

Title	Code design for joint decoding of correlated sources using algebraic network coding over AWGN channels
Author(s)	Shimizu, Takashi; Matsumoto, Tad; Murata, Hidekazu; Yamamoto, Koji; Yoshida, Susumu
Citation	7th International Conference on Information, Communications and Signal Processing, 2009. ICICS 2009.: 1-5
Issue Date	2009-12
Type	Conference Paper
Text version	publisher
URL	http://hdl.handle.net/10119/9103
Rights	Copyright (C) 2009 IEEE. Reprinted from 7th International Conference on Information, Communications and Signal Processing, 2009. ICICS 2009., 1-5. This material is posted here with permission of the IEEE. Such permission of the IEEE does not in any way imply IEEE endorsement of any of JAIST's products or services. Internal or personal use of this material is permitted. However, permission to reprint/republish this material for advertising or promotional purposes or for creating new collective works for resale or redistribution must be obtained from the IEEE by writing to pubs-permissions@ieee.org . By choosing to view this document, you agree to all provisions of the copyright laws protecting it.
Description	



Code Design for Joint Decoding of Correlated Sources using Algebraic Network Coding over AWGN Channels

Takashi Shimizu[†], Tad Matsumoto^{††}, Hidekazu Murata[†], Koji Yamamoto[†], and Susumu Yoshida[†]

[†]Graduate School of Informatics, Kyoto University

Yoshida-honmachi, Sakyo-ku, Kyoto, 606-8501 Japan

^{††}Japan Advanced Institute of Science and Technology (JAIST), Japan, and

Center for Wireless Communication (CWC) at University of Oulu, Finland

Email: contact-h21@hanase.kuee.kyoto-u.ac.jp

Abstract— We consider the problem of joint decoding of signals transmitted from two correlated sources at a destination. In order to achieve high spectral efficiency at a high range of the channel signal-to-noise power (SNR), a high transmission rate is required. In this paper, we design a class of codes for joint-decoding by exploiting bit interleaved coded modulation with iterative decoding (BICM-ID) technique. Extrinsic information transfer (EXIT) chart analysis is used to evaluate the convergence property as well as to examine the optimality of the system. The simulation results show that the joint decoding with the designed code achieves the performance close to theoretical limits supported by the Slepian-Wolf and Shannon theorems.

I. INTRODUCTION

Joint decoding of signals transmitted from correlated sources can achieve higher performance in comparison to separate decoding of signals from each source because the correlation can well be exploited in decoding. The theoretical limit of compression rates for two correlated source information has been proved by Slepian and Wolf [1]. The practical design of source-channel coding of two binary information sequences using punctured turbo codes over additive white Gaussian noise (AWGN) channels is proposed in [2]. The performance of source-channel coding using punctured turbo codes is very close to the theoretical limits established by the combination of Shannon and Slepian-Wolf theorems [2], [3]. In a higher signal-to-noise power ratio (SNR) value range of the channel, however, higher transmission rates are required to approach the theoretical limit, however, it is still not yet fully examined that whether or not applying the binary source-channel codes to the system with high order modulations can achieve good performance.

Bit interleaved coded modulation with iterative decoding (BICM-ID) techniques have been proposed as high order modulation techniques. Transmitter of BICM-ID systems is a concatenation of an encoder and a mapper, which are separated by an interleaver. Recently, a BICM-ID technique using repetition-based codes and arithmetic extended mappings has been proposed [4]. The BICM-ID scheme can achieve performance close to the capacity of point-to-point communication and also reduce computational complexity of decoding by bit-reduction encoding in comparison to ordinary extended mapping schemes [5]. Ref [4] combines the idea of algebraic network coding [6] and BICM-ID, where the authors call

the technique bit reduction encoding, though. This BICM-ID scheme is exploited for the joint decoding of correlated sources over AWGN channels in this paper.

