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Gentzen Style Sequent Calculi for Some Subsystems of Intuitionistic Logic

by

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Abstract

In this thesis we study subsystems of intuitionistic propositional logic, some of which have not been given satisfactory Gentzen style sequent calculi. We provide appropriate sequent calculi for such logics mainly by the method of *dual-context sequents* and give a new insight into the realm of subsystems of intuitionistic logic.

First we discuss sequent calculi for subintuitionistic logics K^I and BPC which are defined semantically using Kripke models. Extending a known sequent calculus for K^I with ordinary sequents, we introduce a sequent calculus for BPC. These systems are, however, not satisfactory in the respect that the rule for implication involves many premisses. Then we consider dual-context sequents, which have proved popular in the field of linear logic. After giving an interpretation of the sequents in Kripke models, we develop a dual-context sequent calculus, which is closely related to Gentzen's sequent calculus for intuitionistic logic. The completeness theorem of the system with respect to the class of Kripke models for K^I is shown by means of a construction of the canonical model. We also introduce a dual-context sequent calculus that is complete with respect to the class of Kripke models for BPC. The cut-elimination theorem for the dual-context sequent calculi is proved by syntactical methods including more global proof transformation than the ordinary proof of cut-elimination.

Next we investigate relationships between subintuitionistic logics and substructural logics, considering Hilbert style systems that characterize the implicational fragments of subintuitionistic logics and substructural logics. The investigation clarifies the inclusion relationships between the sets of formulas that are provable in each Hilbert style system for these logics.

Finally we discuss sequent calculi for noncommutative substructural logics, particularly the logic BB′I. This logic is important in the respect that it is a noncommutative version of the implicational fragment of linear logic. While the usual sequent calculus for BB′I is defined using merge operation, we introduce a sequent calculus for BB′I without any merge operation. Roughly speaking, the system is obtained from the dual-context sequent calculus for BPC by deleting the structural rules, according to the observation that BB′I is a subsystem of BPC. The cut-elimination theorem for the system is proved using global proof transformation technique analogous to that used in the proof of the cut-elimination theorem for the dual-context sequent calculi for subintuitionistic logics.

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Contents

Chapter 1

Introduction

In the middle of 1930s, Gerhard Gentzen [18] introduced the logical calculi **LJ** for intuitionistic logic and **LK** for classical logic. These calculi consist of rules to manipulate not single formulas but what we call *sequents*: finite sequences of formulas separated by commas and the symbol '$\rightarrow$'. This extension of primitive syntactical concepts enabled Gentzen to enunciate and prove the *Hauptsatz* in a convenient form. In the calculi **LJ** and **LK**, the *Hauptsatz* appears as the *cut-elimination theorem*, which permits solutions of important problems on intuitionistic logic and on classical logic. The cut-elimination theorem for many other sequent calculi introduced after Gentzen's **LJ** and **LK** has also been applied to various studies in logic and computer science.

In this thesis we discuss sequent calculi for subsystems of intuitionistic propositional logic; in particular we deal with some logics for which satisfactory sequent calculi have not been given. In order to provide sequent calculi for such logics, we sometimes consider sequents with one more symbol ';', which are referred to as *dual-context sequents*. The main goal of this thesis is to introduce appropriate sequent calculi for these subsystems of intuitionistic logic and prove the cut-elimination theorem for them.

In the following we explain the background of this research, motivating our approach, and outline the work contained in the thesis.

1.1 Background and motivation

By its constructive nature, intuitionistic logic has played a central role in the study of logic in computer science. Constructive argument induces an algorithmic interpretation of formulas, which is made explicit by the Curry–Howard correspondence [27] between the natural deduction system for intuitionistic logic and the typed λ-calculus; formulas correspond to types, proofs to terms, and reduction of proofs to reduction of terms. (For exposition, see, e.g. Chapter 3 of [10], Chapter 6 of [24].) In [27], Howard also observed a correspondence between cut-free proofs in a Gentzen style sequent calculus and normal forms in the typed λ-calculus. This observation was refined by Herbelin [22] using sequents with the symbol ';', and then the Curry–Howard correspondence for sequent calculi has been one of recent topics in theoretical computer science. (A concise survey of this topic is found in the introduction of [48]; see also [33].)

While natural deduction systems are suited to only a few nonclassical logics, sequent calculi allow us to formalize much more nonclassical logics elegantly. Among others, logics formalized by sequent calculi without some of the structural rules are called *substructural* *logics*. They include linear logic, BCK logic, relevant logic and Lambek calculus, which have been studied separately with their own motivations. (For general information on substructural logics, see [15], [37].) The structural rules, in the intuitionistic case, are given as follows. (A, B, C denote formulas and Γ, Δ denote sequences of formulas.)

$$\frac{\Gamma, A, A, \Delta \to C}{\Gamma, A, \Delta \to C} \text{ (contraction)} \qquad \frac{\Gamma, \Delta \to C}{\Gamma, A, \Delta \to C} \text{ (weakening)}$$

$$\frac{\Gamma, A, B, \Delta \to C}{\Gamma, B, A, \Delta \to C} \text{ (exchange)}$$

An intuitive meaning of a sequent $\Gamma \to C$ is that one can obtain C using the formulas occurring in Γ. The absence of the structural rules puts restrictions on the use of the formulas. Without the contraction rule, each formula cannot be used more than once, and without the weakening rule, each formula must be used at least once. Moreover, without the exchange rule, one must take care on the order of occurrence of the formulas in Γ. Because of these effects, substructural logics are sometimes called *resource-conscious* logics. Although logics obtained from Gentzen's sequent calculus **LJ** by deleting some of the structural rules are subsystems of intuitionistic logic, a restricted form of contraction and weakening is available in intuitionistic linear logic with the modality '!'. Given the Girard translation [19], intuitionistic linear logic retains the constructive aspect of intuitionistic logic, and also it enables more delicate argument on the use of resources. (For an introduction to linear logic, see [21])

Another kind of subsystems of intuitionistic logic, which we call *subintuitionistic logics* following [36], are obtained semantically using Kripke models. While Kripke models for intuitionistic logic are based on preorders $\langle W, \leq \rangle$ with reflexive and transitive relation $\leq$, Kripke models for subintuitionistic logics are often based on $\langle W, R \rangle$ with more general binary relation R. In this thesis we consider particularly the logic K^I defined by Kripke models with any binary relation R, and the logic BPC defined by Kripke models with any transitive relation R. Those Kripke models for subintuitionistic logics do not necessarily force the formulas $(A \supset (A \supset B)) \supset (A \supset B)$ and $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$ which correspond to the contraction and the exchange rules, respectively, in the sequent calculus **LJ**. This means that subintuitionistic logics may qualify as substructural logics and that it is worth investigating the relationships between these logics for the purpose of developing the resource-conscious aspects of subintuitionistic logics. Our first goal is thus to provide sequent calculi for subintuitionistic logics that are suitable to compare with systems for substructural logics.

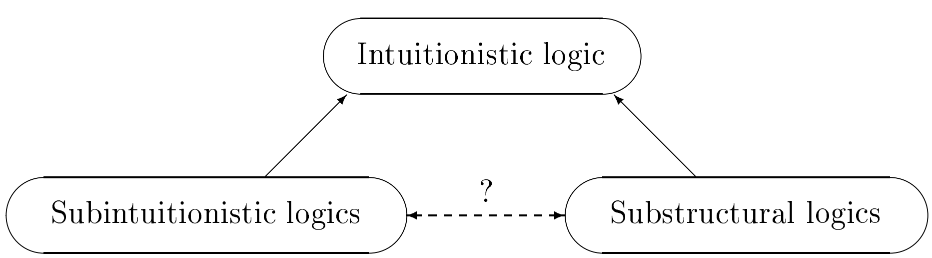


Figure 1.1: Subsystems of intuitionistic logic

The class of subintuitionistic logics was studied by Corsi [12], Došen [14], Restall [36], giving Hilbert style systems for the least subintuitionistic logic K^I and its extensions. Gentzen style sequent calculi for subintuitionistic logics including K^I were also given by Gabbay and Olivetti [17] in the style of labelled deductive system and by Wansing [50] in the style of Display Logic. Their systems are, however, not suitable for our purpose because they are based on sequents involving rather complicated notations and therefore considerably different from Gentzen's sequent calculi **LJ** and **LK**. In contrast to them, Kashima in his unpublished manuscript [28] has introduced sequent calculi for subintuitionistic logics that are based on the same sequents as those of **LK**. We discuss in detail his system for K^I in Section 3.2.

The logic BPC, which is the abbreviated name for Basic Propositional Calculus, is one of subintuitionistic logics, but it has also been studied independently. This logic was first introduced by Visser [49] and recently developed by Ardeshir and Ruitenburg [5], [6]. Other studies on BPC with various motivations are found in [1], [39], [41], [45]. From our viewpoint, BPC is convenient to consider its relationship to substructural logics with the weakening rule such as BCK logic, since the formulas of the form $A \supset (B \supset A)$ always hold in BPC. Gentzen style sequent calculi for BPC have also been given several times, but all of them are not satisfactory because they do not enjoy the subformula property in the usual sense. The first sequent calculus for BPC was given by Ardeshir [4]. The system includes a rule to infer $\Gamma \to A \supset C$ from $\Gamma \to A \supset B$ and $\Gamma \to B \supset C$, which leads to failure of the subformula property. The second system was given by Sasaki [40]. It involves an auxiliary expression $(A \supset B)^+$ which is intended to denote implication in intuitionistic logic. The cut-elimination theorem for the system holds, but yields only a weak form of subformula property in the sense that even $(A \supset B)^+$ is included in the subformulas of $A \supset B$. The third system, which is a slight modification of the second one, was given by Aghaei and Ardeshir [2]. It satisfies only a weak form of subformula property either. Another system is found in the systems for subintuitionistic logics by Wansing [50] mentioned above, which involves more auxiliary expressions.

All studies on subintuitionistic logics so far have never been related to substructural logics, mainly because no satisfactory sequent calculi for subintuitionistic logics have been given. This thesis is the first study that provides adequate sequent calculi for subintuitionistic logics and gives more insight into these subsystems of intuitionistic logic.

1.2 Overview of the thesis

Having seen the background and motivation of this research, we give an overview of the work in the thesis. The figure at the end of this section represents the structure of the thesis, making our contributions clear.

We start with, in Chapter 2, a review of the propositional parts of Gentzen's sequent calculi **LK** for classical logic and **LJ** for intuitionistic logic. In Section 2.1 we show the completeness and cut-elimination theorems for **LK** with the familiar notion of valuation, intending an introduction to semantical proof of cut-elimination. In Section 2.2 we show the completeness theorem of **LJ** with respect to Kripke semantics for intuitionistic logic, which says that the provability in **LJ** is characterized by the mathematical structure. In Section 2.3 we show the cut-elimination theorem for **LJ** in a syntactical way introducing the mix rule instead of the cut rule.

Chapter 3 is devoted to studying subintuitionistic logics. The definitions of the least subintuitionistic logic K^I and the logic BPC are given semantically using Kripke models in Section 3.1. Then in Section 3.2 we introduce Kashima's sequent calculus $\mathbf{GK}^I$ and show the completeness theorem of $\mathbf{GK}^I$ with respect to the class of Kripke models for K^I. The cut-elimination theorem for $\mathbf{GK}^I$ follows from the proof of the completeness theorem as for **LK** in Section 2.1. Next in Section 3.3 we extend the system $\mathbf{GK}^I$ to the sequent calculus **LBP** which is complete with respect to the class of Kripke models for BPC. The system **LBP** enjoys the subformula property in the strict sense unlike the sequent calculi for BPC that have been introduced in the literature.

Although the sequent calculi $\mathbf{GK}^I$ and **LBP** are complete with respect to semantics for K^I and for BPC and enjoy the subformula property, they are not satisfactory in the respect that the rule for implication involves 2^n premisses for n principal formulas in the conclusion. This gives rise to difficulty in comparing $\mathbf{GK}^I$ and **LBP** with sequent calculi for intuitionistic logic and for substructural logics.

In order to provide sequent calculi for K^I and BPC that are suitable to compare with systems for intuitionistic and substructural logics, we consider dual-context sequents, i.e., sequents of the form $\Gamma; \Delta \to A$. Such kind of formulation has proved popular in the field of linear logic; for example, the unified system **LU** [20], linear logic programming [26], and linear type theories [7]. (Some related systems are discussed in Section 3.9 including systems for modal logic.) In Section 3.4 we apply the formulation to the logic K^I and develop the dual-context sequent calculus $\mathbf{DK}^I$. This system enjoys the subformula property and is closely related to Gentzen's sequent calculus **LJ** for intuitionistic logic. Unlike in the system $\mathbf{GK}^I$, each rule for implication in $\mathbf{DK}^I$ holds just one principal formula, having the following form.

$$\frac{\Gamma; \Delta \to A \quad \Pi; B, \Sigma \to C}{\Gamma, A \supset B, \Pi; \Delta, \Sigma \to C}\ (\supset\to) \qquad\qquad \frac{\Gamma; A \to B}{; \Gamma \to A \supset B}\ (\to\supset)$$

Moreover we give a novel kind of interpretation of the sequents in Kripke models, where Γ of a sequent $\Gamma; \Delta \to A$ denotes formulas in a point x of a Kripke model while Δ and A denote formulas in any point y next to x. The soundness theorem of $\mathbf{DK}^I$ is proved easily by virtue of the interpretation. To prove the completeness theorem of $\mathbf{DK}^I$, we introduce a useful notion of x-consistent pair (Definition 3.20), which plays an important role in the construction of the canonical model. In Section 3.5 we also introduce the dual-context sequent calculus **LBP2** by modifying the system $\mathbf{DK}^I$, and show that the system **LBP2** is complete with respect to the class of Kripke models for BPC.

The cut-elimination theorem for $\mathbf{DK}^I$ and **LBP2** is proved by syntactical methods in Sections 3.6 and 3.7, respectively. The main difference from the proof of cut-elimination for **LJ** arises in the case where the right rule over the cut rule is $(\to\supset)$. Since in the cut rule of $\mathbf{DK}^I$ and **LBP2** the cut formula cannot appear on the left hand of ';', it is not possible to push the cut one step up in that case. Instead, we make a more global proof transformation as shown in Figure 1.2, where we assume that the left rule over the cut rule is $(\to\supset)$. This technique is applied also to the proof of the cut-elimination theorem for **LBB′I2** in Section 5.2.

In Chapter 4 we investigate the relationships between subintuitionistic logics and substructural logics from a different perspective. We consider there Hilbert style systems that characterize the implicational fragments of subintuitionistic logics and substructural logics, and clarify the inclusion relationships between the sets of formulas that are provable in each Hilbert style system for these logics.

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\ P_1\\ \Delta; A \to B\end{array}}{; \Delta \to A \supset B}\,(\to\supset) \qquad \dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\vdots\ Q_1\\ \Phi; \Lambda \to A\end{array} \quad \begin{array}{c}\vdots\ Q_2\\ \Psi; B, \Theta \to E\end{array}}{\Phi, A \supset B, \Psi; \Lambda, \Theta \to E}\,(\supset\to)\\ \vdots\ P_2\\ A \supset B, \Sigma; C \to D\end{array}}{; A \supset B, \Sigma \to C \supset D}\,(\to\supset)}{; \Delta, \Sigma \to C \supset D}\,(\text{cut})
$$

$$\Downarrow$$

$$
\dfrac{\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\vdots\ Q_1\\ \Phi; \Lambda \to A\end{array} \quad \begin{array}{c}\vdots\ P_1\\ \Delta; A \to B\end{array}}{\Phi, \Delta; \Lambda \to B}\,(\text{cut}) \qquad \begin{array}{c}\vdots\ Q_2\\ \Psi; B, \Theta \to E\end{array}}{\Phi, \Delta, \Psi; \Lambda, \Theta \to E}\,(\text{cut})\\ \vdots\ P_2'\\ \Delta, \Sigma; C \to D\end{array}}{; \Delta, \Sigma \to C \supset D}\,(\to\supset)
$$

Figure 1.2: Global proof transformation

In Chapter 5 we study sequent calculi for noncommutative substructural logics, particularly the logic BB′I. This logic is important in the respect that it is a noncommutative version of the implicational fragment of linear logic. The usual sequent calculus for BB′I is defined using merge operation (Section 7 of [3], [30]). In Section 5.1 we introduce the sequent calculus **LBB′I2** without any merge operation and show the correspondence between BB′I and **LBB′I2**. Roughly speaking, **LBB′I2** is obtained from the system **LBP2** by deleting the structural rules, according to the observation that BB′I is a subsystem of BPC (cf. Section 4.3). The cut-elimination theorem for **LBB′I2** is proved in Section 5.2 using global proof transformation technique analogous to that used in the proof of the cut-elimination theorem for $\mathbf{DK}^I$ and **LBP2**.

In Chapter 6 we summarize the results of the thesis and indicate directions of further studies derived from our work.

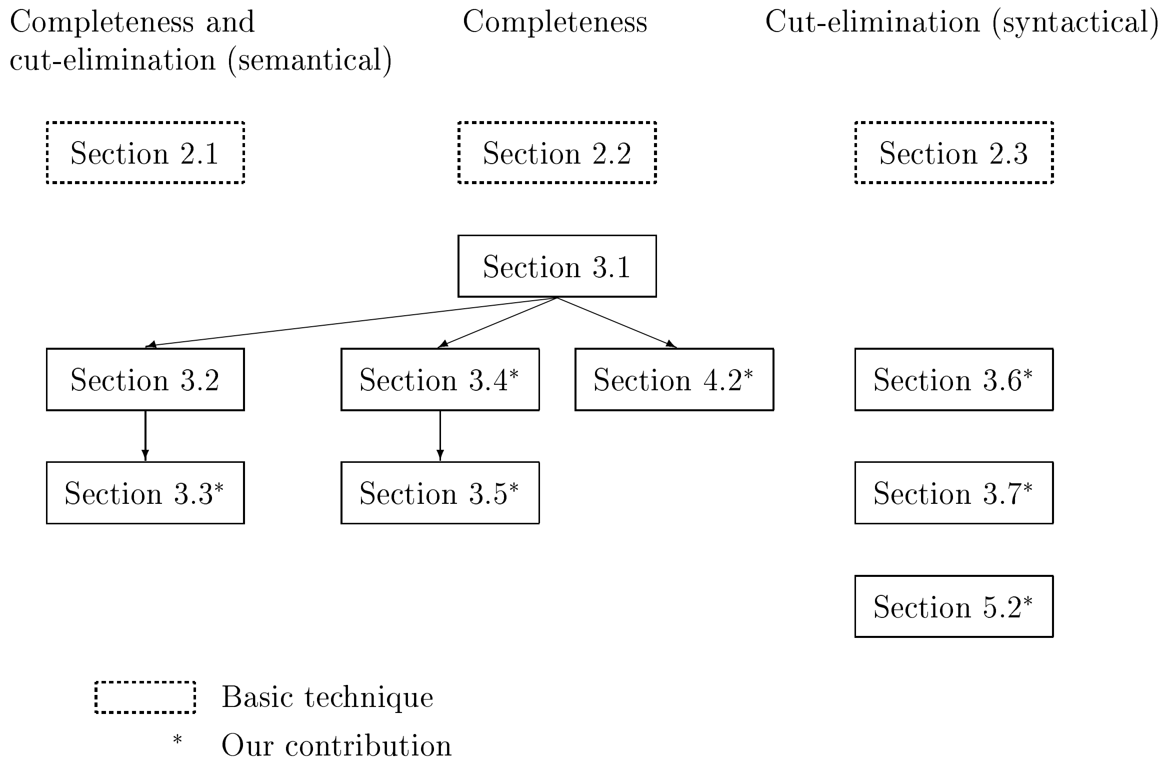


Figure 1.3: Structure of the thesis

Chapter 2

Sequent calculi for classical and intuitionistic logics

In this chapter we review the propositional parts of Gentzen's sequent calculi **LK** for classical logic and **LJ** for intuitionistic logic, concentrating our attention on their completeness and cut-elimination theorems. These materials form the bases of later investigations into sequent calculi for other logics.

2.1 Sequent calculus for classical logic

In this section we study Gentzen's sequent calculus **LK** for classical propositional logic. The completeness theorem with respect to truth-value semantics is proved, followed by the cut-elimination theorem as a corollary. The content of this section will be useful to understand semantical proof of cut-elimination for sequent calculi for other logics. We begin by describing the language and semantics of classical propositional logic.

The language of classical propositional logic has a countable set $\mathcal{PV}$ of propositional variables, the propositional constant $\bot$ and the logical connectives $\wedge$, $\vee$ and $\supset$. The set $\mathcal{F}orm$ of formulas are constructed from these in the usual way. We use $p, q, r, \ldots$ for propositional variables, $A, B, C, \ldots$ for formulas and $\Gamma, \Delta, \Sigma, \ldots$ for finite sequences of formulas separated by commas. For each formula A, $|A|$ denotes the number of occurrences of logical connectives in A, and $\mathrm{Sub}(A)$ denotes the set of all subformulas of A. For each set x of formulas, $\mathrm{Sub}(x)$ denotes the set $\bigcup\{\mathrm{Sub}(A) \mid A \in x\}$.

The semantics of classical propositional logic we consider here is the usual truth-value semantics based on the notion of valuation.

Definition 2.1 A *valuation* is a function from $\mathcal{PV}$ to $\{\mathrm{t}, \mathrm{f}\}$. For a given valuation $v : \mathcal{PV} \to \{\mathrm{t}, \mathrm{f}\}$, the function $\widehat{v} : \mathcal{F}orm \to \{\mathrm{t}, \mathrm{f}\}$ is defined inductively as follows:

$$
\begin{aligned}
&\widehat{v}(p) = v(p) \quad \text{for every } p \in \mathcal{PV},\\
&\widehat{v}(\bot) = \mathrm{f},\\
&\widehat{v}(A \wedge B) = \mathrm{t} \quad \text{iff} \quad \widehat{v}(A) = \widehat{v}(B) = \mathrm{t},\\
&\widehat{v}(A \vee B) = \mathrm{t} \quad \text{iff} \quad \widehat{v}(A) = \mathrm{t} \ \text{ or } \ \widehat{v}(B) = \mathrm{t},\\
&\widehat{v}(A \supset B) = \mathrm{t} \quad \text{iff} \quad \widehat{v}(A) = \mathrm{f} \ \text{ or } \ \widehat{v}(B) = \mathrm{t}.
\end{aligned}
$$

A formula A is a *tautology* if $\widehat{v}(A) = \mathrm{t}$ for every valuation v.

A *sequent* of the system **LK** is defined as an expression of the form $\Gamma \rightarrow \Delta$. The truth of sequents of **LK** is defined in the following way.

Definition 2.2 A sequent $\Gamma \rightarrow \Delta$ of **LK** is *true* for a valuation v if

$$\begin{array}{ll} \widehat{v}(C) = \mathrm{t} \ \text{ for every } C \text{ occurring in } \Gamma & \text{implies} \\ \widehat{v}(D) = \mathrm{t} \ \text{ for some } D \text{ occurring in } \Delta. & \end{array}$$

Proposition 2.3 *For any formula A and any valuation v,*

1. $\widehat{v}(A) = \mathrm{t}$ *if and only if* $\rightarrow A$ *is true for* v,
2. A *is a tautology if and only if* $\rightarrow A$ *is true for every valuation.*

PROOF. Straightforward. ∎

Now we introduce the sequent calculus **LK**. Initial sequents of **LK** are of the following forms:

$$A \rightarrow A,$$

$$\bot \rightarrow \ .$$

Rules of inference of **LK** are the following.

Structural rules:

$$\frac{\Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta}\ (\mathrm{w} \rightarrow) \qquad \frac{\Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, A}\ (\rightarrow \mathrm{w})$$

$$\frac{A, A, \Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta}\ (\mathrm{c} \rightarrow) \qquad \frac{\Gamma \rightarrow \Delta, A, A}{\Gamma \rightarrow \Delta, A}\ (\rightarrow \mathrm{c})$$

$$\frac{\Gamma, A, B, \Pi \rightarrow \Delta}{\Gamma, B, A, \Pi \rightarrow \Delta}\ (\mathrm{e} \rightarrow) \qquad \frac{\Gamma \rightarrow \Delta, A, B, \Sigma}{\Gamma \rightarrow \Delta, B, A, \Sigma}\ (\rightarrow \mathrm{e})$$

Cut rule:

$$\frac{\Gamma \rightarrow \Delta, A \quad A, \Pi \rightarrow \Sigma}{\Gamma, \Pi \rightarrow \Delta, \Sigma}\ (\mathrm{cut})$$

Rules for the logical connectives:

$$\frac{A, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta}\ (\wedge 1 \rightarrow) \qquad \frac{B, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta}\ (\wedge 2 \rightarrow)$$

$$\frac{\Gamma \rightarrow \Delta, A \quad \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \wedge B}\ (\rightarrow \wedge)$$

$$\frac{A, \Gamma \rightarrow \Delta \quad B, \Gamma \rightarrow \Delta}{A \vee B, \Gamma \rightarrow \Delta}\ (\vee \rightarrow)$$

$$\frac{\Gamma \rightarrow \Delta, A}{\Gamma \rightarrow \Delta, A \vee B}\ (\rightarrow \vee 1) \qquad \frac{\Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \vee B}\ (\rightarrow \vee 2)$$

$$\frac{\Gamma \rightarrow \Delta, A \quad B, \Pi \rightarrow \Sigma}{A \supset B, \Gamma, \Pi \rightarrow \Delta, \Sigma}\ (\supset\rightarrow) \qquad \frac{A, \Gamma \rightarrow \Delta, B}{\Gamma \rightarrow \Delta, A \supset B}\ (\rightarrow\supset)$$

Theorem 2.4 (Soundness of LK) *For any sequent $\Gamma \to \Delta$, if $\Gamma \to \Delta$ is provable in* **LK***, then $\Gamma \to \Delta$ is true for every valuation.*

Proof. By induction on the proof of $\Gamma \to \Delta$ in **LK**. ∎

Next let **LK**−(cut) denote the system obtained from **LK** by deleting the cut rule. In the following we show that **LK**−(cut) is sufficient to prove any sequent that is true for every valuation. From this and Theorem 2.4, we can derive the cut-elimination theorem for **LK**.

