mathbf{L}}$ is an orthomodular model, according to Maeda's book [32]. We have to show,

(a) $\langle G(\mathbf{L}), \circ \rangle$ is a semigroup.

(b) $\theta \in G(\mathbf{L})$ and for any $\varphi \in G(\mathbf{L})$, $\theta \circ \varphi = \varphi \circ \theta = \theta$ holds.

(c) For $\varphi, \psi \in G(\mathbf{L})$, $\varphi^{**} = \varphi$ and $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$ holds .

(d) For any $\varphi \in G(\mathbf{L})$, there exists $\varepsilon \in P(G(\mathbf{L}))$ such that $\{\varphi\}^{(r)} = \varepsilon \circ G(\mathbf{L})$.

(e) $u_{\mathbf{L}}(p_i) \in P_c(G(\mathbf{L}))$, that is, $u_{\mathbf{L}}(p_i) \in P(G(\mathbf{L}))$ and $(u_{\mathbf{L}}(p_i))^{\perp\perp} = u_{\mathbf{L}}(p_i)$.

Notice that we use $\alpha, \beta, \gamma, \delta$ and χ as meta-variables for formulas, and especially, the last one is for an input variable of maps. On the other hand, we use $\varphi, \psi, \sigma, \eta, \mu, \nu$ as meta-variables for maps from $[\Phi]$ to $[\Phi]$, and θ for the zero map.

Proposition 6.19 Let $\varphi, \psi \in G(\mathbf{L})$.

(1) The residual map $\varphi^\sharp$ for φ is uniquely determined.

(2) $\varphi^\sharp = \psi^\sharp$ implies $\varphi = \psi$.

(3) $(\varphi \circ \psi)^\sharp = \psi^\sharp \circ \varphi^\sharp$.

Proof : (1): Suppose φ_1, φ_2 are residual maps of φ. Then we have:

$$(a) : \varphi_1 \circ \varphi([\chi]) \geq_{\mathbf{L}} [\chi] \qquad (b) : \varphi \circ \varphi_1([\chi]) \leq_{\mathbf{L}} [\chi]$$

$$(c) : \varphi_2 \circ \varphi([\chi]) \geq_{\mathbf{L}} [\chi] \qquad (d) : \varphi \circ \varphi_2([\chi]) \leq_{\mathbf{L}} [\chi]$$

By (a), we deduce $\varphi_1 \circ \varphi \circ \varphi_2([\chi]) \geq_{\mathbf{L}} \varphi_2([\chi])$, and by (d), $\varphi_1 \circ \varphi \circ \varphi_2([\chi]) \leq_{\mathbf{L}} \varphi_1([\chi])$. Thus we have $\varphi_2([\chi]) \leq_{\mathbf{L}} \varphi_1([\chi])$. Similarly, we have $\varphi_1([\chi]) \leq_{\mathbf{L}} \varphi_2([\chi])$ by (b) and (c). Hence $\varphi_1([\chi]) = \varphi_2([\chi])$ holds for any $[\chi] \in [\Phi]$. Therefore we conclude that $\varphi_1 = \varphi_2$.
(2): Suppose that $\varphi^\sharp([\chi]) = \psi^\sharp([\chi])$ for all $[\chi] \in [\Phi]$. Then, since we have $\psi \circ \psi^\sharp \circ \varphi([\chi]) \leq_{\mathbf{L}} \varphi([\chi])$ and $\psi \circ \varphi^\sharp \circ \varphi([\chi]) \geq_{\mathbf{L}} \psi([\chi])$, $\psi([\chi]) \leq_{\mathbf{L}} \varphi([\chi])$ holds. Similarly, the converse inequality can be shown. Thus we have $\varphi([\chi]) = \psi([\chi])$ for any $[\chi] \in [\Phi]$. Therefore $\varphi = \psi$.
(3): We have $(\psi^\sharp \circ \varphi^\sharp) \circ (\varphi \circ \psi)([\chi]) = \psi^\sharp \circ (\varphi^\sharp \circ \varphi(\psi([\chi]))) \geq_{\mathbf{L}} \psi^\sharp \circ \psi([\chi]) \geq_{\mathbf{L}} [\chi]$ and $(\varphi \circ \psi) \circ (\psi^\sharp \circ \varphi^\sharp)([\chi]) = \varphi \circ (\psi \circ \psi^\sharp(\varphi^\sharp([\chi]))) \leq_{\mathbf{L}} \varphi \circ \varphi^\sharp([\chi]) \leq_{\mathbf{L}} [\chi]$. Thus by the uniqueness of the residual map, we have $(\varphi \circ \psi)^\sharp = \psi^\sharp \circ \varphi^\sharp$. □

(1) and (2) of the above proposition show that $(\cdot)^\sharp$ is an injective operator. It is clear that $\psi^\sharp \circ \varphi^\sharp$ satisfies monotonicity, therefore, by (3), $\varphi, \psi \in G(\mathbf{L})$ implies $\varphi \circ \psi \in G(\mathbf{L})$. This, together with properties of the composition of maps, guarantees that $\langle G(\mathbf{L}), \circ \rangle$ is a semigroup.

Proposition 6.20 For $\varphi, \psi \in G(\mathbf{L})$, the following holds.

(1) $\varphi^* \in G(\mathbf{L})$.

(2) $\varphi^{**} = \varphi$.

(3) $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$.

Proof : (1): Suppose $[\alpha] \leq_{\mathbf{L}} [\beta]$ for $\alpha, \beta \in \Phi$. Then $\neg[\beta] \leq_{\mathbf{L}} \neg[\alpha]$ holds, and so $\varphi^\sharp(\neg[\beta]) \leq_{\mathbf{L}} \varphi^\sharp(\neg[\alpha])$. Therefore we have $\varphi^*([\alpha]) = \neg\varphi^\sharp(\neg[\alpha]) \leq_{\mathbf{L}} \neg\varphi^\sharp(\neg[\beta]) = \varphi^*([\beta])$. This means that φ^* is a monotone map. Let $\sigma([\chi]) := \neg\varphi(\neg[\chi])$. Then σ can be proved to be monotone by similar argument as above and we have $\sigma \circ \varphi^*([\chi]) = \neg\varphi(\neg\neg\varphi^\sharp(\neg[\chi])) = \neg\varphi(\varphi^\sharp(\neg[\chi]) \geq_{\mathbf{L}} [\chi]$, and $\varphi^* \circ \sigma([\chi]) = \neg\varphi^\sharp(\neg\neg\varphi(\neg[\chi])) = \neg\varphi^\sharp(\varphi(\neg[\chi])) \leq_{\mathbf{L}} [\chi]$. Therefore, by the uniqueness of a residual map, $\sigma = (\varphi^*)^\sharp$, which implies that $\varphi^* \in G(\mathbf{L})$.
(2): We have $\varphi^{**}([\chi]) = \neg\varphi^{*\sharp}(\neg[\chi]) = \neg\neg\varphi(\neg\neg[\chi]) = \varphi([\chi])$.
(3):By Proposition 6.19 (3), we have $(\varphi \circ \psi)^*([\chi]) = \neg(\varphi \circ \psi)^\sharp(\neg[\chi]) = \neg\psi^\sharp(\varphi^\sharp(\neg[\chi])) = \neg\psi^\sharp(\neg\neg\varphi^\sharp(\neg[\chi])) = \neg\psi^\sharp(\neg\varphi^*([\chi])) = \psi^* \circ \varphi^*([\chi])$. □

By the above proposition, the operation $*$ is really an operation on $G(\mathbf{L})$ and satisfies the conditions for orthomodular frames.

Proposition 6.21 For a formula α, the map $\mu_\alpha : [\Phi] \to [\Phi]$ is defined by: for any $[\chi] \in [\Phi]$,