Extrinsic information transfer (EXIT) chart [7] has been known as a useful tool for analyzing convergence properties of iterative decoding. EXIT chart, which describes mutual information (MI) transfer characteristics, required when examining the optimality of the system without performing lengthy chain simulations; To evaluate the actual convergence characteristics, decoding trajectory has to be used. In [8], [9], repeat-accumulate (RA) codes and low-density parity-check (LDPC) codes are designed by changing degree distributions to match the EXIT functions. If a system has only one iteration loop, the EXIT functions can be expressed in two-dimension (2D) chart, and the curve-fitting techniques can be used for the code optimization based on a single 2D EXIT chart as in [8], [9]. However, the joint decoding of correlated sources is comprised of more than two iteration loops. Therefore, to analyze the joint decoder, the EXIT projection techniques [10] are used. The code is designed by using curve-fitting method on the projected EXIT chart. The joint decoder with the codes, of which projected EXIT charts are well matched is expected to achieve good performance. The simulation results show that the joint decoding with the designed code can achieve the performance that is 1.7 dB away from the theoretical limits provided by Slepian-Wolf and Shannon theorems.

The remainder of this paper is organized as follows. In Section II, we discuss the system model of joint decoding with correlated sources. In Section III, an example of code design is shown and the performance with the code is evaluated by the simulation. Section IV concludes the paper with summary and our future works.

II. SYSTEM MODEL

In this system model, two source stations S_1 and S_2 , and one destination D are considered as shown in Fig. 1. The source stations S_1 and S_2 generate information sequence U_1 and U_2 , respectively. The sequences are correlated, i.e., $H(U_1, U_2) < H(U_1) + H(U_2)$, where $H(\cdot)$ indicate the entropy function. The correlation model between the sources is given in the following way [2]. U_1 is the binary i.i.d. sequence, $U_1 = \{u_1^{(1)}, u_1^{(2)}, \dots, u_1^{(K)}\} \in \{0, 1\}^K$ and $P(u_1^{(k)} = 0) = 1/2$,

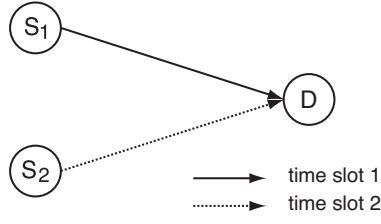


Fig. 1. System model of joint decoding with two correlated sources, where S_1 and S_2 generate correlated information.

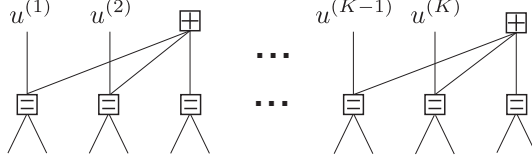


Fig. 2. Graph representation of repetition-based encoder with single parity check bits.

where K is the length of the sequence. The sequence U_2 is defined as $u_2^{(k)} = e^{(k)} \oplus u_1^{(k)}$, where \oplus indicates addition over the Galois field 2 and $e^{(k)}$ is memoryless binary random variable, taking value 1 with probability p_e , i.e., $P(e^{(k)} = 1) = p_e$.

Both source stations transmit information sequences using time-division multiplexing technique over AWGN channels. The information sequences are encoded using repetition-based codes. The encoders add a single parity bit every m_{SPC} bits, then the sequences including single parity bits are repeated ℓ times. Coded sequence is denoted by $C_s = \{c_s^{(1)}, c_s^{(2)}, \dots, c_s^{(N_s)}\}$, where $s = 1$ or 2 indicates the source index, and N_s indicates the length of the coded sequence. The rate of the codes are $R_{c,s} = K/N_s$. Fig. 2 shows the example of graph representation of the codes. Since in practical design of code, the values of ℓ are not constant, the degrees of variable nodes take various values. The ratio between the number of bits which are repeated ℓ times and the number of bits before repetition operations are defined by $d_{v,\ell}$. Here, $\sum_{\ell} d_{v,\ell} = 1$.

Coded bits are permuted by the interleaver π_s , and the interleaved coded bits are denoted by \tilde{C}_s . The coded bits \tilde{C}_s are converted into symbols using extended mapping in which every 16 coded bits $\{b_1, b_2, \dots, b_{16}\}$ are reduced to 8 bits $\{t_1, t_2, \dots, t_8\}$ as follows [4].

$$t_1 = b_1 \oplus b_2 \oplus b_3, \quad t_2 = b_4 \oplus b_5, \quad t_3 = b_6 \oplus b_7, \quad t_4 = b_8, \\ t_5 = b_9 \oplus b_{10} \oplus b_{11}, \quad t_6 = b_{12} \oplus b_{13}, \quad t_7 = b_{14} \oplus b_{15}, \quad t_8 = b_{16}$$

After the bit reductions, $\{t_1, t_2, t_3, t_4\}$ and $\{t_5, t_6, t_7, t_8\}$ are mapped to s_I and s_Q , respectively, by applying mapping rule shown as in Fig. 3. By this mapping rule, 16 bits are allocated to one complex symbol. The transmitted symbol is $s_I + js_Q$. The factor graph representation of the mapping is shown in Fig. 4.