Definition 2.5 Let x, y be sets of formulas. A pair (x, y) is *consistent* (in **LK**−(cut)) if, for any $A_1, \ldots, A_m \in x$ and any $B_1, \ldots, B_n \in y$ $(m, n \geq 0)$,

$$A_1, \ldots, A_m \to B_1, \ldots, B_n$$

is not provable in **LK**−(cut). (x, y) is *saturated* (in **LK**−(cut)) if it is consistent and satisfies the following conditions:

(S1)	$A \wedge B \in x$	implies	$A \in x$	and	$B \in x$,
(S2)	$A \wedge B \in y$	implies	$A \in y$	or	$B \in y$,
(S3)	$A \vee B \in x$	implies	$A \in x$	or	$B \in x$,
(S4)	$A \vee B \in y$	implies	$A \in y$	and	$B \in y$,
(S5)	$A \supset B \in x$	implies	$A \in y$	or	$B \in x$,
(S6)	$A \supset B \in y$	implies	$A \in x$	and	$B \in y$.

Lemma 2.6 *Let x, y be finite sets of formulas. If (x, y) is consistent, then there exists a saturated pair (x', y') such that $x \subseteq x'$ and $y \subseteq y'$.*

Proof. Let $C_1, \ldots, C_n$ be a sequence of all formulas in $\mathrm{Sub}(x \cup y)$ where $|C_i| \geq |C_{i+1}|$ for each $i \in \{1, \ldots, n-1\}$. We define a sequence of consistent pairs (x_i, y_i) $(i = 0, \ldots, n)$ as follows:

1. $(x_0, y_0) = (x, y)$.
2. For $i \geq 1$, the pair (x_i, y_i) is obtained from (x_{i-1}, y_{i-1}) in the following way.

   2.1. If C_i is of the form $A \wedge B$ and $C_i \in x_{i-1}$, then $(x_i, y_i) = (x_{i-1} \cup \{A, B\}, y_{i-1})$. Here (x_i, y_i) is consistent whenever (x_{i-1}, y_{i-1}) is consistent. Indeed, otherwise there exist formulas $D_1, \ldots, D_j \in x_{i-1}$ and $E_1, \ldots, E_k \in y_{i-1}$ such that

   $$D_1, \ldots, D_j, A, B \to E_1, \ldots, E_k$$

   is provable in **LK**−(cut). Then

   $$D_1, \ldots, D_j, A \wedge B \to E_1, \ldots, E_k$$

   is also provable in **LK**−(cut), contrary to the consistency of (x_{i-1}, y_{i-1}).

2.2. If C_i is of the form $A \wedge B$ and $C_i \in y_{i-1}$, then (x_i, y_i) is defined as either $(x_{i-1}, y_{i-1} \cup \{A\})$ or $(x_{i-1}, y_{i-1} \cup \{B\})$. Here we can define (x_i, y_i) to be consistent whenever (x_{i-1}, y_{i-1}) is consistent. Indeed, otherwise there exist formulas $D_1, \ldots, D_j \in x_{i-1}$ and $E_1, \ldots, E_k \in y_{i-1}$ such that

$$D_1, \ldots, D_j \to E_1, \ldots, E_k, A$$
$$D_1, \ldots, D_j \to E_1, \ldots, E_k, B$$

are both provable in **LK**−(cut). Then

$$D_1, \ldots, D_j \to E_1, \ldots, E_k, A \wedge B$$

is also provable in **LK**−(cut), contrary to the consistency of (x_{i-1}, y_{i-1}).

2.3. If C_i is of the form $A \vee B$ and $C_i \in x_{i-1}$, then (x_i, y_i) is defined as either $(x_{i-1} \cup \{A\}, y_{i-1})$ or $(x_{i-1} \cup \{B\}, y_{i-1})$. By an argument symmetric to that in 2.2, we can define (x_i, y_i) to be consistent whenever (x_{i-1}, y_{i-1}) is consistent.

2.4. If C_i is of the form $A \vee B$ and $C_i \in y_{i-1}$, then $(x_i, y_i) = (x_{i-1}, y_{i-1} \cup \{A, B\})$. By an argument symmetric to that in 2.1, the pair (x_i, y_i) is consistent whenever (x_{i-1}, y_{i-1}) is consistent.

2.5. If C_i is of the form $A \supset B$ and $C_i \in x_{i-1}$, then (x_i, y_i) is defined as either $(x_{i-1}, y_{i-1} \cup \{A\})$ or $(x_{i-1} \cup \{B\}, y_{i-1})$. Here we can define (x_i, y_i) to be consistent whenever (x_{i-1}, y_{i-1}) is consistent. Indeed, otherwise there exist formulas $D_1, \ldots, D_j \in x_{i-1}$ and $E_1, \ldots, E_k \in y_{i-1}$ such that

$$D_1, \ldots, D_j \to E_1, \ldots, E_k, A$$
$$D_1, \ldots, D_j, B \to E_1, \ldots, E_k$$

are both provable in **LK**−(cut). Then

$$D_1, \ldots, D_j, A \supset B \to E_1, \ldots, E_k$$

is also provable in **LK**−(cut), contrary to the consistency of (x_{i-1}, y_{i-1}).

2.6. If C_i is of the form $A \supset B$ and $C_i \in y_{i-1}$, then $(x_i, y_i) = (x_{i-1} \cup \{A\}, y_{i-1} \cup \{B\})$. Here (x_i, y_i) is consistent whenever (x_{i-1}, y_{i-1}) is consistent. Indeed, otherwise there exist formulas $D_1, \ldots, D_j \in x_{i-1}$ and $E_1, \ldots, E_k \in y_{i-1}$ such that

$$D_1, \ldots, D_j, A \to E_1, \ldots, E_k, B$$

is provable in **LK**−(cut). Then

$$D_1, \ldots, D_j \to E_1, \ldots, E_k, A \supset B$$

is also provable in **LK**−(cut), contrary to the consistency of (x_{i-1}, y_{i-1}).

2.7. Otherwise, $(x_i, y_i) = (x_{i-1}, y_{i-1})$.

Then (x_i, y_i) is consistent for all $i \in \{0, \ldots, n\}$, and (x_n, y_n) is saturated since the conditions (S1)–(S6) are satisfied with the above constructions 2.1–2.6, respectively. The pair (x_n, y_n) satisfies $x \subseteq x_n$ and $y \subseteq y_n$, so we complete the proof. ∎

Lemma 2.7 *For any sequent* $\Gamma \to \Delta$*, if* $\Gamma \to \Delta$ *is true for every valuation, then* $\Gamma \to \Delta$ *is provable in* **LK**$-$(cut).

PROOF. Suppose that $\Gamma \to \Delta$ is not provable in **LK**$-$(cut). Let γ and δ be the sets of formulas occurring in Γ and Δ, respectively. Then the pair (γ, δ) is consistent, and so by Lemma 2.6, there exists a saturated pair (γ', δ') such that $\gamma \subseteq \gamma'$ and $\delta \subseteq \delta'$. Now define a valuation $v : \mathcal{PV} \to \{\mathrm{t}, \mathrm{f}\}$ as follows:

$$v(p) = \begin{cases} \mathrm{t} & \text{if } p \in \gamma', \\ \mathrm{f} & \text{otherwise.} \end{cases}$$

Our aim is to show that $\Gamma \to \Delta$ is not true for this valuation v. Since $\gamma \subseteq \gamma'$ and $\delta \subseteq \delta'$, it suffices to show that for any formula C,

$$\begin{array}{lll} C \in \gamma' & \text{implies} & \widehat{v}(C) = \mathrm{t}, \quad \text{and} \\ C \in \delta' & \text{implies} & \widehat{v}(C) = \mathrm{f}. \end{array}$$

We prove these by simultaneous induction on the structure of C.

1. C is a propositional variable p. If $p \in \gamma'$ then $\widehat{v}(p) = v(p) = \mathrm{t}$ by the definitions of $\widehat{v}$ and v. If $p \in \delta'$ then $p \notin \gamma'$ by the consistency of (γ', δ'), and so $\widehat{v}(p) = v(p) = \mathrm{f}$.

2. C is the propositional constant $\perp$. If $\perp \in \gamma'$ then (γ', δ') is not consistent, which is a contradiction. If $\perp \in \delta'$ then $\widehat{v}(\perp) = \mathrm{f}$ by the definition of $\widehat{v}$.

3. C is of the form $A \wedge B$. If $A \wedge B \in \gamma'$ then $A \in \gamma'$ and $B \in \gamma'$ by the condition (S1) of the saturated pair (γ', δ'). By the induction hypothesis, $\widehat{v}(A) = \widehat{v}(B) = \mathrm{t}$, and so $\widehat{v}(A \wedge B) = \mathrm{t}$. If $A \wedge B \in \delta'$ then $A \in \delta'$ or $B \in \delta'$ by the condition (S2). By the induction hypothesis, $\widehat{v}(A) = \mathrm{f}$ or $\widehat{v}(B) = \mathrm{f}$, and so $\widehat{v}(A \wedge B) = \mathrm{f}$.

4. C is of the form $A \vee B$. Symmetric to the previous case.

5. C is of the form $A \supset B$. If $A \supset B \in \gamma'$ then $A \in \delta'$ or $B \in \gamma'$ by the condition (S5) of the saturated pair (γ', δ'). By the induction hypothesis, $\widehat{v}(A) = \mathrm{f}$ or $\widehat{v}(B) = \mathrm{t}$, and so $\widehat{v}(A \supset B) = \mathrm{t}$. If $A \supset B \in \delta'$ then $A \in \gamma'$ and $B \in \delta'$ by the condition (S6). By the induction hypothesis, $\widehat{v}(A) = \mathrm{t}$ and $\widehat{v}(B) = \mathrm{f}$, and so $\widehat{v}(A \supset B) = \mathrm{f}$.

■

From Theorem 2.4 and Lemma 2.7, we obtain the following results.

Theorem 2.8 (Completeness of LK) *For any sequent* $\Gamma \to \Delta$, $\Gamma \to \Delta$ *is provable in* **LK** *if and only if* $\Gamma \to \Delta$ *is true for every valuation.*

Theorem 2.9 (Cut-elimination for LK) *For any sequent* $\Gamma \to \Delta$*, if* $\Gamma \to \Delta$ *is provable in* **LK***, then it is provable in* **LK** *without using the cut rule.*

2.2 Sequent calculus for intuitionistic logic

In this section we study Gentzen's sequent calculus **LJ** for intuitionistic propositional logic, in particular its completeness with respect to Kripke semantics. Unlike in the previous section we prove the completeness theorem with the help of the cut rule. The cut-elimination theorem for **LJ** is proved in the next section by syntactical methods.

The language of intuitionistic propositional logic is the same as that of classical propositional logic. The sequent calculus **LJ** is obtained from **LK** by restricting the right hand side of a sequent to at most one formula. More specifically, a *sequent* of **LJ** is an expression of the form $\Gamma \to A$, where A may be empty, and the sequent calculus **LJ** is defined as follows. Initial sequents of **LJ** are of the following forms:

$$A \to A,$$

$$\bot \to \ .$$

Rules of inference of **LJ** are the following.

Structural rules:

$$\frac{\Gamma \to C}{A, \Gamma \to C}\ (\mathrm{w} \to) \qquad\qquad \frac{\Gamma \to}{\Gamma \to A}\ (\to \mathrm{w})$$

$$\frac{A, A, \Gamma \to C}{A, \Gamma \to C}\ (\mathrm{c} \to)$$

$$\frac{\Gamma, A, B, \Pi \to C}{\Gamma, B, A, \Pi \to C}\ (\mathrm{e} \to)$$

Cut rule:

$$\frac{\Gamma \to A \quad A, \Pi \to C}{\Gamma, \Pi \to C}\ (\mathrm{cut})$$

Rules for the logical connectives:

$$\frac{A, \Gamma \to C}{A \wedge B, \Gamma \to C}\ (\wedge 1 \to) \qquad\qquad \frac{B, \Gamma \to C}{A \wedge B, \Gamma \to C}\ (\wedge 2 \to)$$

$$\frac{\Gamma \to A \quad \Gamma \to B}{\Gamma \to A \wedge B}\ (\to \wedge)$$

$$\frac{A, \Gamma \to C \quad B, \Gamma \to C}{A \vee B, \Gamma \to C}\ (\vee \to)$$

$$\frac{\Gamma \to A}{\Gamma \to A \vee B}\ (\to \vee 1) \qquad\qquad \frac{\Gamma \to B}{\Gamma \to A \vee B}\ (\to \vee 2)$$

$$\frac{\Gamma \to A \quad B, \Pi \to C}{A \supset B, \Gamma, \Pi \to C}\ (\supset\to) \qquad\qquad \frac{A, \Gamma \to B}{\Gamma \to A \supset B}\ (\to\supset)$$

The formula C in each of these rules may be empty. The formulas A, B in the structural rules and $A \wedge B$, $A \vee B$, $A \supset B$ in the rules for the logical connectives are called the *principal formulas* of the respective rules.

Next we introduce Kripke semantics for intuitionistic propositional logic, which is considered as truth-value semantics parameterized by preorders. The provability of sequents in **LJ** is characterized by this mathematical structure.

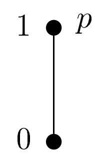
Definition 2.10 Let $\langle W, \leq \rangle$ be a preorder, i.e., $\leq$ be a reflexive and transitive relation on a nonempty set W. An IPC-*valuation* is a function $v : \mathcal{PV} \times W \to \{\mathrm{t}, \mathrm{f}\}$ such that for every $p \in \mathcal{PV}$ and every $x, y \in W$,

$$v(p,x) = \mathrm{t} \quad \text{and} \quad x \leq y \quad \text{imply} \quad v(p,y) = \mathrm{t}.$$

For a given IPC-valuation v, the function $\widehat{v} : \mathcal{F}orm \times W \to \{\mathrm{t}, \mathrm{f}\}$ is defined inductively as follows:

$$\begin{aligned}
&\widehat{v}(p,x) = v(p,x) \quad \text{for every } p \in \mathcal{PV},\\
&\widehat{v}(\bot, x) = \mathrm{f},\\
&\widehat{v}(A \wedge B, x) = \mathrm{t} \quad \text{iff} \quad \widehat{v}(A,x) = \widehat{v}(B,x) = \mathrm{t},\\
&\widehat{v}(A \vee B, x) = \mathrm{t} \quad \text{iff} \quad \widehat{v}(A,x) = \mathrm{t} \ \text{ or } \ \widehat{v}(B,x) = \mathrm{t},\\
&\widehat{v}(A \supset B, x) = \mathrm{t} \quad \text{iff} \quad \forall y \in W[x \leq y \ \text{ and } \ \widehat{v}(A,y) = \mathrm{t} \ \text{ imply } \ \widehat{v}(B,y) = \mathrm{t}].
\end{aligned}$$

Example 2.11 Consider a preorder $\langle \{0,1\}, \leq \rangle$ where $\leq\, = \{(0,0),(0,1),(1,1)\}$, and an IPC-valuation v such that $v(p,0) = \mathrm{f}$ and $v(p,1) = \mathrm{t}$. The situation is graphically represented as follows.



Then $\widehat{v}(p,0) = \mathrm{f}$, and $\widehat{v}(p \supset \bot, 0) = \mathrm{f}$ since $0 \leq 1$ and $\widehat{v}(p,1) = \mathrm{t}$ but $\widehat{v}(\bot,1) = \mathrm{f}$. Hence $\widehat{v}(p \vee (p \supset \bot), 0) = \mathrm{f}$.

Let $\langle W, \leq \rangle$ be a preorder and v an IPC-valuation. Consider a function $V : \mathcal{PV} \to 2^W$ and a relation $\models (\subseteq W \times \mathcal{F}orm)$ such that

$$\begin{aligned}
&V(p) = \{x \in W \mid v(p,x) = \mathrm{t}\} \quad \text{for every } p \in \mathcal{PV},\\
&x \models A \quad \text{iff} \quad \widehat{v}(A,x) = \mathrm{t}.
\end{aligned}$$

Then, Definition 2.10 is rephrased in the following style, which we employ in the sequel.

Definition 2.12 An IPC-*model* is a triple $\langle W, \leq, V \rangle$ where

- W is a nonempty set,
- $\leq$ is a reflexive and transitive relation on W,
- V is a function from $\mathcal{PV}$ to 2^W such that for every $p \in \mathcal{PV}$ and every $x, y \in W$,

$$x \in V(p) \quad \text{and} \quad x \leq y \quad \text{imply} \quad y \in V(p).$$

For a given IPC-model $M = \langle W, \leq, V \rangle$, the truth-relation $\models_M$ ($\subseteq W \times \mathcal{F}orm$) is defined inductively as follows:

$$
\begin{array}{lll}
x \models_M p & \text{iff} & x \in V(p) \quad \text{for every } p \in \mathcal{PV}, \\
x \not\models_M \bot, & & \\
x \models_M A \wedge B & \text{iff} & x \models_M A \ \text{ and } \ x \models_M B, \\
x \models_M A \vee B & \text{iff} & x \models_M A \ \text{ or } \ x \models_M B, \\
x \models_M A \supset B & \text{iff} & \forall y \in W [x \leq y \ \text{ and } \ y \models_M A \ \text{ imply } \ y \models_M B].
\end{array}
$$

If $x \models_M A$ for every A occurring in a sequence Γ, we write $x \models_M \Gamma$. The subscript M may be omitted if understood. A formula A is *true* in an IPC-model $\langle W, \leq, V \rangle$ if $x \models A$ for every $x \in W$.

Lemma 2.13 *For every* IPC-*model* $\langle W, \leq, V \rangle$*, every formula* A *and every* $x, y \in W$*,*

$$x \models A \ \textit{ and } \ x \leq y \ \textit{ imply } \ y \models A.$$

PROOF. By induction on the structure of A.

1. A is a propositional variable p. Follows from the definitions of $\models$ and V.
2. A is the propositional constant $\bot$. $x \models \bot$ is impossible by the definition of $\models$.
3. A is of the form $B \wedge C$. Suppose that $x \models B \wedge C$ and $x \leq y$. Then $x \models B$ and $x \models C$. By the induction hypothesis, $y \models B$ and $y \models C$. Hence $y \models B \wedge C$.
4. A is of the form $B \vee C$. Analogous to the previous case.
5. A is of the form $B \supset C$. Suppose that $x \models B \supset C$ and $x \leq y$. To show $y \models B \supset C$, suppose further that $y \leq z$ and $z \models B$. Then $x \leq z$ by the transitivity of $\leq$. Since $x \models B \supset C$ and $z \models B$, we have $z \models C$ as required.

■

The truth of sequents of **LJ** in IPC-models is defined in the following way.

Definition 2.14 For a given IPC-model M, the truth-relation $\models_M$ for sequents of **LJ** is defined as follows:

$$x \models_M \Gamma \rightarrow A \quad \text{iff} \quad x \models_M \Gamma \ \text{ implies } \ x \models_M A.$$

The case where A is empty is the same as the case where A is $\bot$. The subscript M may be omitted if understood. A sequent $\Gamma \rightarrow A$ is *true* in an IPC-model $\langle W, \leq, V \rangle$ if $x \models \Gamma \rightarrow A$ for every $x \in W$.

Proposition 2.15 *For any formula* A*, any* IPC-*model* $M = \langle W, \leq, V \rangle$ *and any* $x \in W$*,*

1. $x \models_M A$ *if and only if* $x \models_M \rightarrow A$*,*
2. A *is true in* M *if and only if* $\rightarrow A$ *is true in* M*.*

PROOF. Straightforward. ■

Theorem 2.16 (Soundness of LJ) *For any sequent $\Gamma \to A$, if $\Gamma \to A$ is provable in* **LJ**, *then $\Gamma \to A$ is true in every* IPC-*model.*

PROOF. By induction on the proof of $\Gamma \to A$ in **LJ**. Here we consider only the rules ($\supset\to$) and ($\to\supset$). Take any IPC-model $\langle W, \leq, V\rangle$ and any $x \in W$. For the rule ($\supset\to$), we show $x \models A \supset B, \Gamma, \Pi \to C$. Suppose that $x \models A \supset B, \Gamma, \Pi$, i.e., that $x \models A \supset B$, $x \models \Gamma$ and $x \models \Pi$. Since $x \models \Gamma \to A$ by the induction hypothesis, we have $x \models A$, and since $x \models A \supset B$, we have $x \models B$. Moreover, since $x \models B, \Pi \to C$ by the induction hypothesis, we have $x \models C$ as required. For the rule ($\to\supset$), we show $x \models \Gamma \to A \supset B$. Suppose that $x \models \Gamma$. To show $x \models A \supset B$, suppose further that $x \leq y$ and $y \models A$. Then by Lemma 2.13, we have $y \models \Gamma$. Since $y \models A, \Gamma \to B$ by the induction hypothesis, we have $y \models B$ as required. ∎

To prove the completeness theorem of **LJ**, we introduce the following notions.

Definition 2.17 Let x, y be sets of formulas. The pair (x, y) is *consistent* (in **LJ**) if, for any $A_1, \ldots, A_m \in x$ and any $B_1, \ldots, B_n \in y$ $(m, n \geq 0)$,

$$A_1, \ldots, A_m \to B_1 \vee \cdots \vee B_n$$

is not provable in **LJ**. (x, y) is *maximal consistent* (in **LJ**) if it is consistent and for any formula A, $A \in x$ or $A \in y$.

Note that the notation $B_1 \vee \cdots \vee B_n$ above is justified by the cut rule which ensures the associativity of $\vee$ on provability in **LJ**. The cut rule is also used in the proof of the following lemma.

Lemma 2.18 *Let x, y be sets of formulas. If (x, y) is consistent, then there exists a maximal consistent pair (x', y') such that $x \subseteq x'$ and $y \subseteq y'$.*

PROOF. Let $C_1, C_2, \ldots$ be an enumeration of all formulas. We define a sequence of pairs (x_n, y_n) $(n = 0, 1, \ldots)$ as follows:

$$\begin{aligned}(x_0, y_0) &= (x, y),\\ (x_{m+1}, y_{m+1}) &= \begin{cases}(x_m, y_m \cup \{C_{m+1}\}) & \text{if } (x_m, y_m \cup \{C_{m+1}\}) \text{ is consistent,}\\ (x_m \cup \{C_{m+1}\}, y_m) & \text{otherwise.}\end{cases}\end{aligned}$$

Then (x_{m+1}, y_{m+1}) is consistent whenever (x_m, y_m) is consistent. Indeed, otherwise there exist formulas $A_1, \ldots, A_i \in x_m$ and $B_1, \ldots, B_j \in y_m$ such that

$$A_1, \ldots, A_i \to B_1 \vee \cdots \vee B_j \vee C_{m+1},$$
$$A_1, \ldots, A_i, C_{m+1} \to B_1 \vee \cdots \vee B_j$$

are both provable in **LJ**. However, by using the cut rule,

$$A_1, \ldots, A_i \to B_1 \vee \cdots \vee B_j$$

is provable in **LJ**, contrary to the consistency of (x_m, y_m). Thus (x_n, y_n) is consistent for all n, and we obtain a maximal consistent pair $(\bigcup_{n=0}^{\infty} x_n, \bigcup_{n=0}^{\infty} y_n)$. ∎

Theorem 2.19 (Completeness of LJ) *For any sequent* $\Gamma \to A$, $\Gamma \to A$ *is provable in* **LJ** *if and only if* $\Gamma \to A$ *is true in every* IPC*-model.*

PROOF. From left to right, we have Theorem 2.16. For the other direction, suppose that $\Gamma \to A$ is not provable in **LJ**. Let γ be the set of all formulas occurring in Γ. Then the pair $(\gamma, \{A\})$ is consistent, and so by Lemma 2.18, there exists a maximal consistent pair (u, v) such that $\gamma \subseteq u$ and $\{A\} \subseteq v$ (so $A \notin u$ by the consistency of (u, v)). Now define an IPC-model $\langle W^*, \leq^*, V^* \rangle$ as follows:

- W^* is the set of all maximal consistent pairs,
- $(x, w) \leq^* (y, z)$ iff $x \subseteq y$,
- $V^*(p) = \{(x, w) \in W^* \mid p \in x\}$ for every $p \in \mathcal{PV}$.

It is easy to verify that $(u, v) \in W^*$, $\leq^*$ is reflexive and transitive, and if $(x, w) \in V^*(p)$ and $(x, w) \leq^* (y, z)$ then $(y, z) \in V^*(p)$. Thus $\langle W^*, \leq^*, V^* \rangle$ is indeed an IPC-model. Our aim is to show $(u, v) \not\models \Gamma \to A$, which means that $\Gamma \to A$ is not true in this IPC-model. Since $\gamma \subseteq u$ and $A \notin u$, it suffices to show that for any formula B, $B \in u$ if and only if $(u, v) \models B$. (It suffices even when A is empty, since $(u, v) \not\models \bot$.) For this we prove by induction on the structure of B that for any $(x, w) \in W^*$,

$$B \in x \quad \text{if and only if} \quad (x, w) \models B.$$

1. B is a propositional variable p. Follows from the definitions of V^* and $\models$.

2. B is the propositional constant $\bot$. $\bot \in x$ is impossible by the consistency of (x, w).

3. B is of the form $C \wedge D$. Suppose $C \wedge D \in x$. Since $C \wedge D \to C$ is provable in **LJ**, $C \in w$ contradicts the consistency of (x, w). Hence $C \in x$. Similarly, $D \in x$. By the induction hypothesis, $(x, w) \models C$ and $(x, w) \models D$. Thus $(x, w) \models C \wedge D$. For the other direction, suppose $(x, w) \models C \wedge D$, i.e., $(x, w) \models C$ and $(x, w) \models D$. Then $C \in x$ and $D \in x$ by the induction hypothesis. Since $C, D \to C \wedge D$ is provable in **LJ**, $C \wedge D \in w$ contradicts the consistency of (x, w). Hence $C \wedge D \in x$.

4. B is of the form $C \vee D$. Suppose $C \vee D \in x$. Since $C \vee D \to C \vee D$ is provable in **LJ**, $C, D \in w$ contradicts the consistency of (x, w). Hence $C \notin w$ or $D \notin w$, and so $C \in x$ or $D \in x$. By the induction hypothesis, $(x, w) \models C$ or $(x, w) \models D$. Thus $(x, w) \models C \vee D$. For the other direction, suppose $(x, w) \models C \vee D$. Then $(x, w) \models C$ or $(x, w) \models D$, and by the induction hypothesis, $C \in x$ or $D \in x$. Since $C \to C \vee D$ and $D \to C \vee D$ are provable in **LJ**, $C \vee D \in w$ contradicts the consistency of (x, w). Hence $C \vee D \in x$.