$$\mu_\alpha([\chi]) := ([\chi] \vee \neg[\alpha]) \wedge [\alpha]$$

Then, (1): $\mu_\alpha \in G(\mathbf{L})$, and (2): $\mu_\alpha \in P(G(\mathbf{L}))$.
Proof : Note that μ_α depends on $[\alpha]$, not on the formula α itself. Since $\mathbf{L}$ is an orthomodular logic, $(a) : \alpha \wedge (\neg\alpha \vee (\alpha \wedge \beta)) \vdash_{\mathbf{L}} \beta$ and $(b) : \beta \vdash_{\mathbf{L}} \neg\alpha \vee (\alpha \wedge (\neg\alpha \vee \beta))$ hold.
(1): Obviously μ_α satisfies monotonicity. Let $\nu_\alpha([\chi]) := ([\chi] \wedge [\alpha]) \vee \neg[\alpha]$ and we will show that ${\mu_\alpha}^\sharp = \nu_\alpha$. By (a), we have $\mu_\alpha \circ \nu_\alpha([\chi]) = \mu_\alpha(([\chi] \wedge [\alpha]) \vee \neg[\alpha]) = (([\chi] \wedge [\alpha]) \vee \neg[\alpha]) \wedge [\alpha] \leq_{\mathbf{L}} [\chi]$. By (b), we have $\nu_\alpha \circ \mu_\alpha([\chi]) = \nu_\alpha(([\chi] \vee \neg[\alpha]) \wedge [\alpha]) = (([\chi] \vee \neg[\alpha]) \wedge [\alpha]) \vee \neg[\alpha] \geq_{\mathbf{L}} [\chi]$. Thus $\nu_\alpha = {\mu_\alpha}^\sharp$. Of course, ν_α also satisfies monotonicity, we conclude that $\mu_\alpha \in G(\mathbf{L})$.
(2): We will show that ${\mu_\alpha}^* = \mu_\alpha$ and that $\mu_\alpha \circ \mu_\alpha = \mu_\alpha$. We have ${\mu_\alpha}^*([\chi]) = \neg{\mu_\alpha}^\sharp(\neg[\chi]) = \neg((\neg[\chi] \wedge [\alpha]) \vee \neg[\alpha]) = ([\chi] \vee \neg[\alpha]) \wedge [\alpha] = \mu_\alpha([\chi])$. On the other hand, first we have $\mu_\alpha \circ \mu_\alpha([\chi]) = \mu_\alpha(([\chi] \vee \neg[\alpha]) \wedge [\alpha]) = (((([\chi] \vee \neg[\alpha]) \wedge [\alpha]) \vee \neg[\alpha]) \wedge [\alpha]$. By (b), we have $((([\chi] \vee \neg[\alpha]) \wedge [\alpha]) \vee \neg[\alpha] \geq_{\mathbf{L}} [\chi]$, then $((([\chi] \vee \neg[\alpha]) \wedge [\alpha]) \vee \neg[\alpha] \geq_{\mathbf{L}} [\chi] \vee \neg[\alpha]$, and so $\mu_\alpha \circ \mu_\alpha([\chi]) \geq_{\mathbf{L}} \mu_\alpha([\chi])$. Conversely, clearly we have that $([\chi] \vee \neg[\alpha]) \wedge [\alpha] \leq_{\mathbf{L}} [\chi] \vee \neg[\alpha]$, then $((([\chi] \vee \neg[\alpha]) \wedge [\alpha]) \vee \neg[\alpha] \leq_{\mathbf{L}} [\chi] \vee \neg[\alpha]$, and so $\mu_\alpha \circ \mu_\alpha([\chi]) \leq_{\mathbf{L}} \mu_\alpha([\chi])$. Therefore we have $\mu_\alpha \circ \mu_\alpha = \mu_\alpha$. Hence $\mu_\alpha \in P(G(\mathbf{L}))$. □

In particular, $\theta = \mu_\perp \in P(G(\mathbf{L}))$ and $u_{\mathbf{L}}(p_i) = \mu_{p_i} \in P(G(\mathbf{L}))$ hold.

Proposition 6.22 Let $\varphi, \psi \in G(\mathbf{L})$.

(1) For any $[\chi] \in [\Phi]$, $\varphi([\chi]) = [\perp]$ if and only if $\varphi^\sharp([\perp]) \geq_{\mathbf{L}} [\chi]$.

(2) $\theta \circ \varphi = \varphi \circ \theta = \theta$.

(3) $\varphi \circ \psi = \theta$ if and only if $\psi(\neg[\bot]) \leq_{\mathbf{L}} \varphi^\sharp([\bot])$.

Proof : (1): Since $\varphi^\sharp \circ \varphi([\chi]) \geq_{\mathbf{L}} [\chi]$, if $\varphi([\chi]) = [\bot]$, then we have $\varphi^\sharp([\bot]) \geq_{\mathbf{L}} [\chi]$. If $\varphi^\sharp([\bot]) \geq_{\mathbf{L}} [\chi]$, then $[\bot] \geq_{\mathbf{L}} \varphi \circ \varphi^\sharp([\bot]) \geq_{\mathbf{L}} \varphi([\chi])$. Thus, $\varphi([\chi]) = [\bot]$.
(2): Obviously $\theta \circ \varphi([\chi]) = \theta(\varphi([\chi])) = [\bot] = \theta([\chi])$. By replacing $[\chi]$ for $\theta([\chi])$ in (1), we have $\varphi(\theta([\chi])) = [\bot] = \theta([\chi])$ since $\varphi^\sharp([\bot]) \geq_{\mathbf{L}} \theta([\chi]) = [\bot]$ holds. Thus we have $\theta \circ \varphi = \varphi \circ \theta = \theta$.
(3): Suppose $\varphi \circ \psi([\chi]) = \theta([\chi]) = [\bot]$. Take $[\neg\bot]$ for $[\chi]$, then $\varphi \circ \psi([\neg\bot]) = [\bot]$ holds. Therefore $\varphi^\sharp([\bot]) = \varphi^\sharp \circ \varphi(\psi(\neg[\bot])) \geq_{\mathbf{L}} \psi(\neg[\bot])$. Conversely, suppose $\psi(\neg[\bot]) \leq_{\mathbf{L}} \varphi^\sharp([\bot])$. Then $\varphi \circ \psi(\neg[\bot]) \leq_{\mathbf{L}} \varphi \circ \varphi^\sharp([\bot]) \leq_{\mathbf{L}} [\bot]$, and so we have $\varphi \circ \psi(\neg[\bot]) = [\bot]$. Therefore, for any $[\chi] \in [\Phi]$, since we have $[\chi] \leq_{\mathbf{L}} \neg[\bot]$, $\varphi \circ \psi([\chi]) \leq_{\mathbf{L}} \varphi \circ \psi(\neg[\bot]) = [\bot]$. This means that $\varphi \circ \psi([\chi]) = [\bot] = \theta([\bot])$, that is, $\varphi \circ \psi = \theta$. □

Trivially the above proposition guarantees that $\theta = \mu_\bot \in P(G(\mathbf{L}))$ fulfills the condition for $\mathbf{0}$ of an orthomodular frame.

Lemma 6.23 For any $\varphi \in G(\mathbf{L})$, let $[\alpha] := \varphi^\sharp([\bot])$. Then,

$$\{\varphi\}^{(r)} := \{\eta \in G(\mathbf{L}) \mid \varphi \circ \eta = \theta\} = \mu_\alpha \circ G(\mathbf{L})$$

Proof : For $[\chi] \in [\Phi]$, since $\mu_\alpha([\chi]) = ([\chi] \vee \neg[\alpha]) \wedge [\alpha] \leq_{\mathbf{L}} [\alpha] = \varphi^\sharp([\bot])$, we have $\varphi \circ \mu_\alpha([\chi]) \leq_{\mathbf{L}} \varphi \circ \varphi^\sharp([\bot]) \leq_{\mathbf{L}} [\bot]$. Thus, $\varphi \circ \mu_\alpha([\chi]) = [\bot] = \theta([\chi])$. This implies that $\mu_\alpha \circ G(\mathbf{L}) \subseteq \{\varphi\}^{(r)}$. Conversely, take $\eta \in \{\varphi\}^{(r)}$, then $\varphi \circ \eta = \theta$ holds. By Proposition 6.22 (3), $\eta(\neg[\bot]) \leq_{\mathbf{L}} \varphi^\sharp([\bot]) = [\alpha]$, and so $\eta([\chi]) \leq_{\mathbf{L}} \eta(\neg[\bot]) \leq_{\mathbf{L}} [\alpha]$. Thus, we have $((\eta([\chi] \wedge [\alpha]) \vee \neg[\alpha]) \wedge [\alpha] \leq_{\mathbf{L}} \eta([\chi])$. Also, from $\eta([\chi]) \leq_{\mathbf{L}} [\alpha]$, we have $\eta([\chi]) = \eta([\chi]) \wedge [\alpha] \leq_{\mathbf{L}} (\eta([\chi]) \vee \neg[\alpha]) \wedge [\alpha]$. Then $\mu_\alpha \circ \eta([\chi]) = (\eta([\chi]) \vee \neg[\alpha]) \wedge [\alpha] = \eta([\chi])$, which means that $\eta \in \mu_\alpha \circ G(\mathbf{L})$. Therefore $\{\varphi\}^{(r)} = \mu_\alpha \circ G(\mathbf{L})$. □

In the above lemma, since, of course μ_α is in $P(G(\mathbf{L}))$, we have proved so far all the conditions of orthomodular frames, that is, the following theorem holds.

Theorem 6.24 The canonical frame $\mathcal{F}_{\mathbf{L}} = \langle G(\mathbf{L}), \circ, *, \theta \rangle$ for $\mathbf{L}$ is an orthomodular frame. □

We prepare two more lemmas on closed projections in $G(\mathbf{L})$ for establishing the completeness theorem in the next section.