A. Iterative Demapping and Decoding

The destination station jointly decodes the received signals transmitted by S_1 and S_2 based on sum-product algorithm

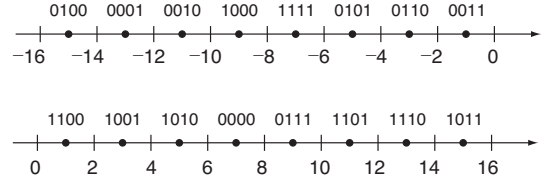


Fig. 3. Base Mapping of 16 ASK.

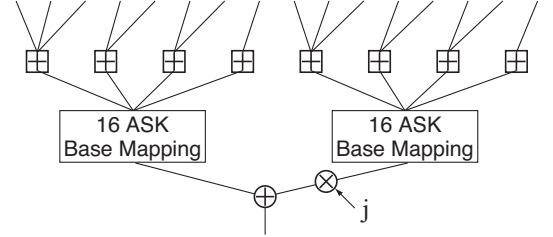


Fig. 4. Graph representation of mapping.

(SPA) on a factor graph [11]. Since real and imaginary parts of the signals preserve orthogonality in this system, the decoding can be performed independently at real and imaginary part. The decoder consists of two demappers (DEM), decoders (DEC), interleavers, and an factor updating log likelihood ratios (LLR) using information of correlation (COR) as shown in Fig. 5. In this paper, the connection of base-demapper and check-node decoder (CND) is referred to as demapper, and that of one variable node (VND) and single-parity check (SPC) decoder as decoder. Since there is no loop on the factor graphs inside the demappers or decoders, only one iteration is needed to improve LLRs, every time the updated *a priori* information is provided. Therefore, iterative decoding is performed only between demappers and decoders, and a decoder and the other decoder via the COR factor.

1) *Iterative Demapping of Base-Mapping*: Let r be the real part or imaginary part of the received signal. The base mapper makes four bits $\{t_1, t_2, t_3, t_4\}$ converted to one symbol of 16 ASK. Constellation points are denoted by ϕ_k , where $k = 1, \dots, 16$, and $\phi(t_m = b)$ indicates a set of constellation points of 16 ASK which are taken bit $t_m = b$ ($m = 0, \dots, 4, b = 0, 1$). The extrinsic LLR of the bit t_m is calculated by

$$L_e(t_m) = \ln \frac{\sum_{\phi_k = \phi(t_m=1)} p(r|\phi_k)p(\phi_k|t_m=1)}{\sum_{\phi_k = \phi(t_m=0)} p(r|\phi_k)p(\phi_k|t_m=0)}, \quad (1)$$

where $p(r|\phi_k)$ is the conditional probability distribution function (pdf), and expressed as,

$$p(r|\phi_k) = \frac{1}{\sqrt{2\pi}\sigma_s} \exp\left(-\frac{|r - \phi_k|^2}{2\sigma_s^2}\right). \quad (2)$$

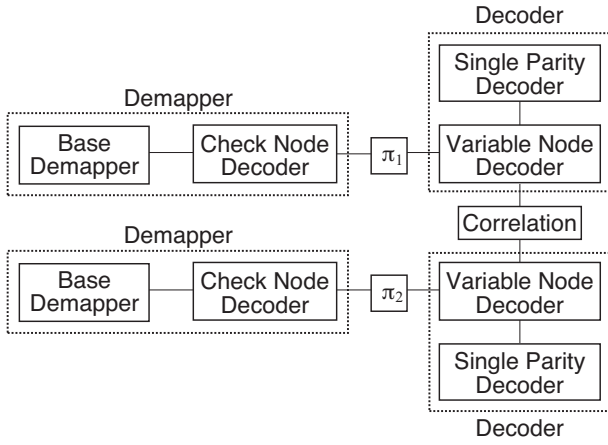


Fig. 5. Decoder structure.