5. B is of the form $C \supset D$. Suppose $C \supset D \in x$. To show $(x, w) \models C \supset D$, suppose further $(x, w) \leq^* (y, z)$ and $(y, z) \models C$. Then $C \supset D \in y$ by the definition of $\leq^*$, and $C \in y$ by the induction hypothesis. Since $C \supset D, C \to D$ is provable in **LJ**, $D \in z$ contradicts the consistency of (y, z). Hence $D \in y$ holds, and by the induction hypothesis, we have $(y, z) \models D$ as required. For the other direction, suppose $C \supset D \notin x$, i.e., $C \supset D \in w$. Then the pair $(x \cup \{C\}, \{D\})$ is consistent. Indeed, otherwise $\Gamma, C \to D$ is provable in **LJ** for some Γ consisting of formulas in x. Then $\Gamma \to C \supset D$ is also provable in **LJ** contrary to the consistency of (x, w). Thus

$(x \cup \{C\}, \{D\})$ is consistent, and by Lemma 2.18, there exists a maximal consistent pair (y, z) such that $x \cup \{C\} \subseteq y$ and $\{D\} \subseteq z$ (so $D \notin y$ by the consistency of (y, z)). Now we have $(x, w) \leq^* (y, z)$, $(y, z) \models C$ and $(y, z) \not\models D$ by the induction hypothesis. This means $(x, w) \not\models C \supset D$.

∎

2.3 Cut-elimination theorem for LJ

In the previous section we proved the completeness theorem of **LJ**, using the cut rule. The present section is devoted to proving the cut-elimination theorem for **LJ** in a syntactical way. For the proof of the cut-elimination theorem, we will introduce the mix rule and show mix-elimination instead of cut-elimination.

First we introduce the notion of height of a proof in **LJ**.

Definition 2.20 The *height* $h(P)$ of a proof P in **LJ** is defined inductively as follows:

1. If P is an initial sequent, then $h(P) = 1$.
2. If P is obtained from the proof Q by applying a one-premiss rule, then $h(P) = h(Q) + 1$.
3. If P is obtained from the proofs Q_1 and Q_2 by applying a two-premiss rule, then $h(P) = \max\{h(Q_1), h(Q_2)\} + 1$.

Now we prove the cut-elimination theorem for **LJ**.

Theorem 2.21 (Cut-elimination for LJ) *For any sequent $\Gamma \to A$, if $\Gamma \to A$ is provable in* **LJ**, *then it is provable in* **LJ** *without using the cut rule.*

Proof. We introduce the following mix rule:

$$\frac{\Gamma \to A \quad \Pi \to C}{\Gamma, \Pi_A \to C} \text{ (mix)}$$

where Π has at least one occurrence of A, and Π_A is obtained from Π by deleting all occurrences of A. The formula A is called the *mix formula* of this inference. It is seen that the cut rule

$$\frac{\Gamma \to A \quad A, \Pi \to C}{\Gamma, \Pi \to C} \text{ (cut)}$$

is derivable from the mix rule and some structural rules as follows.

$$\dfrac{\dfrac{\Gamma \to A \quad A, \Pi \to C}{\Gamma, \Pi_A \to C} \text{ (mix)}}{\Gamma, \Pi \to C}$$

Thus, for the proof of the theorem, it suffices to consider the system with the mix rule instead of the cut rule and show mix-elimination instead of cut-elimination. Our strategy is that of eliminating each mix rule above which any other mix rule does not occur.

Let P be a proof with only one mix rule occurring as the last inference whose mix formula is A. Let P_l and P_r be the subproofs of P whose end-sequents are the left and right premisses of the mix rule, respectively. The proof of eliminating the mix rule in P is by induction on $|A|$, with a subinduction on $h(P_l)+h(P_r)$. Let r_l and r_r be the left and right rules over the mix rule, respectively. We consider the following four cases:

1. At least one of P_l and P_r is an initial sequent.
2. Neither of P_l and P_r is an initial sequent, and the mix formula is principal in both r_l and r_r.
3. The mix formula is not principal in r_l.
4. The mix formula is principal in r_l and not principal in r_r.

Case 1. One of the premisses is an initial sequent $A \to A$, say the left.

$$\dfrac{A \to A \quad \Pi \to C}{A, \Pi_A \to C}\ (\text{mix})$$

The conclusion follows from the right premiss by some structural rules.

The right premiss is the initial sequent $\bot \to$. If r_l is $(\to \text{w})$, the proof looks like

$$\dfrac{\dfrac{\Gamma \to}{\Gamma \to \bot}\ (\to \text{w}) \qquad \bot \to}{\Gamma \to}\ (\text{mix})$$

The conclusion of (mix) is the same as the premiss of $(\to \text{w})$. The case where r_l is one of the other rules is handled as in Case 3 below.

Case 2. Neither of the premisses is an initial sequent, and the mix formula is principal in both r_l and r_r. We first consider the case where r_l is $(\to \wedge)$ and r_r is $(\wedge 1 \to)$.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ P_0\\ \Gamma \to A\end{array} \quad \begin{array}{c}\vdots\ P_1\\ \Gamma \to B\end{array}}{\Gamma \to A \wedge B}\ (\to \wedge) \qquad \dfrac{\begin{array}{c}\vdots\ P_2\\ A, \Pi \to C\end{array}}{A \wedge B, \Pi \to C}\ (\wedge 1 \to)}{\Gamma, \Pi_{A \wedge B} \to C}\ (\text{mix})$$

If $A \wedge B$ does not occur in Π, i.e., $\Pi_{A \wedge B}$ is the same sequence as Π, then we construct the following proof.

$$\begin{array}{c}\dfrac{\begin{array}{c}\vdots\ P_0\\ \Gamma \to A\end{array} \quad \begin{array}{c}\vdots\ P_2\\ A, \Pi \to C\end{array}}{\Gamma, \Pi_A \to C}\ (\text{mix})\\ \hline\hline \Gamma, \Pi \to C\end{array}$$

This (mix) can be eliminated by the induction hypothesis. On the other hand, if $A \wedge B$ occurs in Π, then we construct the following proof.

$$\begin{array}{c}\dfrac{\begin{array}{c}\vdots\ P_0\\ \Gamma \to A\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\ P_0\\ \Gamma \to A\end{array} \quad \begin{array}{c}\vdots\ P_1\\ \Gamma \to B\end{array}}{\Gamma \to A \wedge B}\ (\to \wedge) \qquad \begin{array}{c}\vdots\ P_2\\ A, \Pi \to C\end{array}}{\Gamma, A, \Pi_{A \wedge B} \to C}\ (\text{mix})}{\Gamma, \Gamma_A, (\Pi_{A \wedge B})_A \to C}\ (\text{mix})\\ \hline\hline \Gamma, \Pi_{A \wedge B} \to C\end{array}$$

The upper (mix) can be eliminated by the subinduction hypothesis, and then the lower (mix) can be eliminated by the induction hypothesis. The cases where r_{l} is $(\to\wedge)$ and r_{r} is $(\wedge 2\to)$ and where r_{l} is $(\to\vee 1)$ or $(\to\vee 2)$ and r_{r} is $(\vee\to)$ are proved analogously.

In the case where r_{l} is $(\to\supset)$ and r_{r} is $(\supset\to)$, the proof looks like

$$\dfrac{\dfrac{\begin{matrix}\vdots\, P_0\\ A,\Gamma\to B\end{matrix}}{\Gamma\to A\supset B}\,(\to\supset)\qquad \dfrac{\begin{matrix}\vdots\, P_1\\ \Pi\to A\end{matrix}\quad \begin{matrix}\vdots\, P_2\\ B,\Sigma\to C\end{matrix}}{A\supset B,\Pi,\Sigma\to C}\,(\supset\to)}{\Gamma,\Pi_{A\supset B},\Sigma_{A\supset B}\to C}\,(\text{mix})$$

If $A\supset B$ occurs in both Π and Σ, then we construct the following proof.

$$\begin{array}{c}\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\, P_0\\ A,\Gamma\to B\end{matrix}}{\Gamma\to A\supset B}\,(\to\supset)\quad \begin{matrix}\vdots\, P_1\\ \Pi\to A\end{matrix}}{\Gamma,\Pi_{A\supset B}\to A}\,(\text{mix})\quad \begin{matrix}\vdots\, P_0\\ A,\Gamma\to B\end{matrix}}{\Gamma,\Pi_{A\supset B},\Gamma_A\to B}\,(\text{mix})\qquad \dfrac{\dfrac{\begin{matrix}\vdots\, P_0\\ A,\Gamma\to B\end{matrix}}{\Gamma\to A\supset B}\,(\to\supset)\quad \begin{matrix}\vdots\, P_2\\ B,\Sigma\to C\end{matrix}}{\Gamma,B,\Sigma_{A\supset B}\to C}\,(\text{mix})}{\Gamma,\Pi_{A\supset B},\Gamma_A,\Gamma_B,(\Sigma_{A\supset B})_B\to C}\,(\text{mix})\\ \hline\hline \Gamma,\Pi_{A\supset B},\Sigma_{A\supset B}\to C\end{array}$$

These (mix)'s can be eliminated by the induction and the subinduction hypotheses. If $A\supset B$ does not occur in Π or in Σ, we will dispense with some of the above (mix)'s.

Next we consider the cases where r_{l} or r_{r} is a structural rule whose principal formula is the mix formula. In the case where r_{r} is $(\text{c}\to)$, which led us to introduce the mix rule instead of the cut rule, the proof looks like

$$\dfrac{\begin{matrix}\vdots\, P_0\\ \Gamma\to A\end{matrix}\qquad \dfrac{\begin{matrix}\vdots\, P_1\\ A,A,\Pi\to C\end{matrix}}{A,\Pi\to C}\,(\text{c}\to)}{\Gamma,\Pi_A\to C}\,(\text{mix})$$

This is transformed into

$$\dfrac{\begin{matrix}\vdots\, P_0\\ \Gamma\to A\end{matrix}\quad \begin{matrix}\vdots\, P_1\\ A,A,\Pi\to C\end{matrix}}{\Gamma,\Pi_A\to C}\,(\text{mix})$$

where the (mix) can be eliminated by the subinduction hypothesis. The cases where r_{r} is $(\text{e}\to)$, r_{r} is $(\text{w}\to)$ and r_{l} is $(\to\text{w})$ are proved easily.

Case 3. Neither of the premisses is an initial sequent, and the mix formula is not principal in r_{l}. We consider the case where r_{l} is $(\supset\to)$.

$$\dfrac{\dfrac{\begin{matrix}\vdots\, P_0\\ \Gamma\to A\end{matrix}\quad \begin{matrix}\vdots\, P_1\\ B,\Delta\to C\end{matrix}}{A\supset B,\Gamma,\Delta\to C}\,(\supset\to)\qquad \begin{matrix}\vdots\, P_2\\ \Pi\to D\end{matrix}}{A\supset B,\Gamma,\Delta,\Pi_C\to D}\,(\text{mix})$$

This is transformed into

$$\dfrac{\begin{matrix}\vdots\, P_0\\ \Gamma\to A\end{matrix}\qquad \dfrac{\begin{matrix}\vdots\, P_1\\ B,\Delta\to C\end{matrix}\quad \begin{matrix}\vdots\, P_2\\ \Pi\to D\end{matrix}}{B,\Delta,\Pi_C\to D}\,(\text{mix})}{A\supset B,\Gamma,\Delta,\Pi_C\to D}\,(\supset\to)$$

where the (mix) can be eliminated by the subinduction hypothesis. The other cases are proved analogously.

Case 4. The mix formula is principal in r_{l} and not principal in r_{r}. We consider the case where r_{r} is $(\supset\rightarrow)$.

$$\dfrac{\begin{array}{c}\vdots\, P_0\\ \Gamma \rightarrow C\end{array} \quad \dfrac{\begin{array}{c}\vdots\, P_1\\ \Pi \rightarrow A\end{array} \quad \begin{array}{c}\vdots\, P_2\\ B, \Sigma \rightarrow D\end{array}}{A \supset B, \Pi, \Sigma \rightarrow D}\,(\supset\rightarrow)}{\Gamma, A \supset B, \Pi_C, \Sigma_C \rightarrow D}\,(\mathrm{mix})$$

If C occurs in both Π and Σ, then we construct the following proof.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\, P_0\\ \Gamma \rightarrow C\end{array} \quad \begin{array}{c}\vdots\, P_1\\ \Pi \rightarrow A\end{array}}{\Gamma, \Pi_C \rightarrow A}\,(\mathrm{mix}) \quad \dfrac{\dfrac{\begin{array}{c}\vdots\, P_0\\ \Gamma \rightarrow C\end{array} \quad \begin{array}{c}\vdots\, P_2\\ B, \Sigma \rightarrow D\end{array}}{\Gamma, (B, \Sigma)_C \rightarrow D}\,(\mathrm{mix})}{B, \Gamma, \Sigma_C \rightarrow D}}{A \supset B, \Gamma, \Pi_C, \Gamma, \Sigma_C \rightarrow D}\,(\supset\rightarrow)}{\Gamma, A \supset B, \Pi_C, \Sigma_C \rightarrow D}$$

These (mix)'s can be eliminated by the subinduction hypothesis. If C does not occur in Π or in Σ, we will dispense with some of the above (mix)'s. The other cases than that where r_{r} is $(\supset\rightarrow)$ are proved analogously.

∎

2.4 Notes

Gentzen's LK, LJ and cut-elimination In [18], Gentzen introduced the sequent calculi **LK** and **LJ** for classical and intuitionistic predicate logics, respectively. As far as the propositional part is concerned, our formulation in this chapter differs from Gentzen's original one in that we use the propositional constant $\bot$ whereas Gentzen used the logical connective $\neg$ with the following rules. (Δ is empty in the case of **LJ**.)

$$\dfrac{\Gamma \rightarrow \Delta, A}{\neg A, \Gamma \rightarrow \Delta}\,(\neg\rightarrow) \qquad\qquad \dfrac{A, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg A}\,(\rightarrow\neg)$$

Gentzen proved the cut-elimination theorem for both the calculi **LK** and **LJ** by syntactical methods. He introduced the notion of *rank* to measure the simplicity of a proof rather than height that we used in the previous section. Our proof of the cut-elimination theorem for **LJ** mainly follows Chapter 4 of [47].

Alternative sequent calculus for intuitionistic logic There is another sequent calculus for intuitionistic logic that allows natural semantical proof of cut-elimination. The system is obtained from **LK** by restricting the rule $(\rightarrow\supset)$ to that of **LJ**. The completeness theorem of the system with respect to Kripke semantics for intuitionistic logic can be proved without using the cut rule, where (S1)–(S5) in Definition 2.5 are adopted as the conditions of saturated pairs. Syntactical proof of cut-elimination for the system is also possible with the help of the so-called inversion lemma. For the details, see, e.g. [43].

Chapter 3

Sequent calculi for subintuitionistic logics

In this chapter we study sequent calculi for subintuitionistic logics. We first introduce systems based on sequents of the same form as those of Gentzen's **LK** for classical logic, and then introduce systems based on dual-context style sequents which are close to Gentzen's **LJ** for intuitionistic logic. We prove the completeness and cut-elimination theorems for each of these systems.

3.1 Semantics of subintuitionistic logics

Subintuitionistic logics are defined semantically through Kripke models. While Kripke models for intuitionistic logic are based on preorders $\langle W, \leq \rangle$ with reflexive and transitive relation $\leq$, Kripke models for subintuitionistic logics are often based on $\langle W, R \rangle$ with more general binary relation R. Here we deal with in particular the logic K^I defined by Kripke models with any binary relation R, and the logic BPC defined by Kripke models with any transitive relation R. Those Kripke models for K^I and BPC do not necessarily force the formulas corresponding to the contraction rule and the exchange rule in the sequent calculus **LJ** for intuitionistic logic.

The language of subintuitionistic logics is the same as that of classical propositional logic. Kripke models for K^I and BPC are defined as follows.

Definition 3.1 A *model* is a triple $\langle W, R, V \rangle$ where

- W is a nonempty set,
- R is a binary relation on W,
- V is a function from $\mathcal{PV}$ to 2^W.

A BPC-*model* is a model $\langle W, R, V \rangle$ such that

- R is transitive,
- for every $p \in \mathcal{PV}$ and every $x, y \in W$,

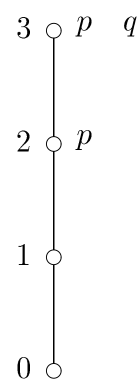
$$x \in V(p) \quad \text{and} \quad xRy \quad \text{imply} \quad y \in V(p).$$

For a given model $M = \langle W, R, V \rangle$, the truth-relation $\models_M$ ($\subseteq W \times \mathcal{F}orm$) is defined inductively as follows:

$$
\begin{array}{lll}
x \models_M p & \text{iff} & x \in V(p) \quad \text{for every } p \in \mathcal{PV}, \\
x \not\models_M \bot, & & \\
x \models_M A \wedge B & \text{iff} & x \models_M A \ \text{ and } \ x \models_M B, \\
x \models_M A \vee B & \text{iff} & x \models_M A \ \text{ or } \ x \models_M B, \\
x \models_M A \supset B & \text{iff} & \forall y \in W[xRy \ \text{ and } \ y \models_M A \ \text{ imply } \ y \models_M B].
\end{array}
$$

If $x \models_M A$ for every A occurring in a sequence Γ, we write $x \models_M \Gamma$. The subscript M may be omitted if understood. A formula A is *true* in a model $\langle W, R, V \rangle$ if $x \models A$ for every $x \in W$.

Example 3.2 Consider a (BPC-)model $\langle \{0, 1, 2, 3\}, <, V \rangle$ where $<$ is the usual strict order on natural numbers and V is a function such that $V(p) = \{2, 3\}$, $V(q) = \{3\}$ and $V(r) = \emptyset$. The situation is graphically represented as follows.



Then $2 \models p \supset q$, and $3 \models p \supset q$ since there is no $x \in \{0, 1, 2, 3\}$ such that $3 < x$. Hence $1 \models p \supset (p \supset q)$. However, $1 \not\models p \supset q$ since $2 \models p$ and $2 \not\models q$. Thus $0 \not\models (p \supset (p \supset q)) \supset (p \supset q)$, which means that $(p \supset (p \supset q)) \supset (p \supset q)$ is not true in this model.

On the other hand, $3 \models p \supset r$ and hence $1 \models q \supset (p \supset r)$. However, since $3 \models q$ and $3 \not\models r$, we have $2 \not\models q \supset r$ and so $1 \not\models p \supset (q \supset r)$. Thus $0 \not\models (q \supset (p \supset r)) \supset (p \supset (q \supset r))$, which means that $(q \supset (p \supset r)) \supset (p \supset (q \supset r))$ is not true in this model either.

Lemma 3.3 *For every* BPC-*model* $\langle W, R, V \rangle$, *every formula* A *and every* $x, y \in W$,

$$x \models A \ \textit{ and } \ xRy \ \textit{ imply } \ y \models A.$$

PROOF. By induction on the structure of A.

1. A is a propositional variable p. Follows from the definition of BPC-model.
2. A is the propositional constant $\bot$. $x \models \bot$ is impossible by the definition of $\models$.
3. A is of the form $B \wedge C$. Suppose that $x \models B \wedge C$ and xRy. Then $x \models B$ and $x \models C$. By the induction hypothesis, $y \models B$ and $y \models C$. Hence $y \models B \wedge C$.

4. A is of the form $B \vee C$. Analogous to the previous case.

5. A is of the form $B \supset C$. Suppose that $x \models B \supset C$ and xRy. To show $y \models B \supset C$, suppose further that yRz and $z \models B$. Then xRz by the transitivity of R. Since $x \models B \supset C$ and $z \models B$, we have $z \models C$ as required.

■

3.2 Sequent calculus for K^I

In this section we study a sequent calculus $\mathbf{GK}^I$ which was introduced by Kashima [28]. The system is based on an analysis of a system for the modal logic K, and differs from it only in the rules for implication and modal operator. We show that $\mathbf{GK}^I$ is complete with respect to the class of Kripke models for K^I, and obtain the cut-elimination theorem for $\mathbf{GK}^I$ as a corollary.

First a *sequent* of $\mathbf{GK}^I$ is defined as an expression of the form $\Gamma \to \Delta$. The truth of sequents of $\mathbf{GK}^I$ in Kripke models is defined in the following way.

Definition 3.4 For a given model M, the truth-relation $\models_M$ for sequents of $\mathbf{GK}^I$ is defined as follows:

$$x \models_M \Gamma \to \Delta \quad \text{iff} \quad x \models_M \Gamma \ \text{ implies } \ x \models_M D \text{ for some } D \text{ occurring in } \Delta.$$

The subscript M may be omitted if understood. A sequent $\Gamma \to \Delta$ is *true* in a model $\langle W, R, V\rangle$ if $x \models \Gamma \to \Delta$ for every $x \in W$.

Proposition 3.5 *For any formula A, any model $M = \langle W, R, V\rangle$ and any $x \in W$,*

1. *$x \models_M A$ if and only if $x \models_M \to A$,*

2. *A is true in M if and only if $\to A$ is true in M.*

PROOF. Straightforward. ■

Now we introduce the sequent calculus $\mathbf{GK}^I$. Initial sequents of $\mathbf{GK}^I$ are of the following forms:

$$A \to A,$$

$$\bot \to \ .$$

Rules of inference of $\mathbf{GK}^I$ are the following.

Structural rules:

$$\frac{\Gamma \to \Delta}{A, \Gamma \to \Delta}\ (\mathrm{w} \to) \qquad \frac{\Gamma \to \Delta}{\Gamma \to \Delta, A}\ (\to \mathrm{w})$$

$$\frac{A, A, \Gamma \to \Delta}{A, \Gamma \to \Delta}\ (\mathrm{c} \to) \qquad \frac{\Gamma \to \Delta, A, A}{\Gamma \to \Delta, A}\ (\to \mathrm{c})$$

$$\frac{\Gamma, A, B, \Pi \to \Delta}{\Gamma, B, A, \Pi \to \Delta}\ (\mathrm{e} \to) \qquad \frac{\Gamma \to \Delta, A, B, \Sigma}{\Gamma \to \Delta, B, A, \Sigma}\ (\to \mathrm{e})$$

Cut rule:

$$\frac{\Gamma \to \Delta, A \quad A, \Pi \to \Sigma}{\Gamma, \Pi \to \Delta, \Sigma} \text{ (cut)}$$

Rules for the logical connectives:

$$\frac{A, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta} (\wedge 1 \to) \qquad \frac{B, \Gamma \to \Delta}{A \wedge B, \Gamma \to \Delta} (\wedge 2 \to)$$

$$\frac{\Gamma \to \Delta, A \quad \Gamma \to \Delta, B}{\Gamma \to \Delta, A \wedge B} (\to \wedge)$$

$$\frac{A, \Gamma \to \Delta \quad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta} (\vee \to)$$

$$\frac{\Gamma \to \Delta, A}{\Gamma \to \Delta, A \vee B} (\to \vee 1) \qquad \frac{\Gamma \to \Delta, B}{\Gamma \to \Delta, A \vee B} (\to \vee 2)$$

$$\frac{\Delta_1, A \to B, \Gamma_1 \quad \Delta_2, A \to B, \Gamma_2 \quad \cdots \quad \Delta_{2^n}, A \to B, \Gamma_{2^n}}{C_1 \supset D_1, \ldots, C_n \supset D_n \to A \supset B} (\supset)$$

where $n \geq 0$, $\Delta_i = D_{i_1}, \ldots, D_{i_k}$, $\Gamma_i = C_{i_{k+1}}, \ldots, C_{i_n}$, and $i_1, \ldots, i_n$ are defined as follows. Enumerate all the *partitions* of $\langle 1, \ldots, n \rangle$:

$$\langle (), (1, \ldots, n) \rangle, \langle (1), (2, \ldots, n) \rangle, \langle (2), (1, 3, \ldots, n) \rangle, \ldots,$$
$$\langle (1, 2), (3, \ldots, n) \rangle, \langle (2, 3), (1, 4, \ldots, n) \rangle, \ldots, \langle (1, \ldots, n), () \rangle.$$

The number of the partitions is 2^n, and $\langle (i_1, \ldots, i_k), (i_{k+1}, \ldots, i_n) \rangle$ is defined to be the ith partition.

For example, when $n = 0, 1, 2$, the rule $(\supset)$ takes the forms

$$\frac{A \to B}{\to A \supset B} (\supset),$$

$$\frac{A \to B, C_1 \quad D_1, A \to B}{C_1 \supset D_1 \to A \supset B} (\supset),$$

and

$$\frac{A \to B, C_1, C_2 \quad D_1, A \to B, C_2 \quad D_2, A \to B, C_1 \quad D_1, D_2, A \to B}{C_1 \supset D_1, C_2 \supset D_2 \to A \supset B} (\supset),$$

respectively.

The rule $(\supset)$ of $\mathbf{GK}^I$ may be well understood via the rule

$$\frac{\Gamma \to A}{\Box\Gamma \to \Box A} (\Box)$$

of the sequent calculus for the modal logic K and the translation of each formula $A \supset B$ of K^I into the formula $\Box(\neg A' \vee B')$ of K. For example, when $n = 2$, the rule $(\supset)$ corresponds to the following inference in the sequent calculus for K.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A' \to B', C'_1, C'_2}{\neg C'_1, A' \to B', C'_2} \quad D'_1, A' \to B', C'_2}{\neg C'_1 \vee D'_1, A' \to B', C'_2}}{\neg C'_1 \vee D'_1, \neg C'_2, A' \to B'} \quad \dfrac{\dfrac{D'_2, A' \to B', C'_1}{\neg C'_1, D'_2, A' \to B'} \quad D'_1, D'_2, A' \to B'}{\neg C'_1 \vee D'_1, D'_2, A' \to B'}}{\neg C'_1 \vee D'_1, \neg C'_2 \vee D'_2, A' \to B'}}{\neg C'_1 \vee D'_1, \neg C'_2 \vee D'_2 \to \neg A', B'}}{\neg C'_1 \vee D'_1, \neg C'_2 \vee D'_2 \to \neg A' \vee B'}}{\Box(\neg C'_1 \vee D'_1), \Box(\neg C'_2 \vee D'_2) \to \Box(\neg A' \vee B')}\ (\Box)$$

Generalizing this analysis to arbitrary n, we arrive at the rule $(\supset)$ of $\mathbf{GK}^I$.