Lemma 6.25 $P_c(G(\mathbf{L})) = \{\mu_\alpha \mid \alpha \in \Phi\}$.
Proof : Suppose $\sigma \in P_c(G(\mathbf{L}))$, then there exists $\varphi \in G(\mathbf{L})$ such that $\{\varphi\}^{(r)} = \sigma \circ G(\mathbf{L})$ holds. By Lemma 6.23, if we put $[\alpha] := \varphi^\sharp([\bot])$, then $\sigma = \mu_\alpha$ due to the uniqueness of the right annihilator. On the other hand, consider μ_α for an arbitrary formula α, and by Lemma 6.9, we have only to

show that ${\mu_\alpha}^{\perp\perp} = \mu_\alpha$. Lemma 6.23 shows how to find $\varphi^\perp$ for $\varphi \in G(\mathbf{L})$, that is, put $[\beta] := \varphi^\sharp([\perp])$, and $\varphi^\perp = \mu_\beta$. It is shown in Proposition 6.21 how to build $\varphi^\sharp$ from φ. So, since ${\mu_\alpha}^\sharp([\perp]) = \neg[\alpha]$, we have that ${\mu_\alpha}^\perp = \mu_{\neg\alpha}$. Also since ${\mu_{\neg\alpha}}^\sharp([\perp]) = [\alpha]$, we have ${\mu_{\neg\alpha}}^\perp = {\mu_\alpha}^{\perp\perp} = \mu_\alpha$. □

Of course, we can take p_i for α above, so $u_{\mathbf{L}}(p_i) = \mu_{p_i} \in P_c(G(\mathbf{L}))$. This shows that the canonical model $\mathfrak{M}_{\mathbf{L}} = \langle \mathcal{F}_{\mathbf{L}}, u_{\mathbf{L}} \rangle$ is indeed an orthomodular model.

Lemma 6.26 For $\alpha, \beta \in \Phi$. Then $\alpha \vdash_{\mathbf{L}} \beta$ if and only if $\mu_\alpha \leq \mu_\beta$, that is, $\mu_\beta \circ \mu_\alpha = \mu_\alpha$.

Proof : Suppose $\alpha \vdash_{\mathbf{L}} \beta$. Because $([\chi] \vee \neg[\alpha]) \wedge [\alpha] \leq_{\mathbf{L}} [\alpha] \leq_{\mathbf{L}} [\beta]$, we have $\mu_\beta \circ \mu_\alpha([\chi]) = (((([\chi] \vee \neg[\alpha]) \wedge [\alpha]) \vee \neg[\beta]) \wedge [\beta] = ([\chi] \vee \neg[\alpha]) \wedge [\alpha] = \mu_\alpha([\chi])$. Thus $\mu_\alpha \leq \mu_\beta$. Conversely, if $\mu_\alpha = \mu_\beta \circ \mu_\alpha$, then $[\alpha] = \mu_\alpha(\neg[\perp]) = \mu_\beta \circ \mu_\alpha(\neg[\perp]) \leq_{\mathbf{L}} \mu_\beta(\neg[\perp]) = [\beta]$. So we have $\alpha \vdash_{\mathbf{L}} \beta$. □

6.2.2 The completeness theorem

Theorem 6.27 (Fundamental theorem for orthomodular logic) Let $\mathbf{L}$ be an orthomodular logic. For any formula α and for any $\varphi \in G(\mathbf{L})$, $\mathfrak{M}_{\mathbf{L}} \models_\varphi \alpha$ if and only if $\varphi \in \mu_\alpha \circ G(\mathbf{L})$.

Proof : Induction on the construction of formula α.

(1): Case $\alpha := p_i$. By definition of truth for model $\mathfrak{M}_{\mathbf{L}}$, $\mathfrak{M}_{\mathbf{L}} \models_\varphi p_i$ if and only if $\varphi \in \mu_{p_i} \circ G(\mathbf{L})$.

(2): Case $\alpha := \beta \wedge \gamma$. We show first that $\mu_\beta \sqcap \mu_\gamma = \mu_{\beta\wedge\gamma}$. By Lemma 6.26, it is obvious that $\mu_{\beta\wedge\gamma} \leq \mu_\beta \sqcap \mu_\gamma$. Take μ_δ such that $\mu_\delta \leq \mu_\beta$ and $\mu_\delta \leq \mu_\gamma$. Then, it is also clear that $\mu_\delta \leq \mu_{\beta\wedge\gamma}$. Thus we have $\mu_{\beta\wedge\gamma} = \mu_\beta \sqcap \mu_\gamma$. Hence by Lemma 6.16, $\mathfrak{M}_{\mathbf{L}} \models_\varphi \beta \wedge \gamma$ if and only if $\varphi \in (\mu_\beta \sqcap \mu_\gamma) \circ G(\mathbf{L})$ if and only if $\varphi \in \mu_{\beta\wedge\gamma} \circ G(\mathbf{L})$.

(3): Case $\alpha := \neg\beta$. Since we have ${\mu_\beta}^\perp = \mu_{\neg\beta}$, by Lemma 6.16, we can conclude that $\mathfrak{M}_{\mathbf{L}} \models_\varphi \neg\beta$ if and only if $\varphi \in {\mu_\beta}^\perp \circ G(\mathbf{L})$ if and only if $\varphi \in \mu_{\neg\beta} \circ G(\mathbf{L})$. □

Corollary 6.28 Let Γ be a non-empty set of formulas and α a formula. Then $\Gamma \vdash_{\mathbf{L}} \alpha$ implies $\mathfrak{M}_{\mathbf{L}} : \Gamma \models \alpha$.

Proof : Suppose $\Gamma \vdash_{\mathbf{L}} \alpha$, then there exist finite number of formulas $\beta_1, \beta_2, \ldots, \beta_n$ such that $\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_n \vdash_{\mathbf{L}} \alpha$. Take any $\varphi \in G(\mathbf{L})$ and suppose $\mathfrak{M}_{\mathbf{L}} \models_\varphi \beta_1 \wedge \cdots \wedge \beta_n$. Then by Theorem 6.27, we have $\varphi \in \mu_{\beta_1\wedge\cdots\wedge\beta_n} \circ G(\mathbf{L})$. Moreover, since $\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_n \vdash_{\mathbf{L}} \alpha$, by Lemma 6.26, $\mu_{\beta_1\wedge\cdots\wedge\beta_n} \circ G(\mathbf{L}) \subseteq \mu_\alpha \circ G(\mathbf{L})$ Thus $\varphi \in \mu_\alpha \circ G(\mathbf{L})$, and so $\mathfrak{M}_{\mathbf{L}} \models_\varphi \alpha$ holds. Therefore we have $\mathfrak{M}_{\mathbf{L}} : \Gamma \models \alpha$. □

Corollary 6.29 Let Γ be a non-empty set of formulas and α a formula. Then $\mathfrak{M}_{\mathbf{L}} : \Gamma \models \alpha$ implies $\Gamma \vdash_{\mathbf{L}} \alpha$.

Proof : Suppose $\Gamma \not\vdash_{\mathbf{L}} \alpha$. Take any finite number of formulas $\beta_1, \beta_2, \ldots, \beta_n \in \Gamma$. Then $\beta_1 \wedge \cdots \wedge \beta_n \not\vdash_{\mathbf{L}} \alpha$ holds. By Lemma 6.26, we have that $\mu_{\beta_1 \wedge \cdots \wedge \beta_n} \circ G(\mathbf{L}) \not\subseteq \mu_\alpha \circ G(\mathbf{L})$, which means that there exists $\varphi \in G(\mathbf{L})$ such that $\varphi \in \mu_{\beta_1 \wedge \cdots \wedge \beta_n} \circ G(\mathbf{L})$ but $\varphi \notin \mu_\alpha \circ G(\mathbf{L})$. Then by Theorem 6.27, $\mathfrak{M}_{\mathbf{L}} \models_\varphi \beta_1 \wedge \cdots \wedge \beta_n$ but $\mathfrak{M}_{\mathbf{L}} \not\models_\varphi \alpha$. Therefore we conclude that $\mathfrak{M}_{\mathbf{L}} : \Gamma \not\models \alpha$. □

Theorem 6.30 (Strong completeness theorem for OML) Let Γ be a non-empty subset of formulas, α a formula, and $\mathcal{C}$ the class of all orthomodular frames. Then for the smallest orthomodular logic **OML**, $\mathcal{C} : \Gamma \models \alpha$ implies $\Gamma \vdash_{\mathbf{OML}} \alpha$.

Proof : Suppose $\mathcal{C} : \Gamma \models \alpha$. Take the canonical frame $\mathcal{F}_{\mathbf{OML}}$ for the logic **OML**, then especially $\mathcal{F}_{\mathbf{OML}} : \Gamma \models \alpha$. This implies, in particular, for the canonical model $\mathfrak{M}_{\mathbf{OML}}$, $\mathfrak{M}_{\mathbf{OML}} : \Gamma \models \alpha$. Thus by the previous corollary, we have that $\Gamma \vdash_{\mathbf{OML}} \alpha$. □

Theorem 6.16 and Theorem 6.30 show that the class of all orthomofular frames can strongly characterizes the quantum logic **OML**.

6.3 General completeness

As for the orthomodular logic **OML**, we will show the completeness theorem of any orthomodular logic, which corresponds to a subvariety of the variety $\mathcal{OML}$. First, we introduce the notion of *terms* to describe a formal system of our logics in a general way.

Let $\mathbf{X}$ be a non-empty set of distinct *variables*. The set $T(\mathbf{X})$ of *terms* over $\mathbf{X}$ of the same type as orthomodular lattices is defined inductively as follows:

(1) $\mathbf{X} \cup \{0, 1\} \subseteq T(\mathbf{X})$.