With *a priori* LLR $L_a(t_n)$, (1) can be expressed as

$$L_e(t_m) = \ln \frac{\sum_{\phi_k=\phi(t_m=1)} p(r|\phi_k) \exp\left(\sum_{n \neq m} \phi_k^{-1}(n) L_a(t_n)\right)}{\sum_{\phi_k=\phi(t_m=0)} p(r|\phi_k) \exp\left(\sum_{n \neq m} \phi_k^{-1}(n) L_a(t_n)\right)}, \quad (3)$$

where $\phi_k^{-1}(n)$ indicates the value of n -th bit corresponding to the constellation point ϕ_k .

2) *Check and Variable Node Decoding Operation*: At the check node of degree n_{COR} , extrinsic LLR of a bit b_m is calculated by,

$$L_e(b_m) = 2 \tanh^{-1} \left(\sum_{k=1, k \neq m}^{n_{\text{COR}}} \tanh \frac{L_a(b_k)}{2} \right), \quad (4)$$

where $L_a(b_k)$ is *a priori* LLR of b_k . The check node decoding is performed at both the CND and SPC.

The extrinsic LLR of a bit b_m from variable node of degree n_{VAR} is given by

$$L_e(b_m) = \sum_{k=1, k \neq j}^{n_{\text{VAR}}} L_a(b_k), \quad (5)$$

where $L_a(b_k)$ is *a priori* LLR of b_k . This variable node decoding is performed at the VND.

3) *LLR updating operation using Correlation Information*: The COR factor updates the LLRs using the correlation information. The probability $P(u_s^{(k)} = 0, 1)$ of $u_s^{(k)} = 0, 1$ is calculated by using the probability $P(u_s^{(k)} = 0, 1)$ of the other information bit $u_s^{(k)} = 0, 1$ and correlation parameter p_e ,

$$P(u_s^{(k)} = 0) = (1 - p_e)P(u_s^{(k)} = 0) + p_e P(u_s^{(k)} = 1), \quad (6)$$

$$P(u_s^{(k)} = 1) = (1 - p_e)P(u_s^{(k)} = 1) + p_e P(u_s^{(k)} = 0). \quad (7)$$

The LLR updating rule at the COR factor is expressed by converting (6) and (7) to LLR domain,

$$L_e(u_s^{(k)}) = \ln \left(\frac{(1 - p_e) \exp(L_a(u_s^{(k)})) + p_e}{(1 - p_e) + p_e \exp(L_a(u_s^{(k)}))} \right). \quad (8)$$

B. Mutual Information Transfer functions

The demapper MI transfer function is denoted by

$$I_{E, \text{DEM}} = T_{\text{DEM}}(I_{A, \text{DEM}}, \gamma_{S_s \text{D}}), \quad (9)$$

where $\gamma_{S_s \text{D}}$ indicates the SNR of $S_s \text{D}$ channel, and $I_{A, \text{DEM}}$ denotes the MI between the transmitted coded bit and the demapper output LLR, fed into the demapper block. The MI transfer function of the VND to the demapper is expressed as

$$I_{\text{VND} \rightarrow \text{DEM}} = T_{\text{VND} \rightarrow \text{DEM}}(I_{E, \text{DEM}}, I_{E, \text{SPC}}, I_{E, \text{COR}}) \quad (10)$$

$$= \frac{m_{\text{SPC}}}{(m_{\text{SPC}} + 1)} \sum_{\ell} \ell \cdot d_{v, \ell} J \left(\sqrt{\sigma_{\text{D}, \text{S}, \text{COR}}^2(\ell - 1)} \right)$$

$$+ \frac{1}{(m_{\text{SPC}} + 1)} \sum_{\ell} \ell \cdot d_{v, \ell} J \left(\sqrt{\sigma_{\text{D}, \text{S}}^2(\ell - 1)} \right), \quad (11)$$

where

$$\sigma_{\text{D}, \text{S}, \text{COR}}^2(\ell) = \ell \cdot J^{-1}(I_{E, \text{DEM}})^2 + J^{-1}(I_{E, \text{SPC}})^2 + J^{-1}(I_{E, \text{COR}})^2 \quad (12)$$

$$\sigma_{\text{D}, \text{S}}^2(\ell) = \ell \cdot J^{-1}(I_{E, \text{DEM}})^2 + J^{-1}(I_{E, \text{SPC}})^2, \quad (13)$$

$J(\cdot)$ indicates the function that converts the standard deviation of LLR to MI in AWGN channels, and $J^{-1}(\cdot)$ is its inverse function [8], [9], $I_{E, \text{DEM}}$, $I_{E, \text{SPC}}$, and $I_{E, \text{COR}}$ are MIs of LLRs from the DEM, SPC, and COR to the VND, respectively.