Theorem 3.6 (Soundness of $\mathbf{GK}^I$) *For any sequent $\Gamma \to \Delta$, if $\Gamma \to \Delta$ is provable in $\mathbf{GK}^I$, then $\Gamma \to \Delta$ is true in every model.*

Proof. By induction on the proof of $\Gamma \to \Delta$ in $\mathbf{GK}^I$. Here we consider only the rule $(\supset)$. Let $\langle W, R, V\rangle$ be any model. By the induction hypothesis, each upper sequent $\Delta_i, A \to B, \Gamma_i$ of the rule $(\supset)$ is true in the model $\langle W, R, V\rangle$. Our aim is to show $x \models C_1 \supset D_1, \ldots, C_n \supset D_n \to A \supset B$ for any $x \in W$. Suppose $x \models C_l \supset D_l$ for every l with $1 \le l \le n$. To show $x \models A \supset B$, suppose further xRy and $y \models A$. Then consider the set $\delta = \{j \mid y \models D_j,\ 1 \le j \le n\}$ and take Δ_i consisting of the formulas D_j with $j \in \delta$. Since $y \models \Delta_i, A \to B, \Gamma_i$, we have $y \models B$ or $y \models C_l$ for some C_l occurring in Γ_i, where $l \notin \delta$ by the partitioning of Δ_i and Γ_i. However, $y \models C_l$ is impossible, because it together with the supposition $x \models C_l \supset D_l$ implies $y \models D_l$, which contradicts the definition of δ. Therefore we have $y \models B$. ∎

Next let $\mathbf{GK}^I-$(cut) denote the system obtained from $\mathbf{GK}^I$ by deleting the cut rule. In the following we show that $\mathbf{GK}^I-$(cut) is sufficient to prove any sequent that is true in every model. From this and Theorem 3.6, we obtain the cut-elimination theorem for $\mathbf{GK}^I$.

Definition 3.7 Let x, y be sets of formulas. A pair (x, y) is *consistent* (in $\mathbf{GK}^I-$(cut)) if, for any $A_1, \ldots, A_m \in x$ and any $B_1, \ldots, B_n \in y$ $(m, n \ge 0)$,

$$A_1, \ldots, A_m \to B_1, \ldots, B_n$$

is not provable in $\mathbf{GK}^I-$(cut). (x, y) is *saturated* (in $\mathbf{GK}^I-$(cut)) if it is consistent and satisfies the following conditions:

$$\begin{array}{lllclcl}
\text{(S1)} & A \wedge B \in x & \text{implies} & A \in x & \text{and} & B \in x, \\
\text{(S2)} & A \wedge B \in y & \text{implies} & A \in y & \text{or} & B \in y, \\
\text{(S3)} & A \vee B \in x & \text{implies} & A \in x & \text{or} & B \in x, \\
\text{(S4)} & A \vee B \in y & \text{implies} & A \in y & \text{and} & B \in y.
\end{array}$$

Lemma 3.8 *Let x, y be finite sets of formulas. If (x, y) is consistent, then there exists a saturated pair (x', y') such that $x \subseteq x' \subseteq \mathrm{Sub}(x \cup y)$ and $y \subseteq y' \subseteq \mathrm{Sub}(x \cup y)$.*

PROOF. Let $C_1, \ldots, C_n$ be a sequence of all formulas in $\mathrm{Sub}(x \cup y)$ where $|C_i| \geq |C_{i+1}|$ for each $i \in \{1, \ldots, n-1\}$. We define a sequence of consistent pairs (x_i, y_i) $(i = 0, \ldots, n)$ as follows:

1. $(x_0, y_0) = (x, y)$.
2. For $i \geq 1$, the pair (x_i, y_i) is obtained from (x_{i-1}, y_{i-1}) in the following way.

   2.1. If C_i is of the form $A \wedge B$ and $C_i \in x_{i-1}$, then $(x_i, y_i) = (x_{i-1} \cup \{A, B\}, y_{i-1})$. Here (x_i, y_i) is consistent whenever (x_{i-1}, y_{i-1}) is consistent. Indeed, otherwise there exist formulas $D_1, \ldots, D_j \in x_{i-1}$ and $E_1, \ldots, E_k \in y_{i-1}$ such that

   $$D_1, \ldots, D_j, A, B \to E_1, \ldots, E_k$$

   is provable in $\mathbf{GK}^I-(\mathrm{cut})$. Then

   $$D_1, \ldots, D_j, A \wedge B \to E_1, \ldots, E_k$$

   is also provable in $\mathbf{GK}^I-(\mathrm{cut})$, contrary to the consistency of (x_{i-1}, y_{i-1}).

   2.2. If C_i is of the form $A \wedge B$ and $C_i \in y_{i-1}$, then (x_i, y_i) is defined as either $(x_{i-1}, y_{i-1} \cup \{A\})$ or $(x_{i-1}, y_{i-1} \cup \{B\})$. Here we can define (x_i, y_i) to be consistent whenever (x_{i-1}, y_{i-1}) is consistent. Indeed, otherwise there exist formulas $D_1, \ldots, D_j \in x_{i-1}$ and $E_1, \ldots, E_k \in y_{i-1}$ such that

   $$D_1, \ldots, D_j \to E_1, \ldots, E_k, A$$
   $$D_1, \ldots, D_j \to E_1, \ldots, E_k, B$$

   are both provable in $\mathbf{GK}^I-(\mathrm{cut})$. Then

   $$D_1, \ldots, D_j \to E_1, \ldots, E_k, A \wedge B$$

   is also provable in $\mathbf{GK}^I-(\mathrm{cut})$, contrary to the consistency of (x_{i-1}, y_{i-1}).

   2.3. If C_i is of the form $A \vee B$ and $C_i \in x_{i-1}$, then (x_i, y_i) is defined as either $(x_{i-1} \cup \{A\}, y_{i-1})$ or $(x_{i-1} \cup \{B\}, y_{i-1})$. By an argument symmetric to that in 2.2, we can define (x_i, y_i) to be consistent whenever (x_{i-1}, y_{i-1}) is consistent.

   2.4. If C_i is of the form $A \vee B$ and $C_i \in y_{i-1}$, then $(x_i, y_i) = (x_{i-1}, y_{i-1} \cup \{A, B\})$. By an argument symmetric to that in 2.1, the pair (x_i, y_i) is consistent whenever (x_{i-1}, y_{i-1}) is consistent.

   2.5. Otherwise, $(x_i, y_i) = (x_{i-1}, y_{i-1})$.

Then (x_i, y_i) is consistent for all $i \in \{0, \ldots, n\}$, and (x_n, y_n) is saturated since the conditions (S1)–(S4) are satisfied with the above constructions 2.1–2.4, respectively. The pair (x_n, y_n) satisfies $x \subseteq x_n \subseteq \mathrm{Sub}(x \cup y)$ and $y \subseteq y_n \subseteq \mathrm{Sub}(x \cup y)$, so we complete the proof. ∎

Lemma 3.9 *For any sequent $\Gamma \to \Delta$, if $\Gamma \to \Delta$ is true in every model, then $\Gamma \to \Delta$ is provable in* $\mathbf{GK}^I$–(cut).

Proof. Suppose that $\Gamma \to \Delta$ is not provable in $\mathbf{GK}^I$–(cut). Let γ and δ be the sets of formulas occurring in Γ and Δ, respectively. Then the pair (γ, δ) is consistent, and so by Lemma 3.8, there exists a saturated pair (u, v) such that $\gamma \subseteq u \subseteq \mathrm{Sub}(\gamma \cup \delta)$ and $\delta \subseteq v \subseteq \mathrm{Sub}(\gamma \cup \delta)$. Now define a model $\langle W, R, V \rangle$ as follows:

- W is the set of all saturated pairs (x, w) such that $x \cup w \subseteq \mathrm{Sub}(\gamma \cup \delta)$,
- $(x, w)R(y, z)$ iff $\forall (A \supset B) \in x[A \in z \text{ or } B \in y]$,
- $V(p) = \{(x, w) \in W \mid p \in x\}$ for every $p \in \mathcal{PV}$.

Our aim is to show $(u, v) \not\models \Gamma \to \Delta$, which means that $\Gamma \to \Delta$ is not true in this model. Since $\gamma \subseteq u$ and $\delta \subseteq v$, it suffices to show that for any $(x, w) \in W$,

$$\begin{array}{lll} C \in x & \text{implies} & (x, w) \models C, \quad \text{and} \\ C \in w & \text{implies} & (x, w) \not\models C. \end{array}$$

We prove this by induction on the structure of C.

1. C is a propositional variable p. If $p \in x$ then $(x, w) \in V(p)$ by the definition of V, and so $(x, w) \models p$. If $p \in w$ then $p \notin x$ by the consistency of (x, w), and so $(x, w) \notin V(p)$, i.e., $(x, w) \not\models p$.

2. C is the propositional constant $\bot$. If $\bot \in x$ then (x, w) is not consistent, which is a contradiction. If $\bot \in w$ then $(x, w) \not\models \bot$ by the definition of $\models$.

3. C is of the form $A \wedge B$. If $A \wedge B \in x$ then $A \in x$ and $B \in x$ by the condition (S1) of the saturated pair (x, w). By the induction hypothesis, $(x, w) \models A$ and $(x, w) \models B$, and so $(x, w) \models A \wedge B$. If $A \wedge B \in w$ then $A \in w$ or $B \in w$ by the condition (S2). By the induction hypothesis, $(x, w) \not\models A$ or $(x, w) \not\models B$, and so $(x, w) \not\models A \wedge B$.

4. C is of the form $A \vee B$. Symmetric to the previous case.

5. C is of the form $A \supset B$. Suppose $A \supset B \in x$. By the definition of R, if $(x, w)R(y, z)$ then $A \in z$ or $B \in y$, and so $(y, z) \not\models A$ or $(y, z) \models B$ by the induction hypothesis. This means $(x, w)R(y, z)$ and $(y, z) \models A$ imply $(y, z) \models B$. Hence $(x, w) \models A \supset B$. For the latter claim, suppose $A \supset B \in w$. Let $C_1 \supset D_1, \ldots, C_n \supset D_n$ be a sequence of all formulas of the form $E \supset F$ in x. Then, there exists at least one consistent pair $(\delta_i \cup \{A\}, \{B\} \cup \gamma_i)$ with $1 \leq i \leq 2^n$ where δ_i and γ_i are the sets of formulas in Δ_i and Γ_i in the rule ($\supset$), respectively. Indeed, otherwise $C_1 \supset D_1, \ldots, C_n \supset D_n \to A \supset B$ is provable in $\mathbf{GK}^I$–(cut), which contradicts the consistency of (x, w). Applying Lemma 3.8 to such a consistent pair $(\delta_i \cup \{A\}, \{B\} \cup \gamma_i)$, we obtain a saturated pair (y, z) such that $\delta_i \cup \{A\} \subseteq y$, $\{B\} \cup \gamma_i \subseteq z$ and $y \cup z \subseteq \mathrm{Sub}(\delta_i \cup \{A\} \cup \{B\} \cup \gamma_i) \subseteq \mathrm{Sub}(x \cup w) \subseteq \mathrm{Sub}(\gamma \cup \delta)$ (so we have $(y, z) \in W$). By the definition of R and the induction hypothesis, we have $(x, w)R(y, z)$, $(y, z) \models A$ and $(y, z) \not\models B$. This means $(x, w) \not\models A \supset B$.

∎

From Theorem 3.6 and Lemma 3.9, we obtain the following results.

Theorem 3.10 (Completeness of $\mathbf{GK}^I$) *For any sequent $\Gamma \to \Delta$, $\Gamma \to \Delta$ is provable in $\mathbf{GK}^I$ if and only if $\Gamma \to \Delta$ is true in every model.*

Theorem 3.11 (Cut-elimination for $\mathbf{GK}^I$) *For any sequent $\Gamma \to \Delta$, if $\Gamma \to \Delta$ is provable in $\mathbf{GK}^I$, then it is provable in $\mathbf{GK}^I$ without using the cut rule.*

3.3 Sequent calculus for BPC

In this section we introduce a sequent calculus **LBP** by modifying the system $\mathbf{GK}^I$ in the previous section. We show that **LBP** is complete with respect to the class of Kripke models for BPC, and obtain the cut-elimination theorem as a corollary.

The notions of sequents of **LBP** and their truth in Kripke models are defined in the same way as those of $\mathbf{GK}^I$. The sequent calculus **LBP** is obtained from $\mathbf{GK}^I$ by replacing the rule $(\supset)$ by the following one:

$$\frac{\Delta_1, \Sigma, A \to B, \Gamma_1 \quad \Delta_2, \Sigma, A \to B, \Gamma_2 \quad \cdots \quad \Delta_{2^n}, \Sigma, A \to B, \Gamma_{2^n}}{\Sigma, C_1 \supset D_1, \ldots, C_n \supset D_n \to A \supset B} \ (\supset)$$

where $n \geq 0$, and Δ_i and Γ_i are as in the rule $(\supset)$ of $\mathbf{GK}^I$.

Theorem 3.12 (Soundness of LBP) *For any sequent $\Gamma \to \Delta$, if $\Gamma \to \Delta$ is provable in* **LBP**, *then $\Gamma \to \Delta$ is true in every* BPC-*model.*

Proof. By induction on the proof of $\Gamma \to \Delta$ in **LBP**. Here we consider only the rule $(\supset)$. Let $\langle W, R, V \rangle$ be any BPC-model. By the induction hypothesis, each upper sequent $\Delta_i, \Sigma, A \to B, \Gamma_i$ of the rule $(\supset)$ is true in the BPC-model $\langle W, R, V \rangle$. Our aim is to show $x \models \Sigma, C_1 \supset D_1, \ldots, C_n \supset D_n \to A \supset B$ for any $x \in W$. Suppose $x \models \Sigma$ and $x \models C_l \supset D_l$ for every l with $1 \leq l \leq n$. To show $x \models A \supset B$, suppose further xRy and $y \models A$. Then $y \models \Sigma$ by Lemma 3.3. Here consider the set $\delta = \{j \mid y \models D_j,\ 1 \leq j \leq n\}$ and take Δ_i consisting of the formulas D_j with $j \in \delta$. Since $y \models \Delta_i, \Sigma, A \to B, \Gamma_i$, we have $y \models B$ or $y \models C_l$ for some C_l occurring in Γ_i, where $l \notin \delta$ by the partitioning of Δ_i and Γ_i. However, $y \models C_l$ is impossible, because it together with the supposition $x \models C_l \supset D_l$ implies $y \models D_l$, which contradicts the definition of δ. Therefore we have $y \models B$. ∎

Next let **LBP**$-$(cut) denote the system obtained from **LBP** by deleting the cut rule. We show that **LBP**$-$(cut) is sufficient to prove any sequent that is true in every BPC-model. The notions of consistent and saturated pairs (in **LBP**$-$(cut)) are defined in the same way as in Definition 3.7.

Lemma 3.13 *Let x, y be finite sets of formulas. If (x, y) is consistent, then there exists a saturated pair (x', y') such that $x \subseteq x' \subseteq \mathrm{Sub}(x \cup y)$ and $y \subseteq y' \subseteq \mathrm{Sub}(x \cup y)$.*

Proof. Similar to the proof of Lemma 3.8. ∎

Lemma 3.14 *For any sequent* $\Gamma \to \Delta$*, if* $\Gamma \to \Delta$ *is true in every* BPC*-model, then* $\Gamma \to \Delta$ *is provable in* **LBP**$-$(cut).

Proof. Suppose that $\Gamma \to \Delta$ is not provable in **LBP**$-$(cut). Let γ and δ be the sets of formulas occurring in Γ and Δ, respectively. Then the pair (γ, δ) is consistent, and so by Lemma 3.13, there exists a saturated pair (u, v) such that $\gamma \subseteq u \subseteq \mathrm{Sub}(\gamma \cup \delta)$ and $\delta \subseteq v \subseteq \mathrm{Sub}(\gamma \cup \delta)$. Now define a BPC-model $\langle W, R, V\rangle$ as follows:

- W is the set of all saturated pairs (x, w) such that $x \cup w \subseteq \mathrm{Sub}(\gamma \cup \delta)$,
- $(x, w)R(y, z)$ iff $x \subseteq y$ and $\forall (A \supset B) \in x[A \in z$ or $B \in y]$,
- $V(p) = \{(x, w) \in W \mid p \in x\}$ for every $p \in \mathcal{PV}$.

It is easy to verify that R is transitive and that $(x, w) \in V(p)$ and $(x, w)R(y, z)$ imply $(y, z) \in V(p)$. Thus $\langle W, R, V\rangle$ is indeed a BPC-model. Our aim is to show $(u, v) \not\models \Gamma \to \Delta$, which means that $\Gamma \to \Delta$ is not true in this BPC-model. Since $\gamma \subseteq u$ and $\delta \subseteq v$, it suffices to show that for any $(x, w) \in W$,

$$\begin{array}{lll} C \in x & \text{implies} & (x, w) \models C, \quad \text{and} \\ C \in w & \text{implies} & (x, w) \not\models C. \end{array}$$

This is proved by induction on the structure of C. Here we consider only the case where C is of the form $A \supset B$. The other cases are proved in the same way as those in the proof of Lemma 3.9. Suppose $A \supset B \in x$. By the definition of R, if $(x, w)R(y, z)$ then $A \in z$ or $B \in y$, and so $(y, z) \not\models A$ or $(y, z) \models B$ by the induction hypothesis. This means $(x, w)R(y, z)$ and $(y, z) \models A$ imply $(y, z) \models B$. Hence $(x, w) \models A \supset B$. For the latter claim, suppose $A \supset B \in w$. Let $C_1 \supset D_1, \ldots, C_n \supset D_n$ be a sequence of all formulas of the form $E \supset F$ in x. Then, there exists at least one consistent pair $(\delta_i \cup x \cup \{A\}, \{B\} \cup \gamma_i)$ with $1 \leq i \leq 2^n$ where δ_i and γ_i are the sets of formulas in Δ_i and Γ_i in the rule $(\supset)$, respectively. Indeed, otherwise $\Sigma, C_1 \supset D_1, \ldots, C_n \supset D_n \to A \supset B$ is provable in **LBP**$-$(cut) for some Σ consisting of formulas in x, which contradicts the consistency of (x, w). Applying Lemma 3.13 to such a consistent pair $(\delta_i \cup x \cup \{A\}, \{B\} \cup \gamma_i)$, we obtain a saturated pair (y, z) such that $\delta_i \cup x \cup \{A\} \subseteq y$, $\{B\} \cup \gamma_i \subseteq z$ and $y \cup z \subseteq \mathrm{Sub}(\delta_i \cup x \cup \{A\} \cup \{B\} \cup \gamma_i) \subseteq \mathrm{Sub}(x \cup w) \subseteq \mathrm{Sub}(\gamma \cup \delta)$ (so we have $(y, z) \in W$). By the definition of R and the induction hypothesis, we have $(x, w)R(y, z)$, $(y, z) \models A$ and $(y, z) \not\models B$. This means $(x, w) \not\models A \supset B$. ∎

From Theorem 3.12 and Lemma 3.14, we obtain the following results.

Theorem 3.15 (Completeness of LBP) *For any sequent* $\Gamma \to \Delta$*,* $\Gamma \to \Delta$ *is provable in* **LBP** *if and only if* $\Gamma \to \Delta$ *is true in every* BPC*-model.*

Theorem 3.16 (Cut-elimination for LBP) *For any sequent* $\Gamma \to \Delta$*, if* $\Gamma \to \Delta$ *is provable in* **LBP***, then it is provable in* **LBP** *without using the cut rule.*

3.4 Dual-context sequent calculus for K^I

In Section 3.2 we showed that the sequent calculus $\mathbf{GK}^I$ is complete with respect to semantics for K^I and that the cut-elimination theorem holds for $\mathbf{GK}^I$. However, the system $\mathbf{GK}^I$ is not satisfactory in the respect that the rule for implication involves 2^n premisses for n principal formulas in the conclusion. This gives rise to difficulty in comparing $\mathbf{GK}^I$ with sequent calculi for intuitionistic logic and for substructural logics.

The system we introduce in this section avoids the problem on $\mathbf{GK}^I$ by splitting the left hand side of a sequent and making just one principal formula in the rules for implication. This dual-context style system, which we call $\mathbf{DK}^I$, is close to Gentzen's sequent calculus **LJ** for intuitionistic logic and suitable to compare with systems for intuitionistic logic and for substructural logics. We show that $\mathbf{DK}^I$ is complete with respect to the class of Kripke models for K^I, making good use of the structure of the sequent. The cut-elimination theorem for $\mathbf{DK}^I$ is proved by syntactical methods in Section 3.6.

Formally, a *sequent* of $\mathbf{DK}^I$ is defined as an expression of the form $\Gamma; \Delta \to A$, where A may be empty. The interpretation of the sequents in Kripke models is defined in the following way.

Definition 3.17 For a given model $M = \langle W, R, V\rangle$, the truth-relation $\models_M$ for sequents of $\mathbf{DK}^I$ is defined as follows:

$$x \models_M \Gamma; \Delta \to A \quad \text{iff} \quad x \models_M \Gamma \quad \text{implies} \quad \forall y \in W[xRy \text{ and } y \models_M \Delta \text{ imply } y \models_M A].$$

The case where A is empty is the same as the case where A is $\bot$. The subscript M may be omitted if understood. A sequent $\Gamma; \Delta \to A$ is *true* in a model $\langle W, R, V\rangle$ if $x \models \Gamma; \Delta \to A$ for every $x \in W$.

Proposition 3.18 *For any formula A, A is true in every model if and only if $; \to A$ is true in every model.*

PROOF. The implication from left to right is straightforward. For the other direction, suppose that A is not true in a model $M = \langle W, R, V\rangle$, i.e., $x \not\models_M A$ for some $x \in W$. Then consider a model $M' = \langle W \cup \{y\}, R \cup \{(y, x)\}, V\rangle$ where $y \notin W$. We can show by induction on the structure of B that for any $w \in W$, $w \models_M B$ if and only if $w \models_{M'} B$. Hence $x \not\models_{M'} A$, and so $y \not\models_{M'} ; \to A$, which means that $; \to A$ is not true in M'. ∎

Now we introduce the sequent calculus $\mathbf{DK}^I$. Initial sequents of $\mathbf{DK}^I$ are of the following forms:

$$; A \to A,$$
$$; \bot \to \ .$$

Rules of inference of $\mathbf{DK}^I$ are the following.

Structural rules:

$$\frac{\Gamma; \Delta \to C}{A, \Gamma; \Delta \to C}\,(\mathrm{w}\,; \to) \qquad \frac{\Gamma; \Delta \to C}{\Gamma; A, \Delta \to C}\,(; \mathrm{w} \to) \qquad \frac{\Gamma; \Delta \to}{\Gamma; \Delta \to A}\,(\to \mathrm{w})$$

$$\frac{A, A, \Gamma; \Delta \to C}{A, \Gamma; \Delta \to C}\ (\mathrm{c}\,;\to) \qquad \frac{\Gamma; A, A, \Delta \to C}{\Gamma; A, \Delta \to C}\ (;\mathrm{c}\to)$$

$$\frac{\Gamma, A, B, \Pi; \Delta \to C}{\Gamma, B, A, \Pi; \Delta \to C}\ (\mathrm{e}\,;\to) \qquad \frac{\Gamma; \Delta, A, B, \Sigma \to C}{\Gamma; \Delta, B, A, \Sigma \to C}\ (;\mathrm{e}\to)$$

Cut rule:

$$\frac{\Gamma; \Delta \to A \quad \Pi; A, \Sigma \to C}{\Gamma, \Pi; \Delta, \Sigma \to C}\ (\mathrm{cut})$$

Rules for the logical connectives:

$$\frac{\Gamma; A, \Delta \to C}{\Gamma; A \wedge B, \Delta \to C}\ (\wedge 1 \to) \qquad \frac{\Gamma; B, \Delta \to C}{\Gamma; A \wedge B, \Delta \to C}\ (\wedge 2 \to)$$

$$\frac{\Gamma; \Delta \to A \quad \Gamma; \Delta \to B}{\Gamma; \Delta \to A \wedge B}\ (\to \wedge)$$

$$\frac{\Gamma; A, \Delta \to C \quad \Gamma; B, \Delta \to C}{\Gamma; A \vee B, \Delta \to C}\ (\vee \to)$$

$$\frac{\Gamma; \Delta \to A}{\Gamma; \Delta \to A \vee B}\ (\to \vee 1) \qquad \frac{\Gamma; \Delta \to B}{\Gamma; \Delta \to A \vee B}\ (\to \vee 2)$$

$$\frac{\Gamma; \Delta \to A \quad \Pi; B, \Sigma \to C}{\Gamma, A \supset B, \Pi; \Delta, \Sigma \to C}\ (\supset\to) \qquad \frac{\Gamma; A \to B}{; \Gamma \to A \supset B}\ (\to\supset)$$

The formula C in each of these rules may be empty. The formulas A, B in the structural rules and $A \wedge B$, $A \vee B$, $A \supset B$ in the rules for the logical connectives are called the *principal formulas* of the respective rules.

Theorem 3.19 (Soundness of $\mathbf{DK}^I$) *For any sequent $\Gamma; \Delta \to A$, if $\Gamma; \Delta \to A$ is provable in $\mathbf{DK}^I$, then $\Gamma; \Delta \to A$ is true in every model.*

Proof. By induction on the proof of $\Gamma; \Delta \to A$ in $\mathbf{DK}^I$. Here we consider only the rules $(\supset\to)$ and $(\to\supset)$. Take any model $\langle W, R, V \rangle$ and any $x \in W$. For the rule $(\supset\to)$, we show $x \models \Gamma, A \supset B, \Pi; \Delta, \Sigma \to C$. Suppose $x \models \Gamma, A \supset B, \Pi$ and xRy and $y \models \Delta, \Sigma$. Since $x \models \Gamma; \Delta \to A$ by the induction hypothesis, we have $y \models A$, and since $x \models A \supset B$, we have $y \models B$. Moreover, since $x \models \Pi; B, \Sigma \to C$ by the induction hypothesis, we have $y \models C$ as required. For the rule $(\to\supset)$, we show $x \models\, ; \Gamma \to A \supset B$. Suppose xRy and $y \models \Gamma$. Since $y \models \Gamma; A \to B$ by the induction hypothesis, we have that yRz and $z \models A$ imply $z \models B$. This means $y \models A \supset B$. Therefore we have $x \models\, ; \Gamma \to A \supset B$. ∎

To prove the completeness theorem of $\mathbf{DK}^I$, we introduce the following notions.