(2) If $t \in T(\mathbf{X})$, then $t' \in T(\mathbf{X})$.

(3) If $t, s \in T(\mathbf{X})$, then $(t \cap s) \in T(\mathbf{X})$.

We denote a term in which some of variables in $\{x_1, \ldots, x_n\}$ will appear, by $t(x_1, \ldots, x_n)$. As is easily seen, we can associate any term with a formula of our propositional language, that is, we write $\overline{t}(\alpha_1, \ldots, \alpha_n)$ to express the formula which is obtained by substituting each α_i for x_i, $\wedge$ for $\cap$, and $\neg$ for $'$, where each α_i is a formula.

Let $\mathcal{V}$ be a subvariety of $\mathcal{OML}$ and the finite set of its additional defining identities relative to those of $\mathcal{OML}$ be the following:

$$t_1(x_1, \ldots, x_n) = s_1(x_1, \ldots, x_n),$$
$$t_2(x_1, \ldots, x_n) = s_2(x_1, \ldots, x_n),$$
$$\vdots$$
$$t_k(x_1, \ldots, x_n) = s_k(x_1, \ldots, x_n)$$

Let $\mathbf{L}(\mathcal{V})$ the orthomodular logic which corresponds to $\mathcal{V}$. Clearly, it is sufficient to consider the following set $\mathbf{Z}$ of axiom schemes and we may put $\mathbf{L}(\mathcal{V}) = \mathbf{OML} \oplus (\mathbf{Z})$.

$$\overline{t_1}(\alpha_1,\ldots,\alpha_n) \vdash \overline{s_1}(\alpha_1,\ldots,\alpha_n), \quad \overline{s_1}(\alpha_1,\ldots,\alpha_n) \vdash \overline{t_1}(\alpha_1,\ldots,\alpha_n)$$

$$\overline{t_2}(\alpha_1,\ldots,\alpha_n) \vdash \overline{s_2}(\alpha_1,\ldots,\alpha_n), \quad \overline{s_2}(\alpha_1,\ldots,\alpha_n) \vdash \overline{t_2}(\alpha_1,\ldots,\alpha_n)$$

$$\vdots$$

$$\overline{t_n}(\alpha_1,\ldots,\alpha_n) \vdash \overline{s_n}(\alpha_1,\ldots,\alpha_n), \quad \overline{s_n}(\alpha_1,\ldots,\alpha_n) \vdash \overline{t_n}(\alpha_1,\ldots,\alpha_n)$$

Here, each α_i is a meta-variable of formulas. It may sometimes happen that some axiom schemes above are redundant. Above list can be taken account of the most general case.

As the class of orthomodular frames which corresponds to the subvariety $\mathcal{V}$, we define $\mathcal{C}_\mathcal{V}$ by:

$$\mathcal{C}_\mathcal{V} := \{\mathcal{F} : \text{orthomodular frame} \mid P_c(G) \text{ is a member of } \mathcal{V}\}$$

For any orthomodular frame $\mathcal{F} = \langle G, \cdot, *, \mathbf{0}\rangle$, $P_c(G)$ has operations $\sqcap$, $(\cdot)^\perp$, and $\mathbf{0}$ which are discussed in 6.1.2, and they correspond to $\cap$, $(\cdot)'$, and 0 in an orthomodular lattice respectively. So "$P_c(G)$ is a member of $\mathcal{V}$" means that all the additional defining identities of $\mathcal{V}$ hold for any elements in $P_c(G)$.

The orthomodular logic $L_\mathcal{V}$ can be strongly characterized by the class $\mathcal{C}_\mathcal{V}$, i.e. the following theorem holds.

Theorem 6.31 (Strong characterization of $\mathbf{L}_\mathcal{V}$) Let Γ be a non-empty set of formulas and α a formula. Then, $\Gamma \vdash_{\mathbf{L}_\mathcal{V}} \alpha$ if and only if $\mathcal{C}_\mathcal{V} : \Gamma \models \alpha$.

Proof : Quite similar argument to those of Theorem 6.17 and Theorem 6.30 works for this theorem. □

Note that we can define "$\mathcal{F} \in \mathcal{C}_\mathcal{V}$" by first-order statement of the language in $\mathcal{F}$ for any subvariety $\mathcal{V}$ in the following way. A translation function E from $T(\mathbf{X})$ to G is defined:

(1) $E(0) = \mathbf{0}$, $E(1) = \mathbf{0}^\perp$, and $E(x_i) = e_i$.

(2) $E(t') = E(t)^\perp$.

(3) $E(t \cap s) = E(t) \sqcap E(s) = E(t) \cdot (E(s)^\perp \cdot E(t))^\perp$.

Then, if $\mathcal{V}$ has k additional defining identities as in the previous argument, we can define "a frame $\mathcal{F}$ is a member of $\mathcal{C}_\mathcal{V}$" as follows: for any elements $e_1, \ldots, e_n \in G$,

$$\bigwedge_{i=1}^{n} (e_i{}^{\perp\perp} = e_i) \implies \bigwedge_{j=1}^{k} \{E(t_j(x_1,\ldots,x_n)) = E(s_j(x_1,\ldots,x_n))\}$$

6.4 The classical logic and commutativity of its frame

As is seen in Chapter 2, the classical logic **CL** can be formulated by: $\mathbf{CL} = \mathbf{OL} \oplus (\mathbf{Dis})$, where $(\mathbf{Dis})$ represents the distributive law. Since $\mathbf{OML} = \mathbf{OL} \oplus (\mathbf{Oml})$ and the orthomodular law follows from the distributive law, the classical logic is also formulated as $\mathbf{CL} = \mathbf{OML} \oplus (\mathbf{Dis})$, that is, we can say that the classical logic is an orthomodular logic which satisfies the distributive law.

Furthermore, Proposition 2.5 guarantees that the distributive law can be rewritten in the following simpler form in orthomodular lattices:

$$x \cap (x' \cup y) = x \cap y \quad \cdots\cdots\cdots \quad \text{(commutative law)}$$

Therefore, here we introduce a new axiom scheme $(\mathbf{Com})$ below, and we will treat the classical logic as: $\mathbf{CL} = \mathbf{OML} \oplus (\mathbf{Com})$.

$(\mathbf{Com}) : \alpha \wedge (\neg\alpha \vee \beta) \vdash \beta$

From this point of view, we can provide a simpler model for the classical logic than for usual orthomodular logics. The reason why we call that identity the *commutative law* will be clear in the subsection below.

6.4.1 Commutative orthomodular frames

The orthomodular frames which can characterize the logic $\mathbf{OML} \oplus (\mathbf{Com})$ is introduced here. An orthomodular frame $\mathcal{F} = \langle G, \cdot, *, \mathbf{0}\rangle$ is *commutative* if $x \cdot y = y \cdot x$ holds for any $x, y \in G$. For commutative orthomodular frames, the following lemma holds.

Lemma 6.32 Let $\mathcal{F} = \langle G, \cdot, *, \mathbf{0}\rangle$ be a commutative orthomodular frame. Then $\mathcal{F}' = \langle P(G), \cdot, *, \mathbf{0}\rangle$ is also a frame.
Proof : Let G' be $P(G)$. The operations $\cdot$ and $*$ in $\mathcal{F}'$ are the same as those in $\mathcal{F}$. We must check the conditions in Definition 6.1.
(1): Take $x, y \in G' = P(G)$. Then we have $(x \cdot y) \cdot (x \cdot y) = (x \cdot y) \cdot (y \cdot x) = x \cdot y \cdot x = x \cdot x \cdot y = x \cdot y$, and $(x \cdot y)^* = y^* \cdot x^* = y \cdot x = x \cdot y$. Therefore $(x \cdot y) \in P(G) = G'$ and so, $\langle G', \cdot\rangle$ is a semigroup.
(2): Since $P(G') = P(G) = G'$, we have $\mathbf{0} \in P(G')$.
(3): Of course $*$ in $\mathcal{F}$ also satisfies the condition (3) for $\mathcal{F}'$. Indeed, for $x \in G'$, $x^* = x$ holds.
(4):Take any $x \in G' \subseteq G$, and there exists $e \in P(G) = G'$ such that $\{x\}^{(r)} = e \cdot G$. Put $S = \{z \in G' \mid x \cdot z = \mathbf{0}\}$. Then $S \subseteq \{x\}^{(r)}$. Thus, for any $z \in S$, there exists $t \in G$ such that $z = e \cdot t$. Therefore we have $e \cdot z = e \cdot e \cdot t = e \cdot t = z$, and so $z \in e \cdot G'$. Conversely, since we have $e \cdot G' \subseteq e \cdot G$, for $y \in e \cdot G' \subseteq G'$, we have $x \cdot y = \mathbf{0}$. Thus $y \in S$. We have shown that $S = e \cdot G'$. □

In the proof of the condition (4) above, it is easily seen that the operation $(\cdot)^{\perp}$ in $\mathcal{F}'$ is the same as that in $\mathcal{F}$. So we conclude that $P_c(G') = P_c(G)$. It is also clear that in a commutative frame, $\{x\}^{(r)} = \{x\}^{(\ell)}$. By Lemma 6.14, we have that $e, f \in P_c(G)$ in a commutative frame, $e \sqcap f = e \cdot f$. A commutative orthomodular model is defined in a similar way as usual orthomodular model.