The MI transfer function from the VND to the SPC is similarly expressed as

$$I_{\text{VND} \rightarrow \text{SPC}} = T_{\text{VND} \rightarrow \text{SPC}}(I_{E, \text{DEM}}, I_{E, \text{SPC}}, I_{E, \text{COR}}) \quad (14)$$

$$= \frac{m_{\text{SPC}}}{m_{\text{SPC}} + 1} \cdot \sum_{\ell} d_{v, \ell} J \left(\sqrt{\sigma_{\text{D}, \text{COR}}^2(\ell)} \right)$$

$$+ \frac{1}{m_{\text{SPC}} + 1} \cdot \sum_{\ell} d_{v, \ell} J \left(\sqrt{\sigma_{\text{D}}^2(\ell)} \right), \quad (15)$$

where

$$\sigma_{\text{D}, \text{COR}}^2 = \ell \cdot J^{-1}(I_{E, \text{DEM}})^2 + J^{-1}(I_{E, \text{COR}})^2 \quad (16)$$

$$\sigma_{\text{D}}^2 = \ell \cdot J^{-1}(I_{E, \text{DEM}})^2. \quad (17)$$

The MI transfer function from the SPC decoder to the VND is approximated by

$$I_{E, \text{SPC}} = T_{\text{SPC}}(I_{\text{VND} \rightarrow \text{SPC}}) \quad (18)$$

$$\approx 1 - J \left(\sqrt{m_{\text{SPC}}} J^{-1}(1 - I_{\text{VND} \rightarrow \text{SPC}}) \right). \quad (19)$$

The MI transfer function of the VND to the COR factor is expressed as

$$I_{\text{VND} \rightarrow \text{COR}} = \sum_{\ell} d_{v, \ell} J \left(\sqrt{\sigma_{\text{D}, \text{S}}^2(\ell)} \right) \quad (20)$$

The MI transfer function of LLR updating factor which uses correlation information is denoted by

$$I_{E,COR}(L_{E,COR,s}; U_s) = T_{COR}(I_{A,COR}, p_e), \quad (21)$$

where (s, \hat{s}) is the source pair, (1, 2) or (2, 1), and $I_{A,COR}$ indicates the MI of *a priori* LLR given by,

$$I_{A,COR} = I(L_{A,COR,s}; U_{\hat{s}}). \quad (22)$$

By the sufficient activations (9), (10), (14), and (18) with the fixed $I_{A,COR}$, $I_{VND \rightarrow COR}$ converges fixed point $\tilde{I}_{VND \rightarrow COR}$. The EXIT function between $I_{A,COR}$ and $\tilde{I}_{VND \rightarrow COR}$ is defined by $T_{DEM+DEC}$,

$$\tilde{I}_{VND \rightarrow COR} = T_{DEM+DEC}(I_{A,COR}) \quad (23)$$

The EXIT function $T_{DEM+DEC}$ can be seen as the projected function which indicates the limit of MI improvement by iterations between demapper and decoder with fixed $I_{A,COR}$.

C. Rate limit based on Slepian-Wolf and Shannon Theorems

The theoretical limits of compression rate of correlated sources are given by Slepian-Wolf theorem [1]. The source compression rates of U_1 and U_2 are denoted by $R_{s,1}$ and $R_{s,2}$, respectively, and the achievable rates can be expressed as:

$$R_{s,1} \geq H(U_1|U_2) \quad (24)$$

$$R_{s,2} \geq H(U_2|U_1) \quad (25)$$

$$R_{s,1} + R_{s,2} \geq H(U_1, U_2). \quad (26)$$

Channel coding with rates $R_{c,1}$ and $R_{c,2}$ is used by S_1 and S_2 , respectively. The source stations S_1 and S_2 transmit signals using extended mapping with $R_{m,1}$ and $R_{m,2}$ bits per symbol. The capacities of the S_1D and S_2D channels are denoted by C_{S_1D} and C_{S_2D} , respectively. The relation between achievable rates and capacities is expressed following inequalities:

$$R_{s,1}R_{c,1}R_{m,1} \leq C_{S_1D} \quad (27)$$

$$R_{s,2}R_{c,2}R_{m,2} \leq C_{S_2D}. \quad (28)$$

Combining the inequalities (24–28), we get,

$$H(U_1|U_2) \leq \frac{C_{S_1D}}{R_{c,1}R_{m,1}} \quad (29)$$

$$H(U_2|U_1) \leq \frac{C_{S_2D}}{R_{c,2}R_{m,2}} \quad (30)$$

$$H(U_1, U_2) \leq \frac{C_{S_1D}}{R_{c,1}R_{m,1}} + \frac{C_{S_2D}}{R_{c,2}R_{m,2}}. \quad (31)$$

The theoretical limits of rates are given by (29), (30), and (31).

D. Problem Definition

To improve the spectrum efficiency, high transmission rates are preferable. The transmission rate is given by $R_{c,s}R_{m,s}$, where $s = 0$ or 1. Since rate $R_{m,s} = 16$ bits/symbol is constant in the system, coding rate $R_{c,s}$ has to be high with guaranteeing that the EXIT functions have the way to converge

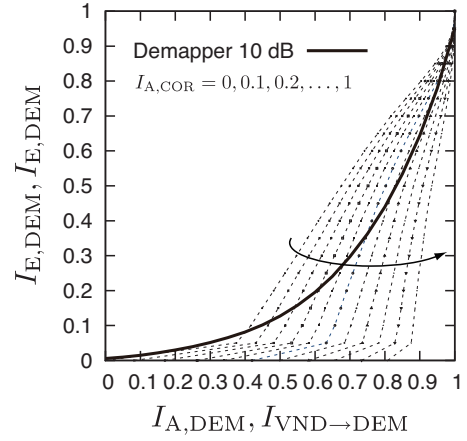


Fig. 6. EXIT functions of demapper and decoder.

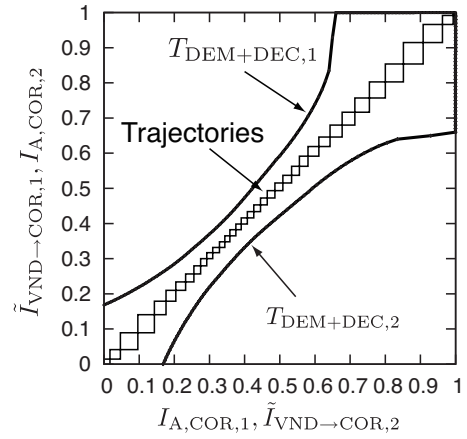


Fig. 7. Projected EXIT function and decoding trajectories.

to the point where mutual information is unity. This problem can be written as following.

$$\max_{d_{v,\ell}} R_{c,s} = \frac{1}{\sum_{\ell} \ell \cdot d_{v,\ell}} \cdot \frac{m_{SPC}}{m_{SPC} + 1} \quad (32)$$

$$\text{subject to } \tilde{I}_{VND \rightarrow COR} > I_{A,COR} + \epsilon_{\text{margin}}, \quad (33)$$

where ϵ_{margin} denotes to the margin to keep EXIT function having the tunnel to reach perfect MI in practical realization.

III. EXAMPLE OF CODE DESIGN

The systematic designs are treated in this paper as $R_{c,1} = R_{c,2} = R_c$, and $\gamma_{S_1D} = \gamma_{S_2D} = 10$ dB. The SNR of both S_1D and S_2D are the same. An example of the distribution of repetition times $d_{v,\ell}$ designed keeping (33) is $d_{v,2} = 0.9875$ and $d_{v,86} = 0.01187$ and single parity check bits are appended every 31 bits, i.e., $m_{SPC} = 31$, when the correlation parameter is $p_e = 0.025$ and length of information sequences are 50865. The coding rate of this code is $R_c = 0.3179$. The demapper EXIT function at 10 dB and the decoder functions when $I_{A,COR} = 0, 0.1, \dots, 1$ are shown in Fig. 6 with the designed codes. In Fig. 7, the projected EXIT function and snapshots