Definition 3.20 Let x, y, z be sets of formulas. A pair (y, z) is *x-consistent* (in $\mathbf{DK}^I$) if, for any $A_1, \ldots, A_l \in x$, any $B_1, \ldots, B_m \in y$, and any $C_1, \ldots, C_n \in z$ $(l, m, n \geq 0)$,

$$A_1, \ldots, A_l; B_1, \ldots, B_m \to C_1 \vee \cdots \vee C_n$$

is not provable in $\mathbf{DK}^I$. (y, z) is *maximal x-consistent* (in $\mathbf{DK}^I$) if it is x-consistent and for any formula A, $A \in y$ or $A \in z$.

Lemma 3.21 *Let x, y, z be sets of formulas. If (y, z) is x-consistent, then there exists a maximal x-consistent pair (y', z') such that $y \subseteq y'$ and $z \subseteq z'$.*

Proof. Let $D_1, D_2, \ldots$ be an enumeration of all formulas. We define a sequence of pairs (y_n, z_n) $(n = 0, 1, \ldots)$ as follows:

$$\begin{aligned} (y_0, z_0) &= (y, z), \\ (y_{m+1}, z_{m+1}) &= \begin{cases} (y_m, z_m \cup \{D_{m+1}\}) & \text{if } (y_m, z_m \cup \{D_{m+1}\}) \text{ is } x\text{-consistent}, \\ (y_m \cup \{D_{m+1}\}, z_m) & \text{otherwise.} \end{cases} \end{aligned}$$

Then (y_{m+1}, z_{m+1}) is x-consistent whenever (y_m, z_m) is x-consistent. Indeed, otherwise there exist formulas $A_1, \ldots, A_i \in x$, $B_1, \ldots, B_j \in y_m$ and $C_1, \ldots, C_k \in z_m$ such that

$$\begin{gathered} A_1, \ldots, A_i; B_1, \ldots, B_j \to C_1 \vee \cdots \vee C_k \vee D_{m+1}, \\ A_1, \ldots, A_i; B_1, \ldots, B_j, D_{m+1} \to C_1 \vee \cdots \vee C_k \end{gathered}$$

are both provable in $\mathbf{DK}^I$. However, by using the cut rule,

$$A_1, \ldots, A_i; B_1, \ldots, B_j \to C_1 \vee \cdots \vee C_k$$

is provable in $\mathbf{DK}^I$, contrary to the x-consistency of (y_m, z_m). Thus (y_n, z_n) is x-consistent for all n, and we obtain a maximal x-consistent pair $(\bigcup_{n=0}^{\infty} y_n, \bigcup_{n=0}^{\infty} z_n)$. ∎

Theorem 3.22 (Completeness of $\mathbf{DK}^I$) *For any formula A, $; \to A$ is provable in $\mathbf{DK}^I$ if and only if A is true in every model.*

Proof. The implication from left to right is immediate by Theorem 3.19 and Proposition 3.18. For the other direction, suppose that $; \to A$ is not provable in $\mathbf{DK}^I$. Then the pair $(\emptyset, \{A\})$ is $\emptyset$-consistent, and so by Lemma 3.21, there exists a maximal $\emptyset$-consistent pair (u, v) such that $\{A\} \subseteq v$ (so $A \notin u$ by the $\emptyset$-consistency of (u, v)). Now define a model $\langle W^*, R^*, V^* \rangle$ as follows:

- W^* is the set of all maximal $\emptyset$-consistent pairs,
- $(x, w) R^* (y, z)$ iff (y, z) is x-consistent,
- $V^*(p) = \{(x, w) \in W^* \mid p \in x\}$ for every $p \in \mathcal{PV}$.

Our aim is to show $(u, v) \not\models A$, which means that A is not true in this model. For this purpose, we prove by induction on the structure of B that for any $(x, w) \in W^*$,

$$B \in x \quad \text{if and only if} \quad (x, w) \models B.$$

1. B is a propositional variable p. Follows from the definitions of V^* and $\models$.

2. B is the propositional constant $\bot$. $\bot \in x$ is impossible by the $\emptyset$-consistency of (x, w).

3. B is of the form $C \wedge D$. Suppose $C \wedge D \in x$. Since $; C \wedge D \to C$ is provable in $\mathbf{DK}^I$, $C \in w$ contradicts the $\emptyset$-consistency of (x, w). Hence $C \in x$. Similarly, $D \in x$. By the induction hypothesis, $(x, w) \models C$ and $(x, w) \models D$. Thus $(x, w) \models C \wedge D$. For the other direction, suppose $(x, w) \models C \wedge D$, i.e., $(x, w) \models C$ and $(x, w) \models D$. Then $C \in x$ and $D \in x$ by the induction hypothesis. Since $; C, D \to C \wedge D$ is provable in $\mathbf{DK}^I$, $C \wedge D \in w$ contradicts the $\emptyset$-consistency of (x, w). Hence $C \wedge D \in x$.

4. B is of the form $C \vee D$. Suppose $C \vee D \in x$. Since $; C \vee D \to C \vee D$ is provable in $\mathbf{DK}^I$, $C, D \in w$ contradicts the $\emptyset$-consistency of (x, w). Hence $C \notin w$ or $D \notin w$, and so $C \in x$ or $D \in x$. By the induction hypothesis, $(x, w) \models C$ or $(x, w) \models D$. Thus $(x, w) \models C \vee D$. For the other direction, suppose $(x, w) \models C \vee D$. Then $(x, w) \models C$ or $(x, w) \models D$, and by the induction hypothesis, $C \in x$ or $D \in x$. Since $; C \to C \vee D$ and $; D \to C \vee D$ are provable in $\mathbf{DK}^I$, $C \vee D \in w$ contradicts the consistency of (x, w). Hence $C \vee D \in x$.

5. B is of the form $C \supset D$. Suppose $C \supset D \in x$. To show $(x, w) \models C \supset D$, suppose further $(x, w) R^* (y, z)$ and $(y, z) \models C$. Then (y, z) is x-consistent by the definition of R^*, and $C \in y$ holds by the induction hypothesis. Since $C \supset D; C \to D$ is provable in $\mathbf{DK}^I$, $D \in z$ contradicts the x-consistency of (y, z). Hence $D \in y$ holds, and by the induction hypothesis, we have $(y, z) \models D$ as required. For the other direction, suppose $C \supset D \notin x$, i.e., $C \supset D \in w$. Then the pair $(\{C\}, \{D\})$ is x-consistent. Indeed, otherwise $\Gamma; C \to D$ is provable in $\mathbf{DK}^I$ for some Γ consisting of formulas in x. Then $; \Gamma \to C \supset D$ is also provable in $\mathbf{DK}^I$ contrary to the $\emptyset$-consistency of (x, w). Hence $(\{C\}, \{D\})$ is x-consistent, and by Lemma 3.21, there exists a maximal x-consistent pair (y, z) such that $\{C\} \subseteq y$ and $\{D\} \subseteq z$ (so $D \notin y$ by the x-consistency of (y, z)). Now we have $(x, w) R^* (y, z)$, $(y, z) \models C$ and $(y, z) \not\models D$ by the induction hypothesis. This means $(x, w) \not\models C \supset D$.

■

3.5 Dual-context sequent calculus for BPC

It is possible to introduce a dual-context sequent calculus also for BPC. In this section we consider a modification of the system $\mathbf{DK}^I$ and show that it is complete with respect to the class of Kripke models for BPC. The modified system is called **LBP2** after **LBP** in Section 3.3.

Sequents of the system **LBP2** and their interpretation in Kripke models are defined in the same way as those of $\mathbf{DK}^I$. The sequent calculus **LBP2** is obtained from $\mathbf{DK}^I$ by replacing the rule $(\to\supset)$ by the following one.

$$\frac{\Gamma; \Delta, A \to B}{; \Gamma, \Delta \to A \supset B} \ (\to\supset)$$

Proposition 3.23 *For any formula A, A is true in every* BPC*-model if and only if* $; \to A$ *is true in every* BPC*-model.*

PROOF. The implication from left to right is straightforward. For the other direction, suppose that A is not true in a BPC-model $M = \langle W, R, V\rangle$, i.e., $x \not\models_M A$ for some $x \in W$. Then consider a model $M' = \langle W', R', V\rangle$ where $W' = W \cup \{y\}$, $y \notin W$ and $R' = R \cup \{(y, x)\} \cup \{(y, z) \mid xRz\}$. It is easy to see that R' is transitive, and so M' is a BPC-model. Then we can show by induction on the structure of B that for any $w \in W$, $w \models_M B$ if and only if $w \models_{M'} B$. Since $x \in W$ and $x \not\models_M A$, we have $x \not\models_{M'} A$. Hence $y \not\models_{M'} ; \to A$, which means that $; \to A$ is not true in M'. ∎

Theorem 3.24 (Soundness of LBP2) *For any sequent* $\Gamma; \Delta \to A$*, if* $\Gamma; \Delta \to A$ *is provable in* **LBP2***, then* $\Gamma; \Delta \to A$ *is true in every* BPC*-model.*

PROOF. By induction on the proof of $\Gamma; \Delta \to A$ in **LBP2**. We consider only the rule ($\to\supset$). Take any BPC-model $\langle W, R, V\rangle$ and any $x \in W$. To show $x \models ; \Gamma, \Delta \to A \supset B$, suppose that xRy and $y \models \Gamma, \Delta$. Then by Lemma 3.3, yRz implies $z \models \Delta$. Since $y \models \Gamma; \Delta, A \to B$ by the induction hypothesis, we have that yRz and $z \models A$ imply $z \models B$. This means $y \models A \supset B$. Hence we have $x \models ; \Gamma, \Delta \to A \supset B$. ∎

To prove the completeness theorem of **LBP2**, we define the notions of x-consistent and maximal x-consistent pairs (in **LBP2**) in the same way as in Definition 3.20.

Lemma 3.25 *Let x, y, z be sets of formulas. If (y, z) is x-consistent, then there exists a maximal x-consistent pair (y', z') such that $y \subseteq y'$ and $z \subseteq z'$.*

PROOF. Similar to the proof of Lemma 3.21. ∎

Theorem 3.26 (Completeness of LBP2) *For any formula A,* $; \to A$ *is provable in* **LBP2** *if and only if A is true in every* BPC*-model.*

PROOF. The implication from left to right is immediate by Theorem 3.24 and Proposition 3.23. For the other direction, suppose that $; \to A$ is not provable in **LBP2**. Then the pair $(\emptyset, \{A\})$ is $\emptyset$-consistent, and so by Lemma 3.25, there exists a maximal $\emptyset$-consistent pair (u, v) such that $\{A\} \subseteq v$ (so $A \notin u$ by the $\emptyset$-consistency of (u, v)). Now define a BPC-model $\langle W^*, R^*, V^*\rangle$ as follows:

- W^* is the set of all maximal $\emptyset$-consistent pairs,
- $(x, w)R^*(y, z)$ iff $x \subseteq y$ and (y, z) is x-consistent,
- $V^*(p) = \{(x, w) \in W^* \mid p \in x\}$ for every $p \in \mathcal{PV}$.

It is easy to verify that R^* is transitive and that $(x, w) \in V^*(p)$ and $(x, w)R^*(y, z)$ imply $(y, z) \in V^*(p)$. Thus $\langle W^*, R^*, V^*\rangle$ is indeed a BPC-model. Our aim is to show $(u, v) \not\models A$, which means that A is not true in this BPC-model. To this end, we prove by induction on the structure of B that for any $(x, w) \in W^*$,

$$B \in x \quad \text{if and only if} \quad (x, w) \models B.$$

Here we consider only the case where B is of the form $C \supset D$. The other cases are proved in the same way as those in the proof of Theorem 3.22. Suppose $C \supset D \in x$. To show $(x, w) \models C \supset D$, suppose further $(x, w)R^*(y, z)$ and $(y, z) \models C$. Then (y, z) is x-consistent by the definition of R^*, and $C \in y$ holds by the induction hypothesis. Since $C \supset D; C \to D$ is provable in **LBP2**, $D \in z$ contradicts the x-consistency of (y, z). So $D \in y$ holds, and by the induction hypothesis, we have $(y, z) \models D$ as required. For the other direction, suppose $C \supset D \notin x$, i.e., $C \supset D \in w$. Then the pair $(x \cup \{C\}, \{D\})$ is x-consistent. Indeed, otherwise $\Gamma; \Delta, C \to D$ is provable in **LBP2** for some Γ and Δ consisting of formulas in x. Then $; \Gamma, \Delta \to C \supset D$ is also provable in **LBP2** contrary to the $\emptyset$-consistency of (x, w). Hence $(x \cup \{C\}, \{D\})$ is x-consistent, and by Lemma 3.25, there exists a maximal x-consistent pair (y, z) such that $x \cup \{C\} \subseteq y$ and $\{D\} \subseteq z$ (so $D \notin y$ by the x-consistency of (y, z)). Now we have $(x, w)R^*(y, z)$, $(y, z) \models C$ and $(y, z) \not\models D$ by the induction hypothesis. This means $(x, w) \not\models C \supset D$. ∎

3.6 Cut-elimination theorem for $\mathbf{DK}^I$

In Section 3.4 we proved the completeness theorem of $\mathbf{DK}^I$, using the cut rule. The present section is devoted to proving the cut-elimination theorem for $\mathbf{DK}^I$ in a syntactical way. Our proof of cut-elimination for $\mathbf{DK}^I$ includes more global proof transformation than the ordinary syntactical proof of cut-elimination for other systems.

First we introduce the notion of height of a proof in $\mathbf{DK}^I$.

Definition 3.27 The *height* $h(P)$ of a proof P in $\mathbf{DK}^I$ is defined inductively as follows:

1. If P is an initial sequent, then $h(P) = 1$.
2. If P is obtained from the proof Q by applying a one-premiss rule, then $h(P) = h(Q) + 1$.
3. If P is obtained from the proofs Q_1 and Q_2 by applying a two-premiss rule, then $h(P) = \max\{h(Q_1), h(Q_2)\} + 1$.

Now we prove the cut-elimination theorem for $\mathbf{DK}^I$. In the following proof, the main difference from the proof of cut-elimination for **LJ** arises in Case 4.

Theorem 3.28 (Cut-elimination for $\mathbf{DK}^I$) *For any sequent $\Gamma; \Delta \to A$, if $\Gamma; \Delta \to A$ is provable in $\mathbf{DK}^I$, then it is provable in $\mathbf{DK}^I$ without using the cut rule.*

PROOF. We introduce the following mix rule:

$$\frac{\Gamma; \Delta \to A \quad \Pi; \Sigma \to C}{\Gamma, \Pi; \Delta, \Sigma_A \to C} \text{ (mix)}$$

where Σ has at least one occurrence of A, and Σ_A is obtained from Σ by deleting all occurrences of A. The formula A is called the *mix formula* of this inference. It is seen that the cut rule

$$\frac{\Gamma; \Delta \to A \quad \Pi; A, \Sigma \to C}{\Gamma, \Pi; \Delta, \Sigma \to C} \text{ (cut)}$$

is derivable from the mix rule and some structural rules as follows.

$$\dfrac{\dfrac{\Gamma; \Delta \to A \quad \Pi; A, \Sigma \to C}{\Gamma, \Pi; \Delta, \Sigma_A \to C}\,(\text{mix})}{\Gamma, \Pi; \Delta, \Sigma \to C}$$

Thus, for the proof of the theorem, it suffices to consider the system with the mix rule instead of the cut rule and show mix-elimination instead of cut-elimination. Our strategy is that of eliminating each mix rule above which any other mix rule does not occur.

Let P be a proof with only one mix rule occurring as the last inference whose mix formula is A. Let P_{l} and P_{r} be the subproofs of P whose end-sequents are the left and right premisses of the mix rule, respectively. The proof of eliminating the mix rule in P is by induction on $|A|$, with a subinduction on $h(P_{\mathrm{l}}) + h(P_{\mathrm{r}})$. Let r_{l} and r_{r} be the left and right rules over the mix rule, respectively. We consider the following four cases:

1. At least one of P_{l} and P_{r} is an initial sequent.
2. Neither of P_{l} and P_{r} is an initial sequent, and the mix formula is principal in both r_{l} and r_{r}.
3. The mix formula is not principal in r_{l}.
4. The mix formula is principal in r_{l} and not principal in r_{r}.

Case 1. One of the premisses is an initial sequent $; A \to A$, say the left.

$$\dfrac{; A \to A \quad \Pi; \Sigma \to C}{\Pi; A, \Sigma_A \to C}\,(\text{mix})$$

The conclusion follows from the right premiss by some structural rules.

The right premiss is the initial sequent $; \bot \to$. If r_{l} is $(\to \mathrm{w})$, the proof looks like

$$\dfrac{\dfrac{\Gamma; \Delta \to}{\Gamma; \Delta \to \bot}\,(\to \mathrm{w}) \qquad ; \bot \to}{\Gamma; \Delta \to}\,(\text{mix})$$

The conclusion of (mix) is the same as the premiss of $(\to \mathrm{w})$. The case where r_{l} is one of the other rules is handled as in Case 3 below.

Case 2. Neither of the premisses is an initial sequent, and the mix formula is principal in both r_{l} and r_{r}. Note in this case that r_{l} and r_{r} are not $(\to\supset)$ and $(\supset\to)$, because the mix formula cannot be principal in $(\supset\to)$. Here we consider the case where r_{l} and r_{r} are $(\to \wedge)$ and $(\wedge 1 \to)$.

$$\dfrac{\dfrac{\overset{\vdots\, P_0}{\Gamma; \Delta \to A} \quad \overset{\vdots\, P_1}{\Gamma; \Delta \to B}}{\Gamma; \Delta \to A \wedge B}\,(\to \wedge) \qquad \dfrac{\overset{\vdots\, P_2}{\Pi; A, \Sigma \to C}}{\Pi; A \wedge B, \Sigma \to C}\,(\wedge 1 \to)}{\Gamma, \Pi; \Delta, \Sigma_{A \wedge B} \to C}\,(\text{mix})$$

If $A \wedge B$ does not occur in Σ, i.e., $\Sigma_{A \wedge B}$ is the same sequence as Σ, then we construct the following proof.

$$\dfrac{\dfrac{\overset{\vdots\, P_0}{\Gamma; \Delta \to A} \quad \overset{\vdots\, P_2}{\Pi; A, \Sigma \to C}}{\Gamma, \Pi; \Delta, \Sigma_A \to C}\,(\text{mix})}{\Gamma, \Pi; \Delta, \Sigma \to C}$$

This (mix) can be eliminated by the induction hypothesis. On the other hand, if $A \wedge B$ occurs in Σ, then we construct the following proof.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, P_0 \\ \Gamma; \Delta \to A\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\, P_0 \\ \Gamma; \Delta \to A\end{array} \quad \begin{array}{c}\vdots\, P_1 \\ \Gamma; \Delta \to B\end{array}}{\Gamma; \Delta \to A \wedge B}(\to \wedge) \qquad \begin{array}{c}\vdots\, P_2 \\ \Pi; A, \Sigma \to C\end{array}}{\Gamma, \Pi; \Delta, A, \Sigma_{A \wedge B} \to C}(\text{mix})}{\Gamma, \Gamma, \Pi; \Delta, \Delta_A, (\Sigma_{A\wedge B})_A \to C}(\text{mix})}{\Gamma, \Pi; \Delta, \Sigma_{A \wedge B} \to C}$$

The upper (mix) can be eliminated by the subinduction hypothesis, and then the lower (mix) can be eliminated by the induction hypothesis.

Case 3. Neither of the premisses is an initial sequent, and the mix formula is not principal in r_l. Here we consider the case where r_l is $(\supset\to)$.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, P_0 \\ \Gamma; \Delta \to A\end{array} \quad \begin{array}{c}\vdots\, P_1 \\ \Phi; B, \Lambda \to C\end{array}}{\Gamma, A \supset B, \Phi; \Delta, \Lambda \to C}(\supset\to) \qquad \begin{array}{c}\vdots\, P_2 \\ \Pi; \Sigma \to D\end{array}}{\Gamma, A \supset B, \Phi, \Pi; \Delta, \Lambda, \Sigma_C \to D}(\text{mix})$$

This is transformed into

$$\dfrac{\begin{array}{c}\vdots\, P_0 \\ \Gamma; \Delta \to A\end{array} \qquad \dfrac{\begin{array}{c}\vdots\, P_1 \\ \Phi; B, \Lambda \to C\end{array} \quad \begin{array}{c}\vdots\, P_2 \\ \Pi; \Sigma \to D\end{array}}{\Phi, \Pi; B, \Lambda, \Sigma_C \to D}(\text{mix})}{\Gamma, A \supset B, \Phi, \Pi; \Delta, \Lambda, \Sigma_C \to D}(\supset\to)$$

where the (mix) can be eliminated by the subinduction hypothesis.

Case 4. The mix formula is principal in r_l and not principal in r_r. The problematic cases are those where r_r is $(\to\supset)$. The other cases are proved in the usual way. Here we consider a few subcases of those problematic cases.

Subcase 4.1. r_l is $(\to \wedge)$ and r_r is $(\to\supset)$.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, P_0 \\ \Gamma; \Delta \to A\end{array} \quad \begin{array}{c}\vdots\, P_1 \\ \Gamma; \Delta \to B\end{array}}{\Gamma; \Delta \to A \wedge B}(\to \wedge) \qquad \dfrac{\begin{array}{c}\vdots\, P_2 \\ \Sigma; C \to D\end{array}}{; \Sigma \to C \supset D}(\to\supset)}{\Gamma; \Delta, \Sigma_{A \wedge B} \to C \supset D}(\text{mix})$$

In this case, each $A \wedge B$ in Σ must be introduced by $(\mathrm{w}\,;\to)$ in the proof P_2. Giving up all such $(\mathrm{w}\,;\to)$'s, we obtain the following proof.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, P_2' \\ \Sigma_{A \wedge B}; C \to D\end{array}}{; \Sigma_{A \wedge B} \to C \supset D}(\to\supset)}{\Gamma; \Delta, \Sigma_{A \wedge B} \to C \supset D}$$

Subcase 4.2. r_l is $(\to\supset)$ and r_r is $(\to\supset)$.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, P_1\\ \Delta; A \to B\end{array}}{;\Delta \to A \supset B}\,(\to\supset) \qquad \dfrac{\begin{array}{c}\vdots\, P_2\\ \Sigma; C \to D\end{array}}{;\Sigma \to C \supset D}\,(\to\supset)}{;\Delta, \Sigma_{A\supset B} \to C \supset D}\,(\text{mix})$$

In this case, $A \supset B$ in Σ may be introduced by $(\supset\to)$ in the proof P_2.

$$\begin{array}{c}\dfrac{\begin{array}{c}\vdots\, Q_1\\ \Phi; \Lambda \to A\end{array} \quad \begin{array}{c}\vdots\, Q_2\\ \Psi; B, \Theta \to E\end{array}}{\Phi, A \supset B, \Psi; \Lambda, \Theta \to E}\,(\supset\to)\\ \vdots\, P_2\\ \dfrac{\Sigma; C \to D}{;\Sigma \to C \supset D}\,(\to\supset)\end{array}$$

Now consider the following proof.

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\, Q_1\\ \Phi; \Lambda \to A\end{array} \quad \begin{array}{c}\vdots\, P_1\\ \Delta; A \to B\end{array}}{\Phi, \Delta; \Lambda \to B}\,(\text{mix}) \qquad \begin{array}{c}\vdots\, Q_2\\ \Psi; B, \Theta \to E\end{array}}{\Phi, \Delta, \Psi; \Lambda, \Theta_B \to E}\,(\text{mix})}{\Phi, \Delta, \Psi; \Lambda, \Theta \to E}$$

These two (mix)'s can be eliminated by the induction hypothesis. Note that Δ in the end-sequent $\Phi, \Delta, \Psi; \Lambda, \Theta \to E$ replaces $A \supset B$ in the sequent $\Phi, A \supset B, \Psi; \Lambda, \Theta \to E$. On the other hand, when $A \supset B$ is introduced by $(\text{w}\,;\to)$ as in Subcase 4.1, we introduce Δ by $(\text{w}\,;\to)$'s instead of $A \supset B$. Replacing these kinds of $(A \supset B)$'s in P_2 by Δ's, we can construct a proof of $\Delta, \Sigma_{A\supset B}; C \to D$, and obtain a proof of $;\Delta, \Sigma_{A\supset B} \to C \supset D$.

■

3.7 Cut-elimination theorem for LBP2

In this section we prove the cut-elimination theorem for **LBP2** in a syntactical way. The proof is more embarrassing than that for $\mathbf{DK}^I$ in the previous section, since it involves the so-called inversion lemma. We will change the way of case-splitting before in the proof of cut-elimination for **LBP2**.

First we introduce the notion of height of a proof in **LBP2**.

Definition 3.29 The *height* $h(P)$ of a proof P in **LBP2** is defined inductively as follows:

1. If P is an initial sequent, then $h(P) = 1$.

2. If P is obtained from the proof Q by applying a one-premiss rule, then $h(P) = h(Q) + 1$.

3. If P is obtained from the proofs Q_1 and Q_2 by applying a two-premiss rule, then $h(P) = \max\{h(Q_1), h(Q_2)\} + 1$.

Next we prove the following lemma, which is needed in the proof of the cut-elimination theorem for **LBP2**.