The class of all commutative orthomodular frames characterizes the logic **CL**. For the soundness, the following theorem holds.

Theorem 6.33 Let Γ be a non-empty set of formulas and α a formula. For the class of all commutative frames $\mathcal{D}$, $\Gamma \vdash_{\mathbf{CL}} \alpha$ implies $\mathcal{D} : \Gamma \models \alpha$.

Proof : Almost the same argument goes through as for Theorem 6.16. We have only to show for the case of the axiom scheme *Com*. In order to show this, it is enough to show that for any commutative orthmodular frame $\mathcal{F}$, we have $e \cdot (e \cdot f^{\perp})^{\perp} \leq f$ for any $e, f \in P(G)$. But by Lemma 6.14, we have $e \sqcap f = e \cdot (f^{\perp} \cdot e)^{\perp}$. Since our frame is commutative, we have also that $e \sqcap f = e \cdot f$, and so $e \cdot (f^{\perp} \cdot e)^{\perp} = e \cdot f \leq f$. □

6.4.2 Completeness for CL

The canonical frame $\mathcal{F}_{\mathbf{CL}} = \langle G(\mathbf{CL}), \circ, *, \theta \rangle$ for the logic **CL** is constructed in the same way as for usual orthomodular logics (Definition 6.18), except for the underlying set. In this case $G(\mathbf{CL}) = \{\mu_\alpha \,|\, \alpha \in \mathbf{\Phi}\}$, where the map is $\mu_\alpha([\chi]) = ([\chi] \vee \neg[\alpha]) \wedge [\alpha]$ for $[\chi] \in [\Phi]$. The canonical model $\mathfrak{M}_{\mathbf{CL}}$ is also defined in the same way as in Definition 6.18.

Lemma 6.34 $\mathcal{F}_{\mathbf{CL}}$ is a commutative orthomodular frame.

Proof : We must check the conditions in Definition 6.1 and commutativeness.

(1): By the axiom scheme (**Com**) and Proposition 2.6, for $\mu_\alpha, \mu_\beta \in G(\mathbf{CL})$, we have $\mu_\alpha \circ \mu_\beta([\chi]) = ((([\chi] \vee \neg[\beta]) \wedge [\beta]) \vee \neg[\alpha]) \wedge [\alpha] = [\chi] \wedge ([\alpha] \wedge [\beta]) = ([\chi] \vee \neg([\alpha] \wedge [\beta])) \wedge ([\alpha] \wedge [\beta])$. Therefore $\mu_\alpha \circ \mu_\beta([\chi]) \in G(\mathbf{CL})$ and $\mu_\alpha \circ \mu_\beta = \mu_\beta \circ \mu_\alpha$. Of course each μ_α has ${\mu_\alpha}^{\sharp}([\chi]) = ([\chi] \wedge [\alpha]) \vee \neg[\alpha]$ as its residual map which is monotone.

(2): It is clear because $\theta([\chi]) = \mu_{\perp}$.

(3): It is also obvious because ${\mu_\alpha}^{*} = \mu_\alpha$ holds and we have the commutativeness.

(4): Since $G(\mathbf{CL}) = \mathbf{P_c}(\mathbf{G}(\mathbf{CL}))$, similar argument in Lemma 6.32 also works for this case. □

The presence of the previous lemma ensures that the similar argument in Subsection 6.2.2 goes through and we can reach the following completeness theorem for the logic **CL**.

Theorem 6.35 (Strong completeness for CL) Let Γ be a non-empty set of formulas and α a formula. For the class of all commutative frames $\mathcal{D}$, $\mathcal{D} : \Gamma \models \alpha$ implies $\Gamma \vdash_{\mathbf{CL}} \alpha$. □

6.5 Infinitary orthomodular logics

Infinitary logics are logics in which connectives whose arity is infinite are allowed to build up formulas. In this section, we will concentrate on the infinitary extension of the smallest orthomodular logic $\mathbf{OML}_{inf}$, in which an infinitary conjunction ($\bigwedge$) appears. General infinitary orthomodular logics can be treated similarly as in Section 6.3.

6.5.1 Syntax of infinitary orthomodular logics

This time, our language consists of a collection of uncountably many propositional symbols $\{p_\lambda \mid \lambda \in \Lambda\}$, a propositional constant $\perp$, a unary connective $\neg$, an infinitary conjunction $\bigwedge$, and a pair of parentheses $(,)$.

The set Ψ of all formulas of this language is defined by the three formation rules:

(1) $\perp$, and each propositional variable p_λ are formulas.

(2) If α is a formula, then so is $(\neg\alpha)$.

(3) If Δ is a non-empty set of formulas, then is $(\bigwedge \Delta)$ is a formula.

Since we have the double negation law, the infinitary disjunction connective $\bigvee$ can be introduced by the definitional abbreviation, i.e., $(\bigvee \Delta)$ is the abbreviation for $\neg(\bigwedge\{\neg\alpha \mid \alpha \in \Delta\})$.

Definition 6.36 (Infinitary orthomodular logics) An *infinitary orthomodular logic* $\mathbf{L}$ on the set Ψ of formulas is a subset of the product $\Psi \times \Psi$ (we write $\alpha \vdash_{\mathbf{L}} \beta$ to mean that the pair of formulas $\langle \alpha, \beta \rangle$ is a member of $\mathbf{L}$) which includes the following axiom schemes and closed under the following inference rules:

Axiom schemes:
Comparing with the axiom schemes in Definition 2.8, the double negation law and the orthomodular law are needed. (Ax3) and (Ax4) are unified to the following form. Others are the same as those in Definition 2.8.

(**Dbn**): $\neg\neg\alpha \vdash_{\mathbf{L}} \alpha$
(**Oml**): $\alpha \wedge (\neg\alpha \vee (\alpha \wedge \beta)) \vdash_{\mathbf{L}} \beta$
(Ax3+4):

$$(\bigwedge \Gamma) \vdash_{\mathbf{L}} \alpha, \quad \text{where} \quad \alpha \in \Gamma$$

Inference Rules:
(R1) and (R4) are the same as those of Definition 2.8. (R2) is changed into the following form.
(R2+):

$$\frac{\alpha \vdash_L \beta \quad (\text{for all} \quad \beta \in \Delta)}{\alpha \vdash_L \bigwedge \Delta}$$

Since we have the double negation law, the Axiom schemes (Ax5) and (Ax6), and the Inference Rules (R3) are derivable.

The intersection of all infinitary orthomodular logics on Ψ, that is called *the smallest infinitary orthomodular logic* is denoted by $\mathbf{OML}_{inf}$. ∎

OML is characterized by the variety of all orthomodular lattices, whereas $\mathbf{OML}_{inf}$ can be characterized by the class of all *complete* orthomodular lattices. This observation means that we need to extend our orthomodular frames for the infinitary quantum logic to be complete with respect to orthomodular frames. This extension will be done in the following section.

6.5.2 An extension of orthomodular frames and soundness theorem

Definition 6.37 (Complete orthomodular model) An orthomodular frame $\mathcal{F} = \langle G, \cdot, *, \mathbf{0} \rangle$ is *complete* if it satisfies also the following condition (4+).

(4+) For any non-empty subset $M \subseteq G$, there exists a projection $e \in P(G)$ such that the right annihilator of M can be expressed as:

$$M^{(r)} = e \cdot G \quad (:= \{ e \cdot y \mid y \in G \})$$

An orthomodular model $\mathfrak{M} = \langle \mathcal{F}, u \rangle$ is *complete* if its frame $\mathcal{F}$ is complete. The truth condition with respect to a complete orthomodular model is defined as follows:

(0) $\mathfrak{M} \models_x \bot$ if and only if $x = \mathbf{0}$

(1) $\mathfrak{M} \models_x p_i$ if and only if $x \in u(p_i) \cdot G$

(2) $\mathfrak{M} \models_x \bigwedge \Gamma$ if and only if $\mathfrak{M} \models_x \alpha$ for all $\alpha \in \Gamma$, where Γ is a non-empty set of formulas.

(3) $\mathfrak{M} \models_x \neg\alpha$ if and only if for any $y \in G$, $\mathfrak{M} \models_y \alpha$ implies $y^* \cdot x = \mathbf{0}$

∎

Let $M := \{x_\lambda \mid \lambda \in \Lambda\}$ be a non-empty subset of G. The condition (4+) guarantees that there exists $e \in P(G)$ such that $M^{(r)} = e \cdot G$. Then by the similar argument as for the proof of Theorem 6.10, we can prove that

$M^{(r)} = e\cdot G = \bigcap_{\lambda\in\Lambda} x_\lambda{}^{\perp}\cdot G$. From this fact, quite the same lemma as Lemma 6.15 also holds for complete orthomodular models.