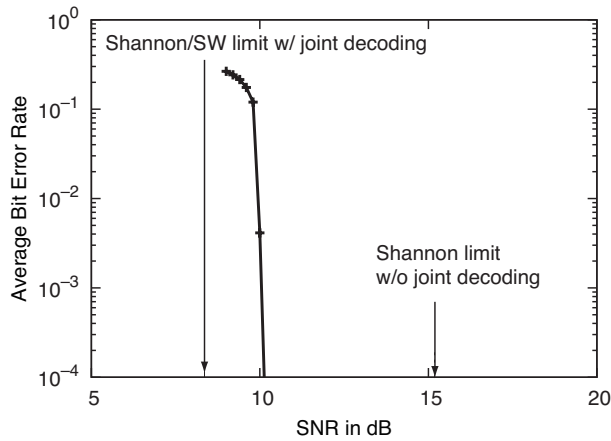


Fig. 8. Projected EXIT function and decoding trajectories.

of decoding trajectories are shown. Since the projected EXIT functions $T_{\text{DEM+DEC},1}$ does not intersect with the function $T_{\text{DEM+DEC},2}$ and trajectories can reach a (1, 1) point, perfect information of U_1 and U_2 can be retrieved by iterations. In Fig. 8, the bit-error ratio (BER) performance is shown. Since the transmission rate is $R_m R_c = 5.086$, The limit of SNR with coding rate $R_c = 0.3179$ is 8.355 dB. With the designed code, about 1.7 dB away from the theoretical limit. If the signals transmitted by sources are independently decoded at the destination station, the Shannon limit of SNR is 15.18 dB. By jointly decoding of both received signal, about 5 dB gain can be achieved.

IV. CONCLUSION

The code design for joint decoding of the signals transmitted from correlated sources are considered. By using curve-fitting method on the projected EXIT chart, the code is proposed to achieve close to the limit established by the Slepian-Wolf and Shannon theorems. For future work, the code design at asymmetric channel gains will be considered.

ACKNOWLEDGMENTS

This work is supported in part by a Grant-in-Aid for Scientific Research (A) (no. 20246067) from the Japan Society for the Promotion of Science (JSPS).

REFERENCES

- [1] D. Slepian and J. K. Wolf, "Noiseless coding of correlated information sources," *IEEE Trans. Inf. Theory*, vol. 19, no. 4, pp. 471–480, July 1973.
- [2] J. Garcia-Frias and Y. Zhao, "Near-Shannon/Slepian-Wolf performance for unknown correlated sources over AWGN channels," *IEEE Trans. Commun.*, vol. 53, no. 4, pp. 555–559, Apr. 2005.
- [3] T. M. Cover and J. A. Thomas, *Element of Information Theory*. Wiley, 2006.
- [4] T. Yano and T. Matsumoto, "Arithmetic extended-mapping for BICM-ID with repetition codes," *Proc. ITG/IEEE Workshop on Smart Antennas*, pp. 16–19, Feb. 2009.
- [5] P. Henkel, "Extended mappings for bit-interleaved coded modulation," *Proc. IEEE PIMRC '06*, pp. 11–14, Sept. 2006.
- [6] R. Zhang and L. Hanzo, "Physical-layer algebraic network coding and superposition coding for the multi-source cooperation aided uplink," *Proc. IEEE VTC 2009-Spring*, 2009.
- [7] A. Ashikhmin, G. Kramer, and S. ten Brink, "Extrinsic information transfer functions: Model and erasure channel properties," *IEEE Trans. Inf. Theory*, vol. 50, no. 11, pp. 2657–2673, Nov. 2004.
- [8] S. ten Brink, "Design of repeat-accumulate codes for iterative detection and decoding," *IEEE Trans. Signal Process.*, vol. 51, no. 11, pp. 2764–2772, Nov. 2003.
- [9] S. ten Brink, G. Kramer, and A. Ashikhmin, "Design of low-density parity-check codes for modulation and detection," *IEEE Trans. Commun.*, vol. 52, no. 4, pp. 670–678, Apr. 2004.
- [10] F. Brännström, L. K. Rasmussen, and A. J. Grant, "Convergence analysis and optimal scheduling for multiple concatenated codes," *IEEE Trans. Inf. Theory*, vol. 51, no. 9, pp. 3354–3364, Sept. 2005.
- [11] F. Kschischang, B. Frey, and H. Loeliger, "Factor graphs and the sum-product algorithm," *IEEE Trans. Inf. Theory*, vol. 47, no. 2, pp. 498–519, Feb. 2001.