Lemma 3.30 *Suppose that a sequent* $\Gamma; \Delta \to C$ *is provable in* **LBP2** *without using the cut rule. Then the sequents* $\Gamma; A, B, \Delta_{A \wedge B} \to C$, $\Gamma; A, \Delta_{A \vee B} \to C$ *and* $\Gamma; B, \Delta_{A \vee B} \to C$ *are provable in* **LBP2** *without using the cut rule, where* $\Delta_{A \wedge B}$ *and* $\Delta_{A \vee B}$ *are the sequences obtained from* Δ *by deleting all occurrences of* $A \wedge B$ *and* $A \vee B$*, respectively.*

PROOF. By induction on the height of the proof of $\Gamma; \Delta \to C$. Here we consider only the case where the last rule applied is $(\to\supset)$.

$$\frac{\Gamma; \Delta, D \to E}{; \Gamma, \Delta \to D \supset E} \ (\to\supset)$$

By the induction hypothesis, $\Gamma; A, B, \Delta_{A \wedge B}, D \to E$ is provable without using the cut rule. In the resulting proof, each $A \wedge B$ in Γ must be introduced by $(\mathrm{w}\,; \to)$. Giving up all such $(\mathrm{w}\,; \to)$'s, we can construct the following proof.

$$\cfrac{\cfrac{\vdots}{\Gamma_{A \wedge B}; A, B, \Delta_{A \wedge B}, D \to E}}{\cfrac{; \Gamma_{A \wedge B}, A, B, \Delta_{A \wedge B} \to D \supset E}{; A, B, \Gamma_{A \wedge B}, \Delta_{A \wedge B} \to D \supset E}} \ (\to\supset)$$

The sequents $; A, \Gamma_{A \vee B}, \Delta_{A \vee B} \to D \supset E$ and $; B, \Gamma_{A \vee B}, \Delta_{A \vee B} \to D \supset E$ are proved in analogous ways to the above. ∎

Now we prove the cut-elimination theorem for **LBP2**.

Theorem 3.31 (Cut-elimination for LBP2) *For any sequent* $\Gamma; \Delta \to A$*, if* $\Gamma; \Delta \to A$ *is provable in* **LBP2***, then it is provable in* **LBP2** *without using the cut rule.*

PROOF. We introduce the following mix rule:

$$\frac{\Gamma; \Delta \to A \quad \Pi; \Sigma \to C}{\Gamma, \Pi; \Delta, \Sigma_A \to C} \ (\text{mix})$$

where Σ has at least one occurrence of A, and Σ_A is obtained from Σ by deleting all occurrences of A. The formula A is called the *mix formula* of this inference. It is seen that the cut rule

$$\frac{\Gamma; \Delta \to A \quad \Pi; A, \Sigma \to C}{\Gamma, \Pi; \Delta, \Sigma \to C} \ (\text{cut})$$

is derivable from the mix rule and some structural rules as follows.

$$\cfrac{\cfrac{\Gamma; \Delta \to A \quad \Pi; A, \Sigma \to C}{\Gamma, \Pi; \Delta, \Sigma_A \to C} \ (\text{mix})}{\Gamma, \Pi; \Delta, \Sigma \to C}$$

Thus, for the proof of the theorem, it suffices to consider the system with the mix rule instead of the cut rule and show mix-elimination instead of cut-elimination. Our strategy is that of eliminating each mix rule above which any other mix rule does not occur.

Let P be a proof with only one mix rule occurring as the last inference whose mix formula is A. Let P_{l} and P_{r} be the subproofs of P whose end-sequents are the left and right premisses of the mix rule, respectively. The proof of eliminating the mix rule in P is by induction on $|A|$, with a subinduction on $h(P_{\mathrm{l}})+h(P_{\mathrm{r}})$. Let r_{l} and r_{r} be the left and right rules over the mix rule, respectively. We consider the following three cases:

1. At least one of P_{l} and P_{r} is an initial sequent.

2. Neither of P_{l} and P_{r} is an initial sequent, and the mix formula is not principal in r_{l}.

3. The mix formula is principal in r_{l}.

Case 1. One of the premisses is an initial sequent $;A \to A$, say the left.

$$\dfrac{;A \to A \quad \Pi; \Sigma \to C}{\Pi; A, \Sigma_A \to C}\ (\mathrm{mix})$$

The conclusion follows from the right premiss by some structural rules.

The right premiss is the initial sequent $;\bot \to$. If r_{l} is $(\to \mathrm{w})$, the proof looks like

$$\dfrac{\dfrac{\Gamma; \Delta \to}{\Gamma; \Delta \to \bot}\ (\to \mathrm{w}) \qquad ;\bot \to}{\Gamma; \Delta \to}\ (\mathrm{mix})$$

The conclusion of (mix) is the same as the premiss of $(\to \mathrm{w})$. The case where r_{l} is one of the other rules is handled as in Case 2 below.

Case 2. Neither of the premisses is an initial sequent, and the mix formula is not principal in r_{l}. Here we consider the case where r_{l} is $(\supset\to)$.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, P_1\\ \Gamma; \Delta \to A\end{array} \quad \begin{array}{c}\vdots\, P_2\\ \Phi; B, \Lambda \to C\end{array}}{\Gamma, A \supset B, \Phi; \Delta, \Lambda \to C}\ (\supset\to) \qquad \begin{array}{c}\vdots\, P_{\mathrm{r}}\\ \Pi; \Sigma \to D\end{array}}{\Gamma, A \supset B, \Phi, \Pi; \Delta, \Lambda, \Sigma_C \to D}\ (\mathrm{mix})$$

This is transformed into

$$\dfrac{\begin{array}{c}\vdots\, P_1\\ \Gamma; \Delta \to A\end{array} \qquad \dfrac{\begin{array}{c}\vdots\, P_2\\ \Phi; B, \Lambda \to C\end{array} \quad \begin{array}{c}\vdots\, P_{\mathrm{r}}\\ \Pi; \Sigma \to D\end{array}}{\Phi, \Pi; B, \Lambda, \Sigma_C \to D}\ (\mathrm{mix})}{\Gamma, A \supset B, \Phi, \Pi; \Delta, \Lambda, \Sigma_C \to D}\ (\supset\to)$$

where the (mix) can be eliminated by the subinduction hypothesis.

Case 3. The mix formula is principal in r_{l}. The case where r_{l} is $(\to \mathrm{w})$ is easy. We assume that r_{l} is one of $(\to \wedge)$, $(\to \vee 1)$, $(\to \vee 2)$ and $(\to\supset)$.

Subcase 3.1. r_{l} is $(\to \wedge)$.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\, P_1\\ \Gamma; \Delta \to A\end{array} \quad \begin{array}{c}\vdots\, P_2\\ \Gamma; \Delta \to B\end{array}}{\Gamma; \Delta \to A \wedge B}\ (\to \wedge) \qquad \begin{array}{c}\vdots\, P_{\mathrm{r}}\\ \Pi; \Sigma \to C\end{array}}{\Gamma, \Pi; \Delta, \Sigma_{A \wedge B} \to C}\ (\mathrm{mix})$$

Applying Lemma 3.30 to the proof P_r, we can construct a proof P'_r of $\Pi; A, B, \Sigma_{A\wedge B} \to C$ without using the mix rule. Then we construct the following proof.

$$\dfrac{\dfrac{\overset{\vdots\; P_2}{\Gamma;\Delta\to B} \qquad \dfrac{\dfrac{\overset{\vdots\; P_1}{\Gamma;\Delta\to A} \quad \overset{\vdots\; P'_\mathrm{r}}{\Pi; A, B,\Sigma_{A\wedge B}\to C}}{\Gamma,\Pi;\Delta,(B,\Sigma_{A\wedge B})_A\to C}\,(\text{mix})}{\Gamma,\Pi;\Delta,B,\Sigma_{A\wedge B}\to C}}{\Gamma,\Gamma,\Pi;\Delta,\Delta_B,(\Sigma_{A\wedge B})_B\to C}\,(\text{mix})}{\Gamma,\Pi;\Delta,\Sigma_{A\wedge B}\to C}$$

These two (mix)'s can be eliminated by the induction hypothesis. (Except for Lemma 3.30 we would have got into trouble when r_r is $(\to\supset)$.)

Subcase 3.2. r_l is $(\to\vee 1)$.

$$\dfrac{\dfrac{\overset{\vdots\; P_1}{\Gamma;\Delta\to A}}{\Gamma;\Delta\to A\vee B}\,(\to\vee 1) \qquad \overset{\vdots\; P_\mathrm{r}}{\Pi;\Sigma\to C}}{\Gamma,\Pi;\Delta,\Sigma_{A\vee B}\to C}\,(\text{mix})$$

Applying Lemma 3.30 to the proof P_r, we can construct a proof P'_r of $\Pi; A, \Sigma_{A\vee B} \to C$ without using the mix rule. Then we construct the following proof.

$$\dfrac{\dfrac{\overset{\vdots\; P_1}{\Gamma;\Delta\to A} \quad \overset{\vdots\; P'_\mathrm{r}}{\Pi; A,\Sigma_{A\vee B}\to C}}{\Gamma,\Pi;\Delta,(\Sigma_{A\vee B})_A\to C}\,(\text{mix})}{\Gamma,\Pi;\Delta,\Sigma_{A\vee B}\to C}$$

This (mix) can be eliminated by the induction hypothesis.

Subcase 3.3. r_l is $(\to\vee 2)$. Similar to the previous case.

Subcase 3.4. r_l is $(\to\supset)$. We consider only the case where r_r is $(\to\supset)$.

$$\dfrac{\dfrac{\overset{\vdots\; P_1}{\Gamma;\Delta, A\to B}}{;\Gamma,\Delta\to A\supset B}\,(\to\supset) \qquad \dfrac{\overset{\vdots\; P_2}{\Pi;\Sigma, C\to D}}{;\Pi,\Sigma\to C\supset D}\,(\to\supset)}{;\Gamma,\Delta,\Pi_{A\supset B},\Sigma_{A\supset B}\to C\supset D}\,(\text{mix})$$

Our first goal is to construct a proof of $\Pi;\Gamma,\Delta,\Sigma_{A\supset B}, C\to D$ without using the mix rule. If $A\supset B$ does not occur in Σ, i.e., $\Sigma_{A\supset B}$ is the same sequence as Σ, then the required sequent follows from the end-sequent of the proof P_2 by $(;\mathrm{w}\to)$. If $A\supset B$ occurs in Σ, then we construct the following proof.

$$\dfrac{\dfrac{\dfrac{\overset{\vdots\; P_1}{\Gamma;\Delta, A\to B}}{;\Gamma,\Delta\to A\supset B}\,(\to\supset) \qquad \overset{\vdots\; P_2}{\Pi;\Sigma, C\to D}}{\Pi;\Gamma,\Delta,(\Sigma, C)_{A\supset B}\to D}\,(\text{mix})}{\Pi;\Gamma,\Delta,\Sigma_{A\supset B}, C\to D}$$

This (mix) can be eliminated by the subinduction hypothesis. In the resulting proof of $\Pi; \Gamma, \Delta, \Sigma_{A \supset B}, C \to D$, each $A \supset B$ in Π must be introduced by either (w ; $\to$) or ($\supset\to$). Let us consider the latter case.

$$\dfrac{\begin{matrix} \vdots\, Q_1 & \vdots\, Q_2 \\ \Phi; \Lambda \to A & \Psi; B, \Theta \to E \end{matrix}}{\Phi, A \supset B, \Psi; \Lambda, \Theta \to E}\,(\supset\to)$$

$$\vdots$$

$$\Pi; \Gamma, \Delta, \Sigma_{A \supset B}, C \to D$$

We then consider the following proof.

$$\dfrac{\dfrac{\dfrac{\begin{matrix} \vdots\, Q_1 & \vdots\, P_1 \\ \Phi; \Lambda \to A & \Gamma; \Delta, A \to B \end{matrix}}{\Phi, \Gamma; \Lambda, \Delta_A \to B}\,(\text{mix}) \quad \begin{matrix} \vdots\, Q_2 \\ \Psi; B, \Theta \to E \end{matrix}}{\Phi, \Gamma, \Psi; \Lambda, \Delta_A, \Theta_B \to E}\,(\text{mix})}{\Phi, \Gamma, \Psi; \Lambda, \Delta, \Theta \to E}$$

These two (mix)'s can be eliminated by the induction hypothesis. Note that Γ and Δ in the end-sequent $\Phi, \Gamma, \Psi; \Lambda, \Delta, \Theta \to E$ replace $A \supset B$ in the sequent $\Phi, A \supset B, \Psi; \Lambda, \Theta \to E$. On the other hand, when $A \supset B$ in Π is introduced by (w ; $\to$), the conclusion $A \supset B, \Phi; \Lambda \to E$ of (w ; $\to$) may be replaced by $\Gamma, \Phi; \Delta, \Lambda \to E$ with (w ; $\to$) and (; w $\to$). Replacing these kinds of sequents as well as those below in the proof of $\Pi; \Gamma, \Delta, \Sigma_{A \supset B}, C \to D$, we can construct the following proof.

$$\dfrac{\dfrac{\begin{matrix} \vdots \\ \Gamma, \Pi_{A \supset B}; \Delta, \Gamma, \Delta, \Sigma_{A \supset B}, C \to D \end{matrix}}{; \Gamma, \Pi_{A \supset B}, \Delta, \Gamma, \Delta, \Sigma_{A \supset B} \to C \supset D}\,(\to\supset)}{; \Gamma, \Delta, \Pi_{A \supset B}, \Sigma_{A \supset B} \to C \supset D}$$

∎

3.8 Hilbert style systems for K^I and BPC

In this section we consider Hilbert style systems for K^I and BPC, and their relationships with the dual-context sequent calculi introduced in the previous sections. Hilbert style systems for subintuitionistic logics have been studied in [12], [14], [36], [41], [44]. Here we show a correspondence between Hilbert style systems and our sequent calculi, which yields the completeness of the Hilbert style systems with respect to the classes of Kripke models for K^I and for BPC.

In the following we will omit parentheses using the convention that $\wedge$ and $\vee$ bind more strongly than $\supset$. We will also use $\top$ as an abbreviation of $p \supset p$ for a fixed propositional variable p, and $\bigwedge \Gamma$ as the formula $(\cdots((A_1 \wedge A_2) \wedge A_3) \cdots \wedge A_{n-1}) \wedge A_n$ if Γ is a nonempty sequence $A_1, \ldots, A_n$, as $\top$ if Γ empty.

Now we introduce a Hilbert style system HK^I, which consists of the following axiom schemes:

$$
\begin{array}{ll}
\text{(A1)} & A \supset A, \\
\text{(A2)} & (A \supset B) \wedge (B \supset C) \supset (A \supset C), \\
\text{(A3)} & A \wedge B \supset A, \\
\text{(A4)} & A \wedge B \supset B, \\
\text{(A5)} & (A \supset B) \wedge (A \supset C) \supset (A \supset B \wedge C), \\
\text{(A6)} & A \supset A \vee B, \\
\text{(A7)} & B \supset A \vee B, \\
\text{(A8)} & (A \supset C) \wedge (B \supset C) \supset (A \vee B \supset C), \\
\text{(A9)} & A \wedge (B \vee C) \supset (A \wedge B) \vee (A \wedge C), \\
\text{(A10)} & \bot \supset A,
\end{array}
$$

and the following rules of inference:

$$
\frac{A \quad A \supset B}{B}\,(\mathrm{MP})\,, \qquad \frac{A}{B \supset A}\,(\mathrm{AF})\,, \qquad \frac{A \quad B}{A \wedge B}\,(\wedge\,\mathrm{I})\,.
$$

Lemma 3.32 *For any formula A, if A is provable in* HK^I*, then* $; \rightarrow A$ *is provable in* $\mathbf{DK}^I$*.*

PROOF. By induction on the proof of A in HK^I. It is straightforward to see that $; \rightarrow A$ is provable in $\mathbf{DK}^I$ for any axiom A of HK^I. For (MP), suppose that both $; \rightarrow A$ and $; \rightarrow A \supset B$ are provable in $\mathbf{DK}^I$. Then by the cut-elimination theorem, there is a cut-free proof of $; \rightarrow A \supset B$ in $\mathbf{DK}^I$. The last applied rule of the cut-free proof must be $(\rightarrow\supset)$, and so $; A \rightarrow B$ is provable in $\mathbf{DK}^I$. From this and $; \rightarrow A$, we obtain $; \rightarrow B$ using the cut rule in $\mathbf{DK}^I$. The cases of (AF) and $(\wedge\,\mathrm{I})$ are proved easily. ∎

Next we consider the converse of Lemma 3.32. One of the difficulties in establishing derivability in Hilbert style systems for subintuitionistic logics is caused by the lack of the deduction theorem. (For this notion, see, e.g. Theorem 1.12 of [11].) In the following we provide some derivable rules and formulas that facilitate inference in HK^I.

Lemma 3.33 *The following rules are derivable in* HK^I*.*

$$
\frac{A \supset B \quad B \supset C}{A \supset C}\,(\mathrm{Tr}) \qquad \frac{A \supset B \quad A \supset C}{A \supset B \wedge C}\,(\supset\wedge\,\mathrm{I})
$$

PROOF. For (Tr), we have the following proof.

$$
\dfrac{\dfrac{A \supset B \qquad B \supset C}{(A \supset B) \wedge (B \supset C)}\,(\wedge\,\mathrm{I}) \qquad \overset{\text{(A2)}}{(A \supset B) \wedge (B \supset C) \supset (A \supset C)}}{A \supset C}\,(\mathrm{MP})
$$

For $(\supset\wedge\,\mathrm{I})$, we have the following proof.

$$
\dfrac{\dfrac{A \supset B \qquad A \supset C}{(A \supset B) \wedge (A \supset C)}\,(\wedge\,\mathrm{I}) \qquad \overset{\text{(A5)}}{(A \supset B) \wedge (A \supset C) \supset (A \supset B \wedge C)}}{A \supset B \wedge C}\,(\mathrm{MP})
$$

∎

Lemma 3.34 *The following rules are derivable in* HK^I.

$$\frac{A \supset (B \supset C) \quad A \supset (C \supset D)}{A \supset (B \supset D)}\,(\supset \mathrm{Tr}) \qquad\qquad \frac{B \supset C \quad A \supset (C \supset D)}{A \supset (B \supset D)}\,(\mathrm{Tr}\,2)$$

Proof. For $(\supset \mathrm{Tr})$, we have the following proof.

$$\cfrac{\cfrac{A \supset (B \supset C) \quad A \supset (C \supset D)}{A \supset (B \supset C) \wedge (C \supset D)}\,(\supset\wedge\mathrm{I}) \qquad \overset{(\mathrm{A2})}{(B \supset C) \wedge (C \supset D) \supset (B \supset D)}}{A \supset (B \supset D)}\,(\mathrm{Tr})$$

For $(\mathrm{Tr}\,2)$, we have the following proof.

$$\cfrac{\cfrac{B \supset C}{A \supset (B \supset C)}\,(\mathrm{AF}) \qquad A \supset (C \supset D)}{A \supset (B \supset D)}\,(\supset \mathrm{Tr})$$

∎

Lemma 3.35 *The following rule is derivable in* HK^I.

$$\frac{C \supset (D \supset A)}{C \wedge (A \supset B) \supset (D \supset B)}$$

Proof. We have the following proof.

$$\cfrac{\cfrac{\overset{(\mathrm{A3})}{C \wedge (A \supset B) \supset C} \quad C \supset (D \supset A)}{C \wedge (A \supset B) \supset (D \supset A)}\,(\mathrm{Tr}) \qquad \overset{(\mathrm{A4})}{C \wedge (A \supset B) \supset (A \supset B)}}{C \wedge (A \supset B) \supset (D \supset B)}\,(\supset \mathrm{Tr})$$

∎

Lemma 3.36 *The formula* $(A \supset B) \supset (A \wedge C \supset B \wedge C)$ *is provable in* HK^I.

Proof. We have the following proof.

$$\cfrac{\cfrac{\overset{(\mathrm{A3})}{A \wedge C \supset A} \quad \overset{(\mathrm{A1})}{(A \supset B) \supset (A \supset B)}}{(A \supset B) \supset (A \wedge C \supset B)}\,(\mathrm{Tr}\,2) \qquad \cfrac{\overset{(\mathrm{A4})}{A \wedge C \supset C}}{(A \supset B) \supset (A \wedge C \supset C)}\,(\mathrm{AF})}{(A \supset B) \supset (A \wedge C \supset B) \wedge (A \wedge C \supset C)}\,(\supset\wedge\mathrm{I})$$

From this and $(A \wedge C \supset B) \wedge (A \wedge C \supset C) \supset (A \wedge C \supset B \wedge C)$, which is an instance of (A5), we obtain $(A \supset B) \supset (A \wedge C \supset B \wedge C)$ by (Tr). ∎

Lemma 3.37 *The formula* $(D \wedge A \supset C) \wedge (D \wedge B \supset C) \supset (D \wedge (A \vee B) \supset C)$ *is provable in* HK^I.

Proof. We derive the formula from $D \wedge (A \vee B) \supset (D \wedge A) \vee (D \wedge B)$, which is an instance of (A9), and $(D \wedge A \supset C) \wedge (D \wedge B \supset C) \supset ((D \wedge A) \vee (D \wedge B) \supset C)$, which is an instance of (A8), using $(\mathrm{Tr}\,2)$. ∎

Now we prove the converse of Lemma 3.32 in the following form.

Lemma 3.38 *If a sequent* $\Gamma; \Delta \to A$ *is provable in* $\mathbf{DK}^I$*, then* $\bigwedge \Gamma \supset (\bigwedge \Delta \supset A)$ *is provable in* HK^I *or* $\bigwedge \Gamma \supset (\bigwedge \Delta \supset \bot)$ *is provable in* HK^I *when* A *is empty.*

Proof. First we assume the associativity and commutativity of $\wedge$ on $\bigwedge \Gamma$ and $\bigwedge \Delta$ in the formulas of the form $\bigwedge \Gamma \supset (\bigwedge \Delta \supset A)$. This is ensured by the rules (Tr) and (Tr 2) and the fact that for any associative commutative replacement $(\bigwedge \Gamma)'$ of $\bigwedge \Gamma$, the formula $(\bigwedge \Gamma)' \supset \bigwedge \Gamma$ is provable in HK^I.

The lemma is shown by induction on the proof in $\mathbf{DK}^I$. Here we consider only a few cases. If the last inference of the proof is

$$\frac{\Gamma; A, \Delta \to C \quad \Gamma; B, \Delta \to C}{\Gamma; A \vee B, \Delta \to C} \; (\vee \to)$$

then we have the following proof in HK^I.

$$\frac{\dfrac{\bigwedge \Gamma \supset (\bigwedge (A, \Delta) \supset C)}{\bigwedge \Gamma \supset (\bigwedge \Delta \wedge A \supset C)} \quad \dfrac{\bigwedge \Gamma \supset (\bigwedge (B, \Delta) \supset C)}{\bigwedge \Gamma \supset (\bigwedge \Delta \wedge B \supset C)}}{\bigwedge \Gamma \supset (\bigwedge \Delta \wedge A \supset C) \wedge (\bigwedge \Delta \wedge B \supset C)} \; (\supset \wedge \mathrm{I})$$

From this and $(\bigwedge \Delta \wedge A \supset C) \wedge (\bigwedge \Delta \wedge B \supset C) \supset (\bigwedge \Delta \wedge (A \vee B) \supset C)$, which is provable by Lemma 3.37, we obtain $\bigwedge \Gamma \supset (\bigwedge \Delta \wedge (A \vee B) \supset C)$ by (Tr). Hence $\bigwedge \Gamma \supset (\bigwedge (A \vee B, \Delta) \supset C)$ is provable in HK^I.

If the last inference is

$$\frac{\Gamma; \Delta \to A \quad \Pi; B, \Sigma \to C}{\Gamma, A \supset B, \Pi; \Delta, \Sigma \to C} \; (\supset \to)$$

then we have the following proof.

$$\frac{\dfrac{\bigwedge \Gamma \supset (\bigwedge \Delta \supset A)}{\bigwedge \Gamma \wedge (A \supset B) \supset (\bigwedge \Delta \supset B)} \text{ Lemma 3.35} \quad \dfrac{\text{Lemma 3.36}}{(\bigwedge \Delta \supset B) \supset (\bigwedge \Delta \wedge \bigwedge \Sigma \supset B \wedge \bigwedge \Sigma)}}{\bigwedge \Gamma \wedge (A \supset B) \supset (\bigwedge \Delta \wedge \bigwedge \Sigma \supset B \wedge \bigwedge \Sigma)} \; (\mathrm{Tr})$$

Then $(\bigwedge \Gamma \wedge (A \supset B)) \wedge \bigwedge \Pi \supset (\bigwedge \Delta \wedge \bigwedge \Sigma \supset B \wedge \bigwedge \Sigma)$ is also provable in HK^I. From this and $(\bigwedge \Gamma \wedge (A \supset B)) \wedge \bigwedge \Pi \supset (B \wedge \bigwedge \Sigma \supset C)$, which is derivable from the induction hypothesis $\bigwedge \Pi \supset (\bigwedge (B, \Sigma) \supset C)$, we obtain $(\bigwedge \Gamma \wedge (A \supset B)) \wedge \bigwedge \Pi \supset (\bigwedge \Delta \wedge \bigwedge \Sigma \supset C)$ by $(\supset \mathrm{Tr})$. ∎

From Lemmas 3.32 and 3.38, we obtain the following.

Theorem 3.39 *For any formula* A*,* A *is provable in* HK^I *if and only if* $; \to A$ *is provable in* $\mathbf{DK}^I$*.*

Proof. From left to right, we have Lemma 3.32. For the other direction, suppose that $; \to A$ is provable in $\mathbf{DK}^I$. Then by Lemma 3.38, $\top \supset (\top \supset A)$ is provable in HK^I. Since $\top$ is an instance of (A1), we see that A is provable in HK^I by applying (MP) twice. ∎

Combining Theorem 3.39 with Theorem 3.22, we have the following result.

Theorem 3.40 (Completeness of $\mathbf{HK}^I$) *For any formula A, A is provable in* HK^I *if and only if A is true in every model.*

Next we consider a Hilbert style system for BPC and its correspondence to **LBP2**. The Hilbert style system HB consists of the axiom schemes (A1)–(A10) of HK^I as well as the following ones:

$$\text{(A11)} \quad A \supset (B \supset A),$$
$$\text{(A12)} \quad A \supset (B \supset A \wedge B),$$

and (MP) as the single rule of inference. Note that the other rules (AF) and ($\wedge$I) of HK^I are derivable in HB by applying (MP) to the axiom schemes (A11) and (A12).

Lemma 3.41 *For any formula A, if A is provable in* HB*, then $; \rightarrow A$ is provable in* **LBP2**.