Lemma 6.38 Let $\mathfrak{M} = \langle \mathcal{F}, u\rangle = \langle G, \cdot, *, \mathbf{0}, u\rangle$ be a complete orthomodular model. Then, for any formula α, there exists the unique closed projection $e \in P_c(G)$ such that

$$\|\alpha\|^{\mathfrak{M}} := \{x \in G \mid \mathfrak{M} \models_x \alpha\} = e\cdot G$$

□

In particular, $e(\bigwedge\{\alpha_\lambda \mid \lambda \in \Lambda\}) = \sqcap_{\lambda\in\Lambda} e(\alpha_\lambda)$ holds. With the help of Lemma 6.38, the following soundness theorem holds.

Theorem 6.39 Let α, β be formulas and let $\mathcal{E}$ be the class of all complete orthomodular frames. Then $\alpha \vdash_{\mathbf{OML}_{inf}} \beta$ implies $\mathcal{E} : \alpha \models \beta$. □

6.5.3 Completeness for the logic $\mathbf{OML}_{inf}$

Canonical model construction for the logic $\mathbf{OML}_{inf}$ is quite the same as in Definition 6.18. Before showing the completeness theorem, we have to check that the canonical model is indeed a complete orthomodular model.

Lemma 6.40 The canonical frame $\mathcal{F}_{\mathbf{OML}_{inf}} = \langle G(\mathbf{OML}_{inf}), \circ, *, \theta\rangle$ is a complete orthomodular frame.
Proof : It is enough to focus only on the condition (4+). Take any non-empty subset $K \subseteq G(\mathbf{OML}_{inf})$. Put $K := \{\varphi_\lambda \mid \lambda \in \Lambda\}$. Since we have already the condition (4), for each φ_λ, $\{\varphi_\lambda\}^{(r)} = \mu_{\alpha_\lambda} \circ G(\mathbf{OML}_{inf})$ holds, where $\alpha_\lambda = \varphi_\lambda{}^\sharp([\perp])$. Put $\beta := \bigwedge_{\lambda\in\Lambda} \alpha_\lambda$. Then, the similar argument as in the proof of Theorem 6.27 ensures that $\mu_\beta = \sqcap_{\lambda\in\Lambda}\mu_{\alpha_\lambda}$. Thus we can prove that $K^{(r)} = \mu_\beta \circ G(\mathbf{OML}_{inf})$. The proof is completed. □

Now we are in a position to prove the completeness of $\mathbf{OML}_{inf}$ with respect to our complete orthomodular frames. Since we have the above lemma, its proof is quite similar to that of Theorem 6.30.

Theorem 6.41 (Completeness theorem for $\mathbf{OML}_{inf}$) Let α, β be formulas, $\mathcal{E}$ the class of all complete orthomodular frames. Then $\mathcal{E} : \alpha \models \beta$ implies $\alpha \vdash_{\mathbf{OML}_{inf}} \beta$. □

6.6 Note

The completeness result for orthomodular logics in the present paper is based on *residuation theory.* This is explained intensively in [5] and for orthomodularlattices in particular, [32] is also very helpful. Our canonical model construction depends heavily on the following facts. Let $\mathfrak{A} = \langle A, \cap, \cup, ', 0, 1\rangle$

be an orthomodular lattice. Define a binary operation $\Rightarrow$ on $\mathfrak{A}$ as $a \Rightarrow x := (x \cup a') \cap a$. This operation $\Rightarrow$ is called *Sasaki projection* ([26]). Then, the operation has the following properties. For $a, b, x \in A$,

(1) $(a \Rightarrow (a \Rightarrow x)) = (a \Rightarrow x)$

(2) The following conditions are equivalent.
(i) $a \leq b$ (ii) $(a \Rightarrow (b \Rightarrow x)) = (b \Rightarrow (a \Rightarrow x)) = (a \Rightarrow x)$.

Due to these two properties, we can define an order relation on the set $\{\mu_a \mid a \in A\}$ by (2), where μ_a is the map, that is, $\mu_a(x) := a \Rightarrow x$. These properties enable us to consider semigroup frames and to construct the canonical frames of the set of maps from A to A, in which the semigroup operation is the map composition. On the other hand, in a Heyting algebra, the *Heyting implication* $\supset$ (it is defined by: $x \cap y \leq z$ if and only if $x \leq y \supset z$.) has the above properties (1), (2). Therefore we can construct a similar semantics for intuitionistic logics.

In chapter 6, we have seen that the classical logic is characterized by the class of orthomodular frames that are commutative. This fact reminds us an analogous proposition in quantum mechanics that between two observables whose corresponding operators $\hat{A}, \hat{B}$ commutes with each other, i.e., the *commutator* $[\hat{A}, \hat{B}] := \hat{A}\hat{B} - \hat{B}\hat{A} = 0$, the uncertainty phenomena would not occur, that is, these two observables behave in a classical mechanical way. In other words, the classical observables are embedded into the set of quantum observables as the set of observables which mutually commute. This relation between the classical observables and the quantum observables are quite similar to the relation between the class of orthomodular frames and the class of commutative orthomodular frames. In this sense, our model construction reflects very well the situation in quantum physics into mathematical logic.

Of course, our model for the classical logic can be made much simpler. Indeed we need only two-point orthomodular frame to characterize the classical logic.

7 Summary and future works

At the end of this thesis, we summarize our results of this research, open questions, and propose an attempt to do with the completion problem of orthomodular lattices.

7.1 Summary

Among eight varieties of algebras which we introduced in Chapter 2, we show that the following three varieties admit completion.

$$\mathcal{OL}, \quad \mathcal{OL}^{(-)}, \quad \mathcal{BA}$$

The above all three varieties admit both the Dedekind-MacNeille completion and the completion by means of dual spaces.

Since the completion technique via dual space works for all three varieties above, every propositional logic which corresponds to each of them has a relational semantics. In Chapter 3, we construct semantics for the logics $\mathbf{OL}^{(-)}$ in particular, and show the completeness theorems with respect to these semantics.

The Dedekind-MacNeille completions of $\mathcal{OL}, \mathcal{OL}^{(-)}, \mathcal{BA}$ imply that we can show the algebraic completeness theorems of the minimum predicate extensions of the logics $\mathbf{OL}, \mathbf{OL}^{(-)}$, and $\mathbf{CL}$. Indeed, we give the completeness theorems of $\mathbf{P}(\mathbf{OL})$ and $\mathbf{P}(\mathbf{OL}^{(-)})$ in Chapter 4.

Craig's interpolation property is one of the marvelous syntactical properties in logics. Semantically, it can be shown immediately that the super amalgamation property of a subvariety of the variety $\mathcal{OL}^{(-)}$ is a sufficient condition for its corresponding logic to have the Craig's interpolation property. The Dedekind-MacNeille completion technique works also well to show the super amalgamation property of some varieties of algebras, though some refinement is needed, and so we can show that the propositional logic $\mathbf{OL}, \mathbf{OL}^{(-)}$ and their predicate extensions $\mathbf{P}(\mathbf{OL})$ and $\mathbf{P}(\mathbf{OL}^{(-)})$ have the Craig's interpolation property. The classical propositional logic $\mathbf{CL}$ and the classical predicate logic have the interpolation property, but it seems difficult to show their Craig's interpolation property by applying our argument.

It is still unknown whether every orthomodular lattice can be embedded into a complete orthomodular lattice. About this problem, we will provide as much information as possible in the next section, and we will introduce an attempt to attack this problem. If this problem will be solved in the affirmative, we can expect a relational semantics for orthomodular logics and algebraic completeness theorem of the minimum predicate extension of the orthomodular logic $\mathbf{OML}$.

In Chapter 6, we give Kripke-style semantics for orthomodular logics. This semantics is based on a representation theorem for orthomodular lattices, but the representation method is not a kind of usual set-theoretic ones. By using this semantics, we discuss orthomodular logics in a general way. In particular, we clarify the relation of the classical logic among orthomodular logics, which is analogous to the relation between the classical physics and the quantum physics. Finally, we extend our semantics to that for the infinitary orthomodular logics and show its completeness theorem.

7.2 The completion problem of orthomodular lattices

Here we pay our attention to a particular relation between ortholattices and orthomodular lattices, and give a new perspective of this problem, using the fixpoint theory.

As seen in Chapter 1, the algebraic structure of orthomodular lattices is an abstraction of some algebraic properties of Hilbert space. Indeed, all closed subspaces of a Hilbert space forms a complete orthomodular lattice (Theorem 1.3). Here, a *subspace* means a non-empty subset which is closed under additive operation and scalar multiplication, whereas *closed* means that any Cauchy sequence in it converges in that subspace. Therefore, each closed subspace turns out to be again a Hilbert space. Similar proposition also holds for an orthomodular lattice (Theorem 7.2), and this yields us a new characterization for an orthomodular lattice in an ortholattice (Lemma 2.1).

As is already seen in Chapter 3, the variety of ortholattices admits completion (Theorem 3.5), which means precisely the following theorem.