Proof. By induction on the proof of A in HB. If A is an instance of the axiom schemes (A11) or (A12), then we easily see that $; \rightarrow A$ is provable in **LBP2**. The other cases are shown as in the proof of Lemma 3.32. ∎

Lemma 3.42 *If a sequent $\Gamma; \Delta \rightarrow A$ is provable in* **LBP2**, *then $\bigwedge \Gamma \supset (\bigwedge \Delta \supset A)$ is provable in* HB *or $\bigwedge \Gamma \supset (\bigwedge \Delta \supset \bot)$ is provable in* HB *when A is empty.*

Proof. We proceed in the same way as in the proof of Lemma 3.38. The only difference is the case where the last inference is of the form

$$\frac{\Gamma; \Delta, A \rightarrow B}{; \Gamma, \Delta \rightarrow A \supset B} \ (\rightarrow\supset)$$

Then we have the following proof in HB.

$$\cfrac{\cfrac{\cfrac{\text{(A12)}}{\bigwedge \Delta \supset (A \supset \bigwedge \Delta \wedge A)}}{\bigwedge(\Gamma, \Delta) \supset (A \supset \bigwedge \Delta \wedge A)} \quad \cfrac{\bigwedge \Gamma \supset (\bigwedge(\Delta, A) \supset B)}{\bigwedge(\Gamma, \Delta) \supset (\bigwedge \Delta \wedge A \supset B)}}{\cfrac{\bigwedge(\Gamma, \Delta) \supset (A \supset B)}{\top \supset (\bigwedge(\Gamma, \Delta) \supset (A \supset B))} \ \text{(AF)}} \ (\supset \text{Tr})$$

∎

From Lemmas 3.41 and 3.42, we obtain the following.

Theorem 3.43 *For any formula A, A is provable in* HB *if and only if $; \rightarrow A$ is provable in* **LBP2**.

Proof. Similar to the proof of Theorem 3.39. ∎

Combining Theorem 3.43 with Theorem 3.26, we have the following result.

Theorem 3.44 (Completeness of HB) *For any formula A, A is provable in* HB *if and only if A is true in every* BPC*-model.*

3.9 Notes

Subintuitionistic logics The class of subintuitionistic logics was studied by Corsi [12], Došen [14], Restall [36], giving Hilbert style systems for the least subintuitionistic logic K^I as well as its extensions defined by Kripke models with various kinds of binary relations. The logic BPC was earlier introduced by Visser [49] and has been recently developed by Ardeshir and Ruitenburg [5], [6]. Other studies on BPC with diverse motivations are found in [1], [39], [41], [45].

Sequent calculi for subintuitionistic logics Gentzen style sequent calculi for subintuitionistic logics including K^I were given by Gabbay and Olivetti [17] in the form of labelled deductive system that is tailored for a particular kind of proof search, and by Wansing [50] in the form of Display Logic, a general scheme for Gentzen style systems. The sequent calculus $\mathbf{GK}^I$ we discussed in Section 3.2 was introduced by Kashima in his unpublished manuscript [28]. His original system is based on sequents consisting of finite sets of formulas rather than finite sequences, and accordingly dispenses with the contraction and the exchange rules. Besides it has no cut rule, which is proved to be admissible through the completeness theorem with respect to the class of Kripke models for K^I. Syntactical proof of cut-elimination for $\mathbf{GK}^I$ (and for **LBP**) is also possible via rather involved discussions.

Sequent calculi for BPC Gentzen style sequent calculi for BPC have been considered several times. The first sequent calculus for BPC was given by Ardeshir [4]. The system includes a rule to infer $\Gamma \to A \supset C$ from $\Gamma \to A \supset B$ and $\Gamma \to B \supset C$, which leads to failure of the subformula property. The second system was given by Sasaki [40]. It involves an auxiliary expression $(A \supset B)^+$ which is intended to denote implication in intuitionistic logic. (Extension of the language of BPC by an additional intuitionistic implication was considered in [45].) The cut-elimination theorem for the system holds, but yields only a weak form of subformula property in the sense that even $(A \supset B)^+$ is included in the subformulas of $A \supset B$. The third system, which is a slight modification of the second one, was given by Aghaei and Ardeshir [2]. It satisfies only a weak form of subformula property either. Another system is found in the systems for subintuitionistic logics by Wansing [50] mentioned above, which involves more auxiliary expressions.

Dual-context sequent calculi for subintuitionistic logics The idea of applying dual-context sequents to formalizing subintuitionistic logics was suggested by the system for BPC in [40] mentioned above. The point is that we did not introduce an additional (and semantically ambiguous) implication but consider implication at the previous world in Kripke models. This enabled us to prove the soundness theorem smoothly and to prove the completeness theorem in a parallel way to that for **LJ**. (The completeness theorem in the form of the converse of Theorem 3.19 is, however, not valid in the present system. For example, the sequent $\bot\,;\to$ is true in every model, but it is not provable in $\mathbf{DK}^I$.) Dual-context sequent calculi for other subintuitionistic logics discussed in [12], [14], [36] may also be defined by modifying the system $\mathbf{DK}^I$.

Dual-context sequent calculi for modal logics Having seen the ability of dual-context sequent calculi to formalize subintuitionistic logics, we also see that the same method works well in formalizing modal logics by considering $\top \supset A$ of subintuitionistic logics as the formula $\Box A$ of modal logics. For example, a dual-context sequent calculus for the modal logic K is defined as the system with the following left and right rules for the modal operator.

$$\frac{\Gamma; A, \Delta \to \Sigma}{\Gamma, \Box A; \Delta \to \Sigma} (\Box \to) \qquad\qquad \frac{\Gamma; \to A}{; \Gamma \to \Box A} (\to \Box)$$

The completeness and cut-elimination theorems for the dual-context sequent calculus for K are proved similarly to those for $\mathbf{DK}^I$ in this chapter. On the other hand, there are some systems based on sequents of this kind in the field of modal logic. Bierman and de Paiva [9], Davies and Pfenning [13] have used sequents with modal and nonmodal contexts to give natural deduction systems for intuitionistic modal logic. Heuerding et al. [23] have introduced one-sided systems based on split sequents to ensure termination of proof search. Our systems $\mathbf{DK}^I$ and **LBP2** are, however, closer to the system based on more general 2-sequents by Masini [31], where a 2-sequent of the form

$$\begin{matrix} \Gamma_1 & & \Delta_1 \\ \vdots & \vdash & \vdots \\ \Gamma_n & & \Delta_n \end{matrix}$$

is interpreted as the formula

$$(\bigwedge \Gamma_1 \supset \bigvee \Delta_1) \vee \Box((\bigwedge \Gamma_2 \supset \bigvee \Delta_2) \vee \Box(\cdots \Box(\bigwedge \Gamma_n \supset \bigvee \Delta_n) \cdots))$$

and the correspondence between the 2-sequent calculus and a Hilbert style system for the modal logic KD is shown. On comparing the above interpretation with ours (Definition 3.17), we find that the sequent $\Gamma_1; \Gamma_2 \to A$ of $\mathbf{DK}^I$ and **LBP2** corresponds to the 2-sequent such that $n = 2$, Δ_1 is empty and $\Delta_2 = A$. This means that dual-context sequent calculi can be viewed as a simplification of 2-sequent calculi, and practically they are sufficient to formalize logics such as K^I and BPC. (An earlier work by Sato [42] gives a cut-free system for the modal logic S5 based on sequents of the form $\Gamma_1; \Gamma_2 \to \Delta_2; \Delta_1$, which are considered as 2-sequents such that $n = 2$.)

Chapter 4

Relationships between subintuitionistic and substructural logics

In this chapter we investigate the relationships between subintuitionistic logics and substructural logics from a different perspective than that in Chapter 3. We consider here Hilbert style systems that characterize the implicational fragments of subintuitionistic logics and substructural logics, and clarify the inclusion relationships between the sets of formulas that are provable in each Hilbert style system for these logics. This investigation together with dual-context sequent calculi in Chapter 3 leads to sequent calculi for noncommutative substructural logics in the next chapter.

4.1 Implicational fragments of substructural logics

Substructural logics are logics defined by Gentzen style sequent calculi in which applications of the structural rules are restricted. They include linear logic and BCK logic, which are sometimes called *resource-conscious* logics because each assumption (i.e., formula on the left hand side of a sequent) cannot be used more than once in the absence of the contraction rule. Here we consider substructural logics obtained from the implicational fragment of the sequent calculus **LJ** by deleting some of the structural rules. We also present Hilbert style systems for these logics to compare them with systems for subintuitionistic logics in the next section.

Throughout this chapter, the language has the only logical connective $\supset$, and formulas are those constructed from the set $\mathcal{PV}$ of propositional variables and the connective $\supset$. We first define the sequent calculus $\mathbf{LJ}_{\supset}$, which is the implicational fragment of the sequent calculus **LJ** for intuitionistic logic. A *sequent* of $\mathbf{LJ}_{\supset}$ is an expression of the form $\Gamma \to A$. Initial sequents of $\mathbf{LJ}_{\supset}$ are of the following form:

$$A \to A.$$

Rules of inference of $\mathbf{LJ}_{\supset}$ are the following.

Structural rules:

$$\frac{\Gamma, \Delta \to C}{\Gamma, A, \Delta \to C}\,(\mathrm{w} \to) \qquad \frac{\Gamma, A, A, \Delta \to C}{\Gamma, A, \Delta \to C}\,(\mathrm{c} \to) \qquad \frac{\Gamma, A, B, \Delta \to C}{\Gamma, B, A, \Delta \to C}\,(\mathrm{e} \to)$$

Cut rule:

$$\frac{\Gamma \to A \quad \Delta, A, \Sigma \to C}{\Delta, \Gamma, \Sigma \to C}\ (\text{cut})$$

Rules for the logical connective:

$$\frac{\Gamma \to A \quad \Delta, B, \Sigma \to C}{\Delta, A \supset B, \Gamma, \Sigma \to C}\ (\supset\to) \qquad\qquad \frac{\Gamma, A \to B}{\Gamma \to A \supset B}\ (\to\supset)$$

We obtain sequent calculi from the above system $\mathbf{LJ}_\supset$ by deleting some of the structural rules. The sequent calculi **LBCK** and **LBCIW** are obtained from $\mathbf{LJ}_\supset$ by deleting the rules (c $\to$) and (w $\to$), respectively. The sequent calculus **LBCI**, which is known to be the implicational fragment of linear logic, is obtained from $\mathbf{LJ}_\supset$ by deleting both the rules (c $\to$) and (w $\to$).

Next we introduce Hilbert style systems corresponding to the sequent calculi defined above. They consist of combinations of the following axiom schemes:

(B) $(B \supset C) \supset ((A \supset B) \supset (A \supset C))$,
(C) $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$,
(I) $A \supset A$,
(K) $A \supset (B \supset A)$,
(W) $(A \supset (A \supset B)) \supset (A \supset B)$,

and the following rule of inference:

$$\frac{A \quad A \supset B}{B}\ (\text{mp})\,.$$

We define the Hilbert style systems BCI, BCK, BCIW and BCKW as the systems consisting of the axiom schemes indicated by the letters in their names and of the rule (mp). It is shown that these systems correspond to the sequent calculi **LBCI**, **LBCK**, **LBCIW** and $\mathbf{LJ}_\supset$, respectively.

Theorem 4.1 *For any formula A, if A is provable in* BCI, BCK, BCIW *and* BCKW, *then $\to A$ is provable in* **LBCI**, **LBCK**, **LBCIW** *and* $\mathbf{LJ}_\supset$*, respectively.*

PROOF. By induction on the proof of A in each Hilbert style system. It is straightforward to see that for any axiom A of the Hilbert style system, $\to A$ is provable in the respective sequent calculus. For (mp), we have the following proof.

$$\frac{\to A \supset B \qquad \dfrac{\to A \quad B \to B}{A \supset B \to B}\ (\supset\to)}{\to B}\ (\text{cut})$$

∎

To prove the converse of Theorem 4.1, we introduce the following notation.

Definition 4.2 For any finite sequence Γ of formulas and any formula A, the formula $\Gamma \supset A$ is defined inductively as follows:

$$\Gamma \supset A \;=\; A \quad \text{if } \Gamma \text{ is the empty sequence,}$$
$$C, \Gamma \supset A \;=\; C \supset (\Gamma \supset A).$$

Lemma 4.3 *The following rules are derivable in* BCI.

$$\frac{A \supset B}{(C \supset A) \supset (C \supset B)}\ (\text{pref}) \qquad\qquad \frac{\Gamma \supset A \quad A \supset B}{\Gamma \supset B}\ (\text{tr}^*)$$

Proof. For (pref), we have the following proof.

$$\frac{A \supset B \quad \overset{(\text{B})}{(A \supset B) \supset ((C \supset A) \supset (C \supset B))}}{(C \supset A) \supset (C \supset B)}\ (\text{mp})$$

For (tr*), we have the following proof.

$$\frac{\Gamma \supset A \quad \begin{array}{c} A \supset B \\ \vdots\ (\text{pref}) \\ (\Gamma \supset A) \supset (\Gamma \supset B) \end{array}}{\Gamma \supset B}\ (\text{mp})$$

■

Lemma 4.4 *The formula* $(A \supset B) \supset ((\Gamma \supset A) \supset (\Gamma \supset B))$ *is provable in* BCI.

Proof. By induction on the length of Γ. When Γ is empty, the formula is an instance of (I). When $\Gamma = C, \Gamma'$, we have the following proof.

$$\frac{\overset{(\text{ind. hyp.})}{(A \supset B) \supset ((\Gamma' \supset A) \supset (\Gamma' \supset B))} \quad \overset{(\text{B})}{((\Gamma' \supset A) \supset (\Gamma' \supset B)) \supset ((\Gamma \supset A) \supset (\Gamma \supset B))}}{(A \supset B) \supset ((\Gamma \supset A) \supset (\Gamma \supset B))}\ (\text{tr}^*)$$

■

Lemma 4.5 *The following rules are derivable in* BCI.

$$\frac{A \supset (B \supset C)}{B \supset (A \supset C)}\ (\text{ex}) \qquad\qquad \frac{\Gamma \supset A}{(A \supset B) \supset (\Gamma \supset B)}\ (\text{suff}^*)$$

Proof. For (ex), we have the following proof.

$$\frac{A \supset (B \supset C) \quad \overset{(\text{C})}{(A \supset (B \supset C)) \supset (B \supset (A \supset C))}}{B \supset (A \supset C)}\ (\text{mp})$$

For (suff*), we have the following proof.

$$\frac{\Gamma \supset A \quad \dfrac{\overset{\text{Lemma 4.4}}{(A \supset B) \supset ((\Gamma \supset A) \supset (\Gamma \supset B))}}{(\Gamma \supset A) \supset ((A \supset B) \supset (\Gamma \supset B))}\ (\text{ex})}{(A \supset B) \supset (\Gamma \supset B)}\ (\text{mp})$$

■

Now we prove the converse of Theorem 4.1 in somewhat more general form.

Theorem 4.6 *If a sequent* $\Gamma \rightarrow A$ *is provable in* **LBCI**, **LBCK**, **LBCIW** *and* $\mathbf{LJ}_{\supset}$, *then* $\Gamma \supset A$ *is provable in* BCI, BCK, BCIW *and* BCKW, *respectively.*

Proof. First note that each instance of the axiom scheme (I) is provable in BCK and BCKW as follows:

$$\dfrac{\top \qquad \dfrac{\overset{\text{(K)}}{A \supset (\top \supset A)} \qquad \overset{\text{(C)}}{(A \supset (\top \supset A)) \supset (\top \supset (A \supset A))}}{\top \supset (A \supset A)}\,\text{(mp)}}{A \supset A}\,\text{(mp)}$$

where $\top$ is any provable formula, e.g., an instance of (K).

The theorem is shown by induction on the proof in each sequent calculus. Here we consider only a few cases. If the last inference of the proof is

$$\dfrac{\Gamma \rightarrow A \qquad \Delta, B, \Sigma \rightarrow C}{\Delta, A \supset B, \Gamma, \Sigma \rightarrow C}\,(\supset\rightarrow)$$

then we have the following proof in the Hilbert style system.

$$\dfrac{\Delta \supset (B \supset (\Sigma \supset C)) \qquad \dfrac{\dfrac{\Gamma \supset A}{(A \supset B) \supset (\Gamma \supset B)}\,(\text{suff}^*)}{(B \supset (\Sigma \supset C)) \supset ((A \supset B) \supset (\Gamma \supset (\Sigma \supset C)))}\,(\text{suff}^*)}{\Delta \supset ((A \supset B) \supset (\Gamma \supset (\Sigma \supset C)))}\,(\text{tr}^*)$$

If the last inference is

$$\dfrac{\Gamma, A, B, \Delta \rightarrow C}{\Gamma, B, A, \Delta \rightarrow C}\,(\text{e} \rightarrow)$$

then we have the following proof.

$$\dfrac{\Gamma \supset (A \supset (B \supset (\Delta \supset C))) \qquad \overset{\text{(C)}}{(A \supset (B \supset (\Delta \supset C))) \supset (B \supset (A \supset (\Delta \supset C)))}}{\Gamma \supset (B \supset (A \supset (\Delta \supset C)))}\,(\text{tr}^*)$$

The cases of the other structural rules are proved similarly. ∎

We close this section with the definitions of some more Hilbert style systems which do not have the axiom scheme (C). The Hilbert style system BB′I is obtained from the system BCI by replacing the axiom scheme (C) by the following one:

$$(\text{B}') \quad (A \supset B) \supset ((B \supset C) \supset (A \supset C)).$$

The Hilbert style systems BB′IK and BB′IW are obtained from BB′I by adding the axiom schemes (K) and (W), respectively. Note that each instance of the axiom scheme (B′) is provable in BCI, and so the systems BB′I, BB′IK and BB′IW are subsystems of BCI, BCK and BCIW, respectively.

4.2 Implicational fragments of subintuitionistic logics

As seen in Example 3.2, each instance of the axiom schemes (W) and (C) is not necessarily true in Kripke models for subintuitionistic logics. This means that subintuitionistic logics may qualify as substructural logics and that it is worth investigating the relationships between these logics.

In this section we introduce Hilbert style systems for the implicational fragments of K^I and BPC to compare them with systems for substructural logics introduced in the previous section. Although Hilbert style systems for K^I and BPC were already given in Section 3.8, they are not suitable for our purpose because to derive some implicational formulas in those systems one needs axioms with other connectives. Here we consider Hilbert style systems that characterize the proper implicational fragments of K^I and of BPC, which facilitate a set-theoretic comparison between theorems of subintuitionistic logics and substructural logics.

First we introduce a Hilbert style system B*IK, which consists of the following axiom schemes:

(I) $A \supset A$,

(K) $A \supset (B \supset A)$,

(B*) $(\Gamma \supset (B \supset C)) \supset ((\Gamma \supset (A \supset B)) \supset (\Gamma \supset (A \supset C)))$,

and the following rule of inference:

$$\frac{A \quad A \supset B}{B}\ (\mathrm{mp})\,.$$

This system turns out to characterize the implicational fragment of BPC. On the other hand, the system for the implicational fragment of K^I consists of the axiom scheme (I), the rule (mp) and the following two additional rules:

$$\frac{A}{B \supset A}\,, \qquad \frac{\Gamma \supset (A \supset B) \quad \Gamma \supset (B \supset C)}{\Gamma \supset (A \supset C)}\,.$$

In the following we show that B*IK is indeed a system for the implicational fragment of BPC. As for the system for the implicational fragment of K^I, we can proceed in a similar way. (See [14] for the details.) Since it is not easy to show the correspondence between B*IK and the implicational fragment of any other system for BPC, we prove again the completeness theorem of B*IK with respect to the class of Kripke models for BPC.

Theorem 4.7 (Soundness of B*IK) *For any formula A, if A is provable in* B*IK *then A is true in every* BPC-*model.*

Proof. By induction on the proof of A in B*IK. Here we show that each instance of the axiom scheme (B*) is true in every BPC-model and that truth in every BPC-model is preserved by (mp).

For (B*), take any BPC-model $\langle W, R, V\rangle$ and any $x \in W$. To show $x \models (\Gamma \supset (B \supset C)) \supset ((\Gamma \supset (A \supset B)) \supset (\Gamma \supset (A \supset C)))$, suppose xRy and $y \models \Gamma \supset (B \supset C)$. To show

$y \models (\Gamma \supset (A \supset B)) \supset (\Gamma \supset (A \supset C))$, suppose further yRz and $z \models \Gamma \supset (A \supset B)$. Then by Lemma 3.3, we have $z \models \Gamma \supset (B \supset C)$. Now our aim is to show $z \models \Gamma \supset (A \supset C)$. For this, it suffices to prove that for every $w \in W$, if $w \models \Gamma \supset (A \supset B)$ and $w \models \Gamma \supset (B \supset C)$ then $w \models \Gamma \supset (A \supset C)$. This is easily seen by induction on the length of Γ.

For (mp), suppose that A and $A \supset B$ are both true in every BPC-model and that B is not true in a BPC-model $M = \langle W, R, V \rangle$. Then there exists an $x \in W$ such that $x \not\models_M B$. Now consider a model $M' = \langle W', R', V \rangle$ where $W' = W \cup \{y\}$, $y \notin W$ and $R' = R \cup \{(y, x)\} \cup \{(y, z) \mid xRz\}$. It is easy to see that R' is transitive, and so M' is a BPC-model. Then we can show by induction on the structure of C that for any $w \in W$, $w \models_M C$ if and only if $w \models_{M'} C$. Since $x \in W$ and $x \not\models_M B$, we have $x \not\models_{M'} B$, which contradicts with $x \models_{M'} A$ and $y \models_{M'} A \supset B$. ∎

Theorem 4.8 (Completeness of B*IK) *For any formula A, A is provable in* B*IK *if and only if A is true in every* BPC-*model.*

Proof. From left to right, we have Theorem 4.7. For the other direction, let e be the set of all formulas that are provable in B*IK. Define a relation R between sets of formulas as follows:

$$xRy \quad \text{iff} \quad x \subseteq y \text{ and if } A \supset B \in x \text{ and } A \in y \text{ then } B \in y.$$

Then we consider a BPC-model $\langle W^*, R^*, V^* \rangle$ where

- W^* is the set of all x such that eRx and if $\Gamma \supset (B \supset C) \in x$ and $\Gamma \supset (A \supset B) \in x$ then $\Gamma \supset (A \supset C) \in x$,
- R^* is the restriction to W^* of R,
- $V^*(p) = \{x \in W^* \mid p \in x\}$ for every $p \in \mathcal{PV}$.

It is easy to verify that $e \in W^*$, R^* is transitive, and if $x \in V^*(p)$ and xR^*y then $y \in V^*(p)$. Thus $\langle W^*, R^*, V^* \rangle$ is indeed a BPC-model. Our aim is to show that if a formula A is true in this BPC-model then $A \in e$, i.e., A is provable in B*IK. For this purpose, it suffices to show that for any $x \in W^*$,

$$x \models A \quad \text{if and only if} \quad A \in x.$$

We prove this by induction on the structure of A. The base case is straightforward. For the induction step, it follows from the definition of R that $x \models A \supset B$ if $A \supset B \in x$. Conversely, suppose $x \models A \supset B$. We show first xRy for $y = \{C \mid A \supset C \in x\}$. Since eRx and $C \supset (A \supset C) \in e$, if $C \in x$ then $A \supset C \in x$ and so $C \in y$. Hence $x \subseteq y$. Also, if $C \supset D \in x$ and $C \in y$, i.e., $A \supset C \in x$ then $A \supset D \in x$ by the condition for $x \in W^*$, and hence $D \in y$. Thus we have xRy, which implies also eRy by the transitivity of R. Moreover $y \in W^*$ holds since we can find that $\Gamma \supset (D \supset E) \in y$ and $\Gamma \supset (C \supset D) \in y$ imply $\Gamma \supset (C \supset E) \in y$. We have also $A \in y$ since $A \supset A \in e \subseteq x$, and hence $y \models A$ by the induction hypothesis. Now we have $x \models A \supset B$, xR^*y and $y \models A$. Thus $y \models B$ holds, and by the induction hypothesis, we have $B \in y$, i.e., $A \supset B \in x$ as required. ∎

4.3 Relationships between subintuitionistic and substructural logics

We have seen that the system B*IK characterizes the implicational fragment of BPC. In B*IK, each instance of the axiom schemes

$$\text{(B)}\quad (B \supset C) \supset ((A \supset B) \supset (A \supset C)),$$
$$\text{(B}'\text{)}\quad (A \supset B) \supset ((B \supset C) \supset (A \supset C))$$

is provable. Indeed, (B) is (B*) with Γ empty. For (B′), we first derive $(A \supset B) \supset ((B \supset C) \supset (B \supset C))$ from (K) and (I), and $(A \supset B) \supset ((B \supset C) \supset (A \supset B))$ from (K). Then we derive (B′) from the two and (B*) with $\Gamma = A \supset B, B \supset C$.

On the other hand, each instance of the axiom schemes

$$\text{(W)}\quad (A \supset (A \supset B)) \supset (A \supset B),$$
$$\text{(C)}\quad (A \supset (B \supset C)) \supset (B \supset (A \supset C))$$

is not in general provable in B*IK, because it is not necessarily true in BPC-models as seen in Example 3.2.

As a result, we have the following scheme, which represents the inclusion relationships between the sets of formulas that are provable in each system for subintuitionistic and substructural logics.

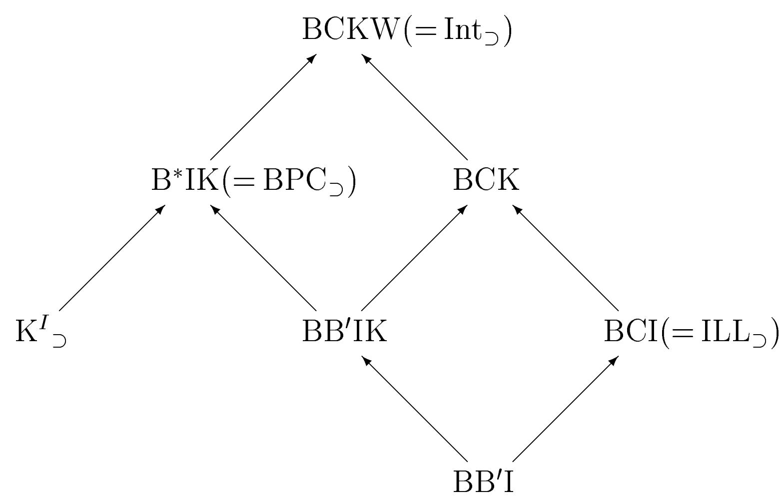


Figure 4.1: Relationships between subintuitionistic and substructural logics

Here $\mathrm{Int}_\supset$, $\mathrm{BPC}_\supset$, $\mathrm{K}^I{}_\supset$ and $\mathrm{ILL}_\supset$ mean systems for the implicational fragments of intuitionistic logic, BPC, K^I and (intuitionistic) linear logic, respectively. Note that the systems B*IK and BCK are incomparable. An example of a formula that is provable in B*IK (and $\mathrm{K}^I{}_\supset$) but not in BCK is $((p \supset q) \supset (q \supset r)) \supset ((p \supset q) \supset (p \supset r))$. In the next chapter we study sequent calculi for subsystems of BPC, in particular the logic BB′I, a noncommutative substructural logic without the axiom scheme (C).