Theorem 7.1 For any ortholattice $\mathfrak{A}$, there exists a complete ortholattice $\mathfrak{L}$ and a map $f : \mathfrak{A} \to \mathfrak{L}$, such that $\mathfrak{A}$ can be embedded into $\mathfrak{L}$ by f. □

The construction of the complete ortholattice $\mathfrak{L}$ is given, either by the Dedekind-MacNeille completion ([6]), or by a completion by way of the dual space of the ortholattice, i.e., a non-empty set with an irreflexive and symmetric binary relation ([16]). Our work is based on the above theorem.

Another basic fact in our approach is on a property for orthomodular lattices, which is shown below. (See [26])

Theorem 7.2 Let $\mathfrak{A} = \langle A, \cap, \cup, ', 0, 1\rangle$ be an orthomodular lattice. Then, for any $a, b \in A$ with $a \leq b$, the *interval* $[a, b] := \{x \in A \mid a \leq x \leq b\}$, which is a sublattice of $\mathfrak{A}$ forms an orthomodular lattice, in which the orthocomplement $(\cdot)^*$ is defined as: $x^* := (x' \cup a) \cap b$ for $x \in [a, b]$. □

We will not discuss the detail of the proof. This theorem seems to reflect the fact in a Hilbert space that each closed subspace of a Hilbert space is again a Hilbert space. We use a part of this theorem, to characterize orthomodular lattices in ortholattices, given below.

The following observation is the key fact in our argument.

Lemma 7.3 Let $\mathfrak{A} = \langle A, \cap, \cup, ', 0, 1 \rangle$ be an ortholattice. Then, the following two conditions are equivalent.

(1) $\mathfrak{A}$ is an orthomodular lattice.

(2) For any $a \in A$, the sublattice $[0, a] := \{x \in A \mid 0 \leq x \leq a\}$ forms an ortholattice, whose orthocomplement $(\cdot)^*$ is defined as: $x^* := x' \cap a$.

Proof : (1)$\Rightarrow$(2): Take any $a \in A$, and consider the interval $[0, a]$. Then, it is trivial that $[0, a]$ is closed under meet, join and $(\cdot)^*$, and hence, it is a bounded lattice. Therefore to show (2), it is enough to check that the operation $(\cdot)^*$ satisfies the conditions in Definition 1.2. First, $x^{**} = (x' \cap a)' \cap a = (x \cup a') \cap a = x$ because $x \leq a$. Next, $x \cap x^* = x \cap x' \cap a = 0$. For the last, if $x \leq y$, then $y' \leq x'$, and so, $y' \cap a \leq x' \cap a$. Thus we have $y^* \leq x^*$.
(2)$\Rightarrow$(1): We have only to show that $\mathfrak{A}$ satisfies the orthomodular law. Take $x, y \in A$ with $x \leq y$. Then, since $[0, y]$ is an ortholattice, we have $y = x \cup x^* = x \cup (x' \cap y)$.

□

If we assume (1), we can say moreover that $[0, a]$ is an orthomodular lattice, as is easily guessed from Theorem 7.2. Similarly, we can show that the same relation holds between *complete* ortholattices and *complete* orthomodular lattice.

Corollary 7.4 Let $\mathfrak{L} = \langle L, \bigcap, \bigcup, ', 0, 1 \rangle$ be a complete ortholattice. Then, the following two conditions are equivalent.

(1) $\mathfrak{L}$ is a complete orthomodular lattice.

(2) For any $a \in L$, the sublattice $[0, a] := \{x \in L \mid 0 \leq x \leq a\}$ is a complete ortholattice, whose orthocomplement $(\cdot)^*$ is defined as: $x^* := x' \cap a$.

□

Let $\mathfrak{L} = \langle L, \bigcap, \bigcup, (\cdot)', 0, 1 \rangle$ be a complete ortholattice. For any subset $X \subseteq L$ and any element $a \in L$, $P_a(X)$ denotes the following property:

1) $0, 1 \in X$.

2) $A := [0, a] \mathbin{\bigcap\!\!\!\!\circ}\ X$ forms an ortholattice, where
$x \cap_A y := x \cap_L y$, $\;x \cup_A y := x \cup_L y$, and $x^* := x' \cap_A a$ for $x, y \in A$.

In the above condition, the meet (join) in A, for example, is denoted by $\cap_A$ $(\cup_A)$. We define an operator $\Psi : \mathcal{P}(L) \to \mathcal{P}(L)$ as:

$$\Psi(X) := \{a \in X \mid P_a(X) \text{ holds}\}$$

Note that Ψ is a decreasing operator, that is, $\Psi(X) \subseteq X$ holds for any subset $X \subseteq L$. A subset Y of L is a *fixpoint* of Ψ if $\Psi(Y) = Y$. Is is easy to see that any finite Boolean subalgera of $\mathfrak{L}$ is a fixpoint of Ψ. On the fixpoints of this operator Ψ, the following holds.

Theorem 7.5 A subset $Y \subseteq L$ is a fixpoint of Ψ, if and only if Y is a suborthomodular lattice in $\mathfrak{L}$.

Proof : Suppose Y is a suborthomodular lattice in $\mathfrak{L}$. Take any $a \in Y$, then $[0, a] \mathbin{\cap\!\!\!\circ} Y$ is an ortholattice by Lemma 7.3, and so $a \in \Psi(Y)$. It is obvious that $\Psi(Y) \subseteq Y$. Thus Y is a fixpoint of the operator Ψ. Conversely, Y is a fixpoint of Ψ. Take any $a \in Y = \Psi(Y)$. Then, $P_a(Y)$ holds, which means that $[0, a] \mathbin{\cap\!\!\!\circ} Y$ forms an ortholattice. Thus, by Lemma 7.3, we conclude that Y is a suborthomodular lattice in $\mathfrak{L}$. □

Consider the case where there is a suborthomodular lattice $\mathfrak{B} = \langle B, \cap, \cup, (\cdot)', 0, 1\rangle$ in $\mathfrak{L}$. Then the following holds.

Theorem 7.6 There is a maximal fixpoint among fixpoints that include the suborthomodular lattice $\mathfrak{B}$.

Proof : We employ Zorn's lemma. Put $\mathcal{F} := \{Y \subseteq L \mid B \subseteq Y, Y = \Psi(Y)\}$. Then $\mathcal{F}$ is not empty. Take any chain $\{C_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{F}$, and put $W := \mathbin{\cup\!\!\!\circ}_{\lambda\in\Lambda} C_\lambda$. We show that $W \in \mathcal{F}$. Of course $B \subseteq W$. So we have to show that W is a fixpoint of Ψ, in other words, that W is a suborthomodular lattice of $\mathfrak{L}$. It is obvious that each C_λ contains $0, 1$, and that the operations $\cap$, $\cup$, and $(\cdot)'$ in each C_λ are just the same as those in $\mathfrak{L}$, and hence, W forms a sublattice of $\mathfrak{L}$ with the same orthocomplementation. Take any $x \in W$. Then, since $\{C_\lambda\}_{\lambda\in\Lambda}$ is a chain, there exists a suborthomodular lattice C_μ ($\mu \in \Lambda$), such that $x \in C_\mu$. Therefore $x \cap x' = 0$ and $x'' = x$ hold. Similarly, take $x, y \in W$, then there exists a suborthomodular lattice C_ν ($\nu \in \Lambda$), such that $x, y \in C_\nu$. Therefore, we have that $x \leq y$ implies $y' \leq x'$ and that $x \leq y$ implies $y = x \cup (x' \cap y)$. Hence W forms a suborthomodular lattice, which means that $W \in \mathcal{F}$. Thus, by Zorn's lemma, we can conclude that $\mathcal{F}$ has a maximal element. □

On the other hand, we can define another operation on $\mathcal{P}(L)$, whose fixpoints, this time, form complete orthomodular lattices. For any subset $X \subseteq L$ and any element $a \in L$, $Q_a(X)$ denotes the following property:

1) $0, 1 \in X$.

2) $A := [0, a] \mathbin{\cap\!\!\!\circ} X$ forms a complete ortholattice, where $\bigcap_A S := \bigcap_L S$, $\bigcup_A S := \bigcup_L S$, and $x^* := x' \cap_A a$ for $S \subseteq A, x \in A$.

The subscripts of $\bigcap$ and $\bigcup$ have the same meaning as in the definition of the condition $P_a(X)$. We define an operator $\Phi : \mathcal{P}(L) \to \mathcal{P}(L)$ as:

$$\Phi(X) := \{a \in X \mid Q_a(X) \text{ holds}\}$$

Trivially, Φ is also a decreasing operator. Similar characterization theorem holds also for the fixpoints of this operator Φ.

Theorem 7.7 A subset $Y \subseteq L$ is a fixpoint of Φ, i.e., $\Phi(Y) = Y$, if and only if Y is a complete suborthomodular lattice in $\mathfrak{L}$ $\square$

The proof is almost the same as that of Theorem 7.5. This time, Corollary 7.4 is essential to it.

Note that the argument using Zorn's lemma does not work for this operator Φ because we have to deal with infinitary elements.

For a subset $X \subseteq L$ such that $0, 1 \in X$, define

$$\begin{aligned} \Phi^0(X) &:= X \\ \Phi^n(X) &:= \Phi(\Phi^{n-1}(X)) \quad (n \geq 1) \\ Z &:= \Cap_{n=0}^{\infty} \Phi^n(X) \end{aligned}$$

Then, the following holds.