4.4 Notes

Implicational fragments of substructural logics For general information on substructural logics, see [15], [37]. The names of the axiom schemes (B), (C), (I), (K), (W) and (B′) come from the principal types of certain specific combinators (or closed λ-terms). See [24]. The cut-elimination theorem holds for the systems **LBCI**, **LBCK**, **LBCIW** and $\mathbf{LJ}_{\supset}$. See, e.g. [34]. In the literature, authors often take as the basic system *full Lambek logic* **FL**, which is, roughly speaking, obtained from **LJ** by deleting all structural rules. The Hilbert style system corresponding to the implicational fragment of **FL** consists of the axiom schemes (B) and (I) and of the rule (mp) as well as the following one:

$$\frac{B \quad A \supset (B \supset C)}{A \supset C}\ (\mathrm{mp2})\,.$$

This correspondence and those of BCI, BCK, BCIW and BCKW (Theorems 4.1 and 4.6) are well-known. Our proof of Theorem 4.6 is mainly based on [29], in which still more substructural implicational logics are discussed.

Implicational fragments of subintuitionistic logics The Hilbert style system for the implicational fragment of K^I in Section 4.2 was given by Došen [14]. Our proof of Theorem 4.8 is mainly based on the proof of the completeness theorem of that system in [14]. One of minor differences in the proof is that Došen showed in the induction that $x \models A \supset B$ implies $A \supset B \in x$ by the contraposition using Zorn's Lemma.

Chapter 5

Sequent calculi for noncommutative substructural logics

In this chapter we study sequent calculi for noncommutative substructural logics, in particular the logic BB′I. This logic is important in the respect that it is a noncommutative version of the implicational fragment of linear logic. We introduce a new sequent calculus for BB′I and prove the cut-elimination theorem for the system.

5.1 Sequent calculus for BB′I

We recall that the Hilbert style system BB′I consists of the following axiom schemes:

$$
\begin{array}{ll}
(\mathrm{B}) & (B \supset C) \supset ((A \supset B) \supset (A \supset C)), \\
(\mathrm{B}') & (A \supset B) \supset ((B \supset C) \supset (A \supset C)), \\
(\mathrm{I}) & A \supset A,
\end{array}
$$

and the following rule of inference:

$$\frac{A \quad A \supset B}{B}\ (\mathrm{mp})\,.$$

The usual sequent calculus (Section 7 of [3], [30]) corresponding to the system BB′I is defined using merge operation. (For the precise definition of the system, see Section 5.3.) In this thesis we introduce a sequent calculus for BB′I without merge operation, and show the cut-elimination theorem for the system. Although BB′I is called a *noncommutative* logic ([8], [25], [32], [35]), our system allows exchange for the left hand side of a sequent except for the rightmost formula.

Here is the definition of the sequent calculus **LBB′I2**. Initial sequents of **LBB′I2** are of the following form:

$$A \to A.$$

Rules of inference of **LBB′I2** are the following.

Structural rule:

$$\frac{\Gamma, A, B, \Delta, C \to D}{\Gamma, B, A, \Delta, C \to D}\ (\mathrm{e} \to)$$

Cut rule:

$$\frac{\Gamma \to A \quad \Delta, A \to D}{\Delta, \Gamma \to D} \text{ (cut)}$$

where Δ is empty if Γ is empty. The formula A is called the *cut formula*.

Rules for the logical connective:

$$\frac{\Gamma, C \to A \quad \Delta, B \to D}{\Delta, A \supset B, \Gamma, C \to D} \text{ } (\supset\to) \qquad\qquad \frac{\Gamma, A \to B}{\Gamma \to A \supset B} \text{ } (\to\supset)$$

We first show the correspondence between the systems BB′I and **LBB′I2**, assuming the cut elimination theorem for **LBB′I2**, which is proved in the next section.

Lemma 5.1 *For any formula A, if A is provable in* BB′I, *then $\to A$ is provable in* **LBB′I2**.

Proof. By induction on the proof of A in BB′I. For (B), we have the proof

$$\frac{\dfrac{A \to A \quad B \to B}{A \supset B, A \to B} \text{ } (\supset\to) \qquad C \to C}{B \supset C, A \supset B, A \to C} \text{ } (\supset\to)$$

followed by three applications of $(\to\supset)$. For (B′), we have the above proof followed by one application of $(\mathrm{e} \to)$ and three applications of $(\to\supset)$. For (mp), suppose that both $\to A$ and $\to A \supset B$ are provable in **LBB′I2**. Then by the cut-elimination theorem, there is a cut-free proof of $\to A \supset B$ in **LBB′I2**. The last applied rule of the cut-free proof must be $(\to\supset)$, and so $A \to B$ is provable in **LBB′I2**. From this and $\to A$, we obtain $\to B$ using the cut rule in **LBB′I2**. ∎

For the converse of Lemma 5.1, we use the notation $\Gamma \supset A$ (Definition 4.2) and the convention of association to the right for omitting parentheses.

Lemma 5.2 *The following rules are derivable in* BB′I.

$$\frac{A \supset B}{(C \supset A) \supset (C \supset B)} \text{ (pref)} \qquad\qquad \frac{\Gamma \supset A \quad A \supset B}{\Gamma \supset B} \text{ (tr}^*\text{)}$$

Proof. For (pref), we have the following proof.

$$\frac{A \supset B \quad \overset{\text{(B)}}{(A \supset B) \supset (C \supset A) \supset (C \supset B)}}{(C \supset A) \supset (C \supset B)} \text{ (mp)}$$

For (tr*), we have the following proof.

$$\frac{\Gamma \supset A \quad \begin{array}{c} A \supset B \\ \vdots \text{ (pref)} \\ (\Gamma \supset A) \supset (\Gamma \supset B) \end{array}}{\Gamma \supset B} \text{ (mp)}$$

∎

Lemma 5.3 *The following rule is derivable in* BB′I.

$$\frac{A \supset B \quad \Gamma \supset (B \supset C)}{\Gamma \supset (A \supset C)}$$

Proof. We have the following proof.

$$\dfrac{\Gamma \supset (B \supset C) \qquad \dfrac{A \supset B \quad \overset{(\mathrm{B}')}{(A \supset B) \supset (B \supset C) \supset (A \supset C)}}{(B \supset C) \supset (A \supset C)}\,(\mathrm{mp})}{\Gamma \supset (A \supset C)}\,(\mathrm{tr}^*)$$

Lemma 5.4 *The following rule is derivable in* BB′I.

$$\frac{A \supset (B \supset C)}{A \supset (D \supset B) \supset (D \supset C)}$$

Proof. We have the following proof.

$$\dfrac{A \supset (B \supset C) \quad \overset{(\mathrm{B})}{(B \supset C) \supset (D \supset B) \supset (D \supset C)}}{A \supset (D \supset B) \supset (D \supset C)}\,(\mathrm{tr}^*)$$

Lemma 5.5 *The following rule is derivable in* BB′I.

$$\frac{A \supset (B \supset C)}{(D \supset B) \supset A \supset (D \supset C)}$$

Proof. We have the following proof.

$$\dfrac{A \supset (B \supset C) \quad \overset{(\mathrm{B}')}{(D \supset B) \supset (B \supset C) \supset (D \supset C)}}{(D \supset B) \supset A \supset (D \supset C)}\,\text{Lemma 5.3}$$

Lemma 5.6 *The following rule is derivable in* BB′I.

$$\frac{\Gamma \supset \Delta \supset B \supset C}{\Gamma \supset (A \supset B) \supset \Delta \supset A \supset C}$$

Proof. We show that the formula $(\Delta \supset B \supset C) \supset (A \supset B) \supset \Delta \supset A \supset C$ is provable in BB′I. The rule is then derivable by (tr*). If Δ is empty, the formula is an instance of (B). If $\Delta = D, \Delta'$, then we have the following proof.

$$\dfrac{\begin{array}{c} \overset{(\mathrm{B}')}{(A \supset B) \supset (B \supset C) \supset A \supset C} \\ \vdots\ \text{Lemma 5.4} \\ (A \supset B) \supset (\Delta' \supset B \supset C) \supset \Delta' \supset A \supset C \end{array}}{(D \supset \Delta' \supset B \supset C) \supset (A \supset B) \supset D \supset \Delta' \supset A \supset C}\,\text{Lemma 5.5}$$

Lemma 5.7 *The following rule is derivable in* BB′I.

$$\frac{\Gamma \supset C \supset A \quad \Delta \supset A \supset D}{\mu(\Gamma, \Delta) \supset C \supset D}$$

where $\mu(\Gamma, \Delta)$ *is any merge of* Γ *and* Δ*, i.e., any sequence consisting of the members of* Γ *and* Δ *as multisets, in which both* Γ *and* Δ *preserve their original orders.*

Proof. By induction on the length of Γ. If Γ is empty, the rule follows from Lemma 5.3. Let $\Gamma = \Gamma_1, C_1$ and $\Delta = \Delta_1, \Delta_2$ and $\mu(\Gamma, \Delta) = \mu_1(\Gamma_1, \Delta_1), C_1, \Delta_2$ where $\mu_1(\Gamma_1, \Delta_1)$ is some merge of Γ_1 and Δ_1. Then we have the following proof.

$$\cfrac{\begin{array}{c}\Gamma \supset C \supset A \\ \Vert \\ \Gamma_1 \supset C_1 \supset (C \supset A)\end{array} \quad \cfrac{\begin{array}{c}\Delta \supset A \supset D \\ \Vert \\ \Delta_1 \supset \Delta_2 \supset A \supset D\end{array}}{\Delta_1 \supset (C \supset A) \supset \Delta_2 \supset C \supset D}\ \text{Lemma 5.6}}{\begin{array}{c}\mu_1(\Gamma_1, \Delta_1) \supset C_1 \supset \Delta_2 \supset C \supset D \\ \Vert \\ \mu(\Gamma, \Delta) \supset C \supset D\end{array}}\ \text{(ind. hyp.)}$$

∎

Now we prove the key lemma, which states how a sequent of **LBB′I2** is interpreted as formulas of BB′I.

Lemma 5.8 *If a sequent* $\Gamma, C \to A$ *is provable in* **LBB′I2***, then* $\sigma(\Gamma) \supset (C \supset A)$ *is provable in* BB′I *for any permutation* σ*.*

Proof. By induction on the proof of $\Gamma, C \to A$ in **LBB′I2**. First note that by virtue of the cut-elimination theorem we need not consider the cut rule. Further note that the premisses of the last inference of the proof are also of the form $\Delta, D \to B$.

The only problematic case in the induction is where the last inference is of the form

$$\frac{\Gamma, C \to A \quad \Delta, B \to D}{\Delta, A \supset B, \Gamma, C \to D}\ (\supset\to)$$

The aim is to show that $\sigma(\Delta, A \supset B, \Gamma) \supset (C \supset D)$ is provable in BB′I for any permutation σ. Let $\sigma(\Delta, A \supset B, \Gamma) = \mu_1(\Delta_1, \Gamma_1), A \supset B, \mu_2(\Delta_2, \Gamma_2)$ where $\Delta_1, \Delta_2 = \sigma'(\Delta)$ and $\Gamma_1, \Gamma_2 = \sigma''(\Gamma)$ for some permutations σ' and σ'', and $\mu_i(\Delta_i, \Gamma_i)$ is some merge of Δ_i and Γ_i for $i = 1, 2$. Then we have the following proof.

$$\cfrac{\begin{array}{c}\text{(ind. hyp.)} \\ \Gamma_1 \supset \Gamma_2 \supset C \supset A\end{array} \quad \cfrac{\begin{array}{c}\text{(ind. hyp.)} \\ \Delta_1 \supset \Delta_2 \supset B \supset D\end{array}}{\Delta_1 \supset (A \supset B) \supset \Delta_2 \supset A \supset D}\ \text{Lemma 5.6}}{\begin{array}{c}\mu_1(\Delta_1, \Gamma_1) \supset (A \supset B) \supset \mu_2(\Delta_2, \Gamma_2) \supset C \supset D \\ \Vert \\ \sigma(\Delta, A \supset B, \Gamma) \supset C \supset D\end{array}}\ \text{Lemma 5.7}$$

∎

From Lemmas 5.1 and 5.8, we obtain the following.

Theorem 5.9 *For any formula A, A is provable in* BB′I *if and only if $\to A$ is provable in* **LBB′I2**.

Proof. From left to right, we have Lemma 5.1. For the other direction, suppose that $\to A$ is provable in **LBB′I2**. Then by the cut-elimination theorem, there is a cut-free proof of $\to A$ in **LBB′I2** and A is of the form $A_1 \supset A_2$. The last applied rule of the cut-free proof must be $(\to\supset)$, and so $A_1 \to A_2$ is provable in **LBB′I2**. Hence by Lemma 5.8, $A_1 \supset A_2$, i.e., A is provable in BB′I. ∎

5.2 Cut-elimination theorem for LBB′I2

In the previous section we showed the correspondence between the systems BB′I and **LBB′I2**, using the cut-elimination theorem for **LBB′I2**. The present section is devoted to proving the cut-elimination theorem for **LBB′I2** in a syntactical way. In the proof of the cut-elimination theorem, we use global proof transformation technique analogous to that used in the proof of the cut-elimination theorem for $\mathbf{DK}^I$ and **LBP2**.

First we introduce the notion of height of a proof in **LBB′I2**.

Definition 5.10 The *height* $h(P)$ of a proof P in **LBB′I2** is defined inductively as follows:

1. If P is an initial sequent, then $h(P) = 1$.
2. If P is obtained from the proof Q by applying a one-premiss rule, then $h(P) = h(Q) + 1$.
3. If P is obtained from the proofs Q_1 and Q_2 by applying a two-premiss rule, then $h(P) = \max\{h(Q_1), h(Q_2)\} + 1$.

Now we prove the cut-elimination theorem for **LBB′I2**.

Theorem 5.11 (Cut-elimination for LBB′I2) *For any sequent $\Gamma \to A$, if $\Gamma \to A$ is provable in* **LBB′I2**, *then it is provable in* **LBB′I2** *without using the cut rule.*

Proof. Our strategy is that of eliminating each cut rule above which any other cut rule does not occur. Let P be a proof with only one cut rule occurring as the last inference whose cut formula is A. Let P_l and P_r be the subproofs of P whose end-sequents are the left and right premisses of the cut rule, respectively. The proof of eliminating the cut rule in P is by induction on $|A|$, with a subinduction on $h(P_l) + h(P_r)$. We proceed on a case-by-case basis.

Case 1. The left premiss of the cut rule is an initial sequent $A \to A$.

$$\frac{A \to A \quad \Delta, A \to D}{\Delta, A \to D}\,(\mathrm{cut})$$

The conclusion is the same as the right premiss.

Case 2. The right premiss of the cut rule is an initial sequent $A \to A$.

$$\frac{\Gamma \to A \quad A \to A}{\Gamma \to A}\,(\mathrm{cut})$$

The conclusion is the same as the left premiss.

Case 3. The left rule over the cut rule is (e $\to$).

$$\dfrac{\dfrac{\overset{\vdots\ P_1}{\Gamma, A, B, \Sigma, C \to D}}{\Gamma, B, A, \Sigma, C \to D}\,(\mathrm{e} \to) \qquad \overset{\vdots\ P_2}{\Delta, D \to E}}{\Delta, \Gamma, B, A, \Sigma, C \to E}\,(\mathrm{cut})$$

This is transformed into

$$\dfrac{\dfrac{\overset{\vdots\ P_1}{\Gamma, A, B, \Sigma, C \to D} \qquad \overset{\vdots\ P_2}{\Delta, D \to E}}{\Delta, \Gamma, A, B, \Sigma, C \to E}\,(\mathrm{cut})}{\Delta, \Gamma, B, A, \Sigma, C \to E}\,(\mathrm{e} \to)$$

where the (cut) can be eliminated by the subinduction hypothesis.

Case 4. The left rule over the cut rule is ($\supset\to$).

$$\dfrac{\dfrac{\overset{\vdots\ P_0}{\Gamma, C \to A} \qquad \overset{\vdots\ P_1}{\Sigma, B \to D}}{\Sigma, A \supset B, \Gamma, C \to D}\,(\supset\to) \qquad \overset{\vdots\ P_2}{\Delta, D \to E}}{\Delta, \Sigma, A \supset B, \Gamma, C \to E}\,(\mathrm{cut})$$

This is transformed into

$$\dfrac{\overset{\vdots\ P_0}{\Gamma, C \to A} \qquad \dfrac{\overset{\vdots\ P_1}{\Sigma, B \to D} \qquad \overset{\vdots\ P_2}{\Delta, D \to E}}{\Delta, \Sigma, B \to E}\,(\mathrm{cut})}{\Delta, \Sigma, A \supset B, \Gamma, C \to E}\,(\supset\to)$$

where the (cut) can be eliminated by the subinduction hypothesis.

Case 5. The left rule over the cut rule is ($\to\supset$). We consider the following subcases.

Subcase 5.1. The right rule over the cut rule is (e $\to$). Note in this case that the left hand side of the left premiss of the cut rule is not empty by the condition of the cut rule.

$$\dfrac{\overset{\vdots\ P_1}{\Gamma, C \to D} \qquad \dfrac{\overset{\vdots\ P_2}{\Delta, A, B, \Sigma, D \to E}}{\Delta, B, A, \Sigma, D \to E}\,(\mathrm{e} \to)}{\Delta, B, A, \Sigma, \Gamma, C \to E}\,(\mathrm{cut})$$

This is transformed into

$$\dfrac{\dfrac{\overset{\vdots\ P_1}{\Gamma, C \to D} \qquad \overset{\vdots\ P_2}{\Delta, A, B, \Sigma, D \to E}}{\Delta, A, B, \Sigma, \Gamma, C \to E}\,(\mathrm{cut})}{\Delta, B, A, \Sigma, \Gamma, C \to E}\,(\mathrm{e} \to)$$

where the (cut) can be eliminated by the subinduction hypothesis.

Subcase 5.2. The right rule over the cut rule is $(\supset\to)$. As in the previous subcase, the left hand side of the left premiss of the cut rule is not empty.

$$\dfrac{\begin{array}{c}\vdots\ P_1\\ \Gamma, C \to D\end{array} \quad \dfrac{\begin{array}{c}\vdots\ P_2\\ \Sigma, D\to A\end{array}\quad\begin{array}{c}\vdots\ P_3\\ \Delta, B\to E\end{array}}{\Delta, A\supset B, \Sigma, D\to E}\,(\supset\to)}{\Delta, A\supset B,\Sigma,\Gamma,C\to E}\,(\text{cut})$$

This is transformed into

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ P_1\\ \Gamma, C \to D\end{array}\quad\begin{array}{c}\vdots\ P_2\\ \Sigma, D\to A\end{array}}{\Sigma,\Gamma,C\to A}\,(\text{cut}) \quad \begin{array}{c}\vdots\ P_3\\ \Delta, B\to E\end{array}}{\Delta, A\supset B,\Sigma,\Gamma,C\to E}\,(\supset\to)$$

where the (cut) can be eliminated by the subinduction hypothesis.

Subcase 5.3. The right rule over the cut rule is $(\to\supset)$.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ P_1\\ \Gamma, A\to B\end{array}}{\Gamma\to A\supset B}\,(\to\supset) \quad \dfrac{\begin{array}{c}\vdots\ P_2\\ \Delta, A\supset B, C\to D\end{array}}{\Delta, A\supset B\to C\supset D}\,(\to\supset)}{\Delta,\Gamma\to C\supset D}\,(\text{cut})$$

In this case, the $A \supset B$ in the sequent $\Delta, A \supset B, C \to D$ must be introduced by $(\supset\to)$ in the proof P_2.

$$\begin{array}{c}\dfrac{\begin{array}{c}\vdots\ Q_1\\ \Pi, E\to A\end{array}\quad\begin{array}{c}\vdots\ Q_2\\ \Sigma, B\to F\end{array}}{\Sigma, A\supset B,\Pi,E\to F}\,(\supset\to)\\ \vdots\ P_2\\ \dfrac{\Delta, A\supset B, C\to D}{\Delta, A\supset B\to C\supset D}\,(\to\supset)\end{array}$$

Now we construct the following proof.

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\vdots\ Q_1\\ \Pi, E\to A\end{array}\quad\begin{array}{c}\vdots\ P_1\\ \Gamma, A\to B\end{array}}{\Gamma,\Pi,E\to B}\,(\text{cut})\quad\begin{array}{c}\vdots\ Q_2\\ \Sigma, B\to F\end{array}}{\Sigma,\Gamma,\Pi,E\to F}\,(\text{cut})\\ \vdots\ P_2'\\ \dfrac{\Delta,\Gamma, C\to D}{\Delta,\Gamma\to C\supset D}\,(\to\supset)\end{array}$$

where the two (cut)'s can be eliminated by the induction hypothesis, and P_2' is obtained from P_2 by replacing the $(A \supset B)$'s by Γ's appropriately.

$\blacksquare$

5.3 Notes

Merge formulation of BB′I A sequent calculus for BB′I is usually defined as a merge formulation (Section 7 of [3], [30]). Here we give the definition of the system **LBB′I** ([8], [30], [32]), a version using *guarded merge*. In the definition below, $\Delta \circ \Gamma$ denotes any sequence consisting of the members of Δ and Γ as multisets, in which Δ and Γ preserve their original orders and in which the rightmost formula is the rightmost formula of Γ. Initial sequents of **LBB′I** are of the following form:

$$p \to p \qquad \text{for any propositional variable } p.$$

Rules of inference of **LBB′I** are the following.

Cut rule:

$$\frac{\Gamma \to A \quad \Delta, A, \Sigma \to C}{\Delta \circ \Gamma, \Sigma \to C}\ (\text{cut})$$

where Δ is empty if Γ is empty.

Rules for the logical connective:

$$\frac{\Gamma \to A \quad \Delta, B, \Sigma \to C}{\Delta \circ (A \supset B \circ \Gamma), \Sigma \to C}\ (\supset\to) \qquad\qquad \frac{\Sigma, A \to B}{\Sigma \to A \supset B}\ (\to\supset)$$

where Γ is not empty.

In [30], the correspondence between the systems BB′I and **LBB′I** is shown as well as the cut-elimination theorem for **LBB′I**. Our proofs of Lemmas 5.3–5.8 are mainly based on the proofs of Lemmas 3.3–3.6 of [30].

LBB′I2 and dual-context sequent calculus The system **LBB′I2** was suggested by consideration of deleting structural rules from the dual-context sequent calculus **LBP2**, according to the observation that BB′I is a subsystem of BPC (cf. Section 4.3). Here a sequent $\Gamma, C \to A$ of **LBB′I2** is read as a dual-context sequent $\Gamma; C \to A$. Then the cut-elimination theorem for **LBB′I2** is proved using global proof transformation technique analogous to that used in the proof of the cut-elimination theorem for $\mathbf{DK}^I$ and **LBP2**.

Other noncommutative substructural logics A concise survey of noncommutative substructural logics is found in [35], where noncommutative substructural logics are defined as substructural logics that have neither exchange rules nor axioms for exchange, in general. If we take the system **LBB′I** with merge operation as a sequent calculus for BB′I, this logic is considered as one of noncommutative substructural logics. Since our sequent calculus **LBB′I2** is obtained from the system **LBP2** by deleting structural rules, further studies of the series of logics between BB′I and BPC may serve to reveal some properties of noncommutative substructural logics.

Chapter 6

Conclusion and further work

In this chapter we summarize the results of the thesis and indicate directions of further studies raised by our work.

The first goal of this thesis was to provide sequent calculi for subintuitionistic logics that are suitable to compare with systems for substructural logics, intending to develop the resource-conscious aspects of subintuitionistic logics. It has been achieved with the following results:

- In Chapter 3 we presented sequent calculi for subintuitionistic logics using dual-context style sequents which have been popular in the field of linear logic.

- In Chapter 5 we presented a sequent calculus for a noncommutative substructural logic BB′I. The system was obtained from a dual-context sequent calculus in Chapter 3 by deleting structural rules, according to the observation in Chapter 4.

Additionally, by virtue of the interpretation of dual-context sequents in Kripke models, we have seen possibilities for applying dual-context sequent calculi to formalizing modal logics (cf. Section 3.9).

Many further studies are derived from our observations both in the logic side and in the computational side. Below we will list some of them. In the logic side, the following problems are to be considered.

- Adding other connectives

  It is not so obvious to extend our sequent calculi to systems for meaningful predicate logics. A predicate logic extending BPC is discussed in [38]. Adding multiplicative connectives to our sequent calculi is also to be investigated.

- Dual-context sequent calculi for other modal logics

  In [42], Sato gives a cut-free system for the modal logic S5 based on sequents of the form $\Gamma; \Pi \to \Sigma; \Delta$, which are considered as an extension of dual-context sequents. These kinds of sequents may be effective in formalizing various other modal logics.

- BB′I as a minimal logic

  The logic BB′I, or equivalently called $T_{\to}-W$, is considered as a *minimal logic* in [3]. Our sequent calculus **LBB′I2** supports this view, and further work on extensions of **LBB′I2** may be important to the study of noncommutative substructural logics.

In the computational side, we have the following interesting subjects.

- Curry–Howard correspondence for dual-context sequent calculi

  Proof terms for dual-context sequent calculi and computational contents of global proof transformation in cut-elimination for the calculi are to be investigated on comparison with ones in, e.g. [22], [16], [48], [33].

- Applying **LBB′I2** to substantial work on linear logic

  Considering **LBB′I2** as a system for a noncommutative version of linear logic, we may apply **LBB′I2** or its extensions with other connectives to substantial work on linear logic, such as proof nets.

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