Theorem 7.8 For any $X \subseteq L$ such that $0, 1 \in X$, Z is a fixpoint of Φ.

Proof : We have only to show that $Z \subseteq \Phi(Z)$. Take any $a \in Z$, then for any natural number $i \geq 0$, $a \in \Phi^{i+1}(X)$, which means that $[0, a] \Cap \Phi^i(X)$ forms a complete ortholattice. Put $Y_i := [0, a] \Cap \Phi^i(X)$ for each i. Take any subset $S \subseteq [0, a] \Cap Z$, then because S is also a subset of Y_i for each i, both $\bigcap_{Y_i} S$ and $\bigcup_{Y_i} S$ exist in Y_i for all i, and the former equals $\bigcap_L S$ for all i, and the latter equals $\bigcup_L S$ for all i. Therefore $[0, a] \Cap Z$ is a complete lattice. It is not difficult to show that $[0, a] \Cap Z$ is an ortholattice, since each Y_i is so. Thus $[0, a] \Cap Z$ is a complete ortholattice, and so $a \in \Phi(Z)$. $\square$

The completion problem of an orthomodular lattice is asking whether a given orthomodular lattice can be embedded into a complete orthomodular lattice or not. We formulate this problem by our fixpoints Ψ and Φ. Consider an arbitrary orthomodular lattice $\mathfrak{A} = \langle A, \cap, \cup, (\cdot)', 0, 1 \rangle$. This is of course an ortholattice, and hence by Theorem 7.1, there are a complete ortholattice $\mathfrak{L}$ and a map f such that $\mathfrak{A}$ can be embedded into $\mathfrak{L}$ by this f. Then, $f(A)$ is a suborthomodular lattice of $\mathfrak{L}$, and by Theorem 7.6, we can say that there exists a maximal fixpoint W of the operator Ψ on $\mathfrak{L}$, which includes $f(A)$ and is an orthomodular lattice.

Question 1: Does W form a complete orthomodular lattice?

If we could answer this question in the affirmative, we solved the problem. However, the author feels that there is a certain gap between the maximality and the completeness of W.

Another approach is by the operator Φ. Put $Z := \Cap_{n=0}^{\infty} \Phi^n(L)$. Then by Theorem 7.8, Z forms a complete orthomodular lattice.

Question 2: Does this Z include $f(A)$?

One of the difficult points to show that $f(A) \subseteq Z$, is the lack of *monotonicity* of the operator Φ, and it may happen that Φ destroys the structure of an orthomodular lattice which is included in a subset $X \subseteq L$.

Anyway, some new point of view is needed to solve the problem even if the answer is positive. If the answer of this problem is negative, then we have to face a new question, asking how can we obtain a semantic structure for the minimum predicate extension of the orthomodular logic **OML**.

Appendix

S.Tamura ([46]) introduced the Gentzen-type formal system **GOL** for the smallest orthologic **OL**, and showed the cut-elimination theorem for **OL**. Here his systems **GOL*** and **GOL** are introduced and a few theorems are presented according to his work.

Applying Maehara method to Tamura's cut-free system, we can reach a syntactical proof of the Craig's interpolation property for the logic **OL**.

First, **GOL*** is obtained from following axiom and rules:
In the following, α and β are formulas and Γ, Δ, Σ and Π are finite sequences of formulas.

Axiom:

$$\alpha \to \alpha$$

Rules:

$$\frac{\Gamma \to \Delta}{\Gamma \to \Delta, \alpha} (\to w) \qquad \frac{\Gamma \to \Delta}{\alpha, \Gamma \to \Delta} (w \to)$$

$$\frac{\Gamma \to \Delta, \alpha, \alpha}{\Gamma \to \Delta, \alpha} (\to c) \qquad \frac{\alpha, \alpha, \Gamma \to \Delta}{\alpha, \Gamma \to \Delta} (c \to)$$

$$\frac{\Gamma \to \Delta, \alpha, \beta, \Pi}{\Gamma \to \Delta, \beta, \alpha, \Pi} (\to e) \qquad \frac{\Sigma, \alpha, \beta, \Gamma \to \Delta}{\Sigma, \beta, \alpha, \Gamma \to \Delta} (e \to)$$

$$(\to \vee) \qquad (\wedge \to)$$

$$\frac{\Gamma \to \Delta, \alpha}{\Gamma \to \Delta, \alpha \vee \beta} \qquad \frac{\Gamma \to \Delta, \alpha}{\Gamma \to \Delta, \beta \vee \alpha} \qquad \frac{\alpha, \Gamma \to \Delta}{\beta \wedge \alpha, \Gamma \to \Delta} \qquad \frac{\alpha, \Gamma \to \Delta}{\alpha \wedge \beta, \Gamma \to \Delta}$$

$$\frac{\Gamma \to \alpha \quad \Gamma \to \beta}{\Gamma \to \alpha \wedge \beta} (\to \wedge) \qquad \frac{\alpha \to \Delta \quad \beta \to \Delta}{\alpha \vee \beta \to \Delta} (\vee \to)$$

$$\frac{\alpha \to \Delta}{\to \Delta, \neg\alpha} (\to \neg) \qquad \frac{\Gamma \to \alpha}{\neg\alpha, \Gamma \to} (\neg \to)$$

$$\frac{\Gamma \to \alpha}{\Gamma \to \neg\neg\alpha} (\to \neg\neg) \qquad \frac{\alpha \to \Delta}{\neg\neg\alpha \to \Delta} (\neg\neg \to)$$

$$\frac{\alpha \to \beta}{\neg\beta \to \neg\alpha} (\neg \to \neg)$$

$$(\to \neg\wedge) \qquad\qquad (\neg\vee \to)$$

$$\frac{\Gamma \to \neg\alpha}{\Gamma \to \neg(\alpha \wedge \beta)} \quad \frac{\Gamma \to \neg\alpha}{\Gamma \to \neg(\beta \wedge \alpha)} \quad \frac{\neg\alpha \to \Delta}{\neg(\beta \vee \alpha) \to \Delta} \quad \frac{\neg\alpha \to \Delta}{\neg(\alpha \vee \beta) \to \Delta}$$

$$\frac{\Gamma \to \neg\alpha \quad \Gamma \to \neg\beta}{\Gamma \to \neg(\alpha \vee \beta)} (\to \neg\vee) \qquad \frac{\neg\alpha \to \Delta \quad \neg\beta \to \Delta}{\neg(\alpha \wedge \beta) \to \Delta} (\neg\wedge \to)$$

$$\frac{\neg\alpha \to \Delta \quad \neg\beta \to \Delta}{\to \Delta, \alpha \wedge \beta} (\neg \to \wedge) \qquad \frac{\Gamma \to \neg\alpha \quad \Gamma \to \neg\beta}{\alpha \vee \beta, \Gamma \to} (\vee \to \neg)$$

The system **GOL** is obtained from **GOL*** by adding the following cut rule.

$$\frac{\Gamma \to \Delta \quad \Sigma \to \Pi}{\Gamma, \Sigma_\alpha \to \Delta_\alpha, \Pi} [\alpha](cut)$$

where both Σ and Δ should contain at least one occurrence of α and Σ_α denotes the sequence that lacks all the occurrences of α in the previous sequence Σ, and this cut rule can apply only when either Σ_α or Δ_α must be empty.

The next three theorems hold for **GOL** and **GOL***. Below we denote, for example, **GOL** $\vdash \Sigma \to \Delta$ to mean that the sequent $\Sigma \to \Delta$ is provable in the system **GOL**. Note that, in these systems, there is not the constant $\perp$.

Theorem A.1 (The completeness theorem)
For any formulas α and β, the following two conditions are equivalent:

(a) $\langle \alpha, \beta \rangle \in$ **OL**.

(b) **GOL** $\vdash \alpha \to \beta$.

□

Theorem A.2 (The cut-elimination theorem for GOL)
For any sequence of formulas Γ and Δ, the following two conditions are equivalent:

(a) **GOL** $\vdash \Gamma \to \Delta$.

(b) **GOL*** $\vdash \Gamma \to \Delta$

□

Theorem A.3 (The Craig's interpolation property for GOL)
For any sequence of formulas Γ, Δ, suppose **GOL** $\vdash \Gamma \to \Delta$. If there are propositional variables which are common to Γ and Δ, then there exists a formula γ which satisfies the following:

(1) Both $\mathbf{GOL} \vdash \Gamma \to \gamma$ and $\mathbf{GOL} \vdash \gamma \to \Delta$ hold.

(2) Only these propositional variables which are common to Γ and Δ appear in γ.

If there is no propositional variable which is common to Γ and Δ, either $\mathbf{GOL} \vdash \Gamma \to$ or $\mathbf{GOL} \vdash\to \Delta$ (possibly both) holds. □

In order to prove Theorem A.3, we have only to apply Maehara's method ([33]) to the system $\mathbf{GOL}^*$. Note that, in this case, we do not need the notion of *partition* of sequences of formulas, that is usually essential in proving Craig's interpolation property for systems which have an implication connectives.

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