Glivenko theorems revisited

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Abstract

Glivenko-type theorems for substructural logics (over **FL**) are comprehensively studied in the paper (GO06b). Arguments used there are fully algebraic, as based on the fact that all substructural logics are *algebraizable* (see (GO06a) and also (GJKO07) for the details).

As a complementary work to the algebraic approach developed in (GO06b), we present here a concise, proof-theoretic approach to Glivenko theorems for substructural logics. This will show different features of these two approaches.

Key words: Glivenko's theorem, substructural logics, proof theoretic methods

1 A proof-theoretic analysis of the original Glivenko's theorem

In 1929, V. Glivenko showed the following in (Gli29). (We are concerned only with propositional logics in the present paper.)

Proposition 1 *For any formula α, α is provable in classical propositional logic iff $\neg\neg\alpha$ is provable in intuitionistic propositional logic.*

A standard proof-theoretic way of obtaining the above proposition is to show the next Proposition 2. Then the above propostion follows immediately from it, by taking the empty sequence of formulas for Γ and α for Δ. Here, **LK** and **LJ** are sequent calculi for classical logic and intuitionistic one, respectively. For more information, see e.g. (TS00) or (GJKO07).

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[1] This is an extended version of my talk "*Glivenko theorems without tears*" presented at the 41st MLG meeting at Kinosaki, Japan, December 2007. The author thanks Antonio Ledda for his helpful comments.

Proposition 2 *For any sequent* $\Gamma \Rightarrow \Delta$ *(of* **LK***), if it is provable in* **LK** *then the sequent* $\neg\Delta, \Gamma \Rightarrow$ *is provable in* **LJ***.*

Let us explain briefly how the standard proof of Proposition 2 goes and what are essential in it. The proof is carried out by using the induction of the length of a proof of $\Gamma \Rightarrow \Delta$ in **LK**. (It does not matter whether the proof contains cuts or not.) Depending on the form of the last inference I of the proof, we need to consider the following three cases.

(1) the inference I is either cut rule or one of *right* structural rules of **LK** (e.g. the right-contraction rule),
(2) I is either one of *left* structural rules or one of *left* rules for logical connectives of **LK** (e.g. the left rule for $\wedge$),
(3) I is one of *right* rules for logical connectives of **LK** (e.g. the right rule for $\wedge$).

For the first case, let I be the cut rule, and suppose that a sequent $\Gamma, \Pi \Rightarrow \Delta, \Sigma$ is obtained from $\Gamma \Rightarrow \Delta, \alpha$ and $\alpha, \Pi \Rightarrow \Sigma$. By the hypothesis of induction, both $\neg\alpha, \neg\Delta, \Gamma \Rightarrow$ and $\neg\Sigma, \alpha, \Pi \Rightarrow$ are provable in **LJ**. From the latter, it follows that $\neg\Sigma, \Pi \Rightarrow \neg\alpha$ is also provable in it. Applying the cut to this and the former sequent, $\neg\Delta, \neg\Sigma, \Gamma, \Pi \Rightarrow$ is also provable in **LJ**. This is the sequent what we want to get.

When I is one of right structural rules of **LK**, the corresponding lower sequent of **LJ** is obtained by applying the *left* structural rule J of **LJ** which corresponds to I. It means that J denotes *left-contraction rule* of **LJ**, if I is for instance the *right-contraction rule* of **LK**. Let us show the induction step when I is the *right-contraction rule*. Suppose that $\Gamma \Rightarrow \Delta, \alpha$ is obtained in **LK** from $\Gamma \Rightarrow \Delta, \alpha, \alpha$ by using I. Then, by the hypothesis of induction, we can assume that $\neg\alpha, \neg\alpha, \neg\Delta, \Gamma \Rightarrow$ is provable in **LJ**. Then, by applying the left-contraction rule, we can show that $\neg\alpha, \neg\Delta, \Gamma \Rightarrow$ is provable in **LJ**. Thus, the induction step holds in this case.

For the second case, the corresponding lower sequent of **LJ** can be obtained by applying the same rule I. Note that for every rule which belongs to this case, **LJ** has the same rule as **LK**. To show the induction step, we discuss here the case where I is the left rule either for $\wedge$ or for $\rightarrow$, for example. First, let us considerthe case where I is the left rule for $\wedge$. We suppose that $\alpha \wedge \beta, \Gamma \Rightarrow \Delta$ is obtained from $\alpha, \Gamma \Rightarrow \Delta$ by I. Then, by the hypothesis of induction, we can assume that $\alpha, \neg\Delta, \Gamma \Rightarrow$ is provable in **LJ**. Then, by applying the left rule for $\wedge$ we can show that $\alpha \wedge \beta, \neg\Delta, \Gamma \Rightarrow$ is provable in **LJ**.

When I is the left rule for $\rightarrow$, the argument becomes slightly complicated. We suppose that the sequent $\alpha \rightarrow \beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi$ is obtained by using I from both $\Gamma \Rightarrow \Delta, \alpha$ and $\beta, \Sigma \Rightarrow \Pi$. Then the hypothesis of induction says that both $\neg\alpha, \neg\Delta, \Gamma \Rightarrow$ and $\beta, \neg\Pi, \Sigma \Rightarrow$ are provable in **LJ**. From these two, we can derive both sequents $\neg\Delta, \Gamma \Rightarrow \neg\neg\alpha$ and $\neg\neg\beta, \neg\Pi, \Sigma \Rightarrow$ in **LJ**. Thus, $\neg\neg\alpha \to \neg\neg\beta, \neg\Delta, \neg\Pi, \Gamma, \Sigma \Rightarrow$, and hence $\neg\Delta, \neg\Pi, \Gamma, \Sigma \Rightarrow \neg(\neg\neg\alpha \to \neg\neg\beta)$ are provable in **LJ**. On the other hand, we can show that the sequent $\neg(\neg\neg\alpha \to \neg\neg\beta), \alpha \to \beta \Rightarrow$ is provable in **LJ**. Now, by applying the cut rule (and the exchange rule) to this sequent and the sequent $\neg\Delta, \neg\Pi, \Gamma, \Sigma \Rightarrow \neg(\neg\neg\alpha \to \neg\neg\beta)$, we get the required result. That is, $\alpha \to \beta, \neg\Delta, \neg\Pi, \Gamma, \Sigma \Rightarrow$ is provable in **LJ**. We note here that the sequent $\neg(\neg\neg\alpha \to \neg\neg\beta), \alpha \to \beta \Rightarrow$ is provable in **LJ** without using any of the contraction rule and the weakening rules.

Next, we consider the third case where I is one of right rules for logical connectives of **LK**. We need to examine how to get the required result for each rule which belongs to this case. We consider here the cases where I is the right rule for $\to$ and for $\wedge$. Remaining cases can be treated more easily.

Let I be the right rule for $\to$, and suppose that $\Gamma \Rightarrow \alpha \to \beta$ is obtained from $\alpha, \Gamma \Rightarrow \beta$. By the hypothesis of induction, $\neg\beta, \alpha, \Gamma \Rightarrow$ is provable in **LJ**. Then by using this, $\Gamma \Rightarrow \neg\neg\alpha \to \neg\neg\beta$ is shown to be provable. On the other hand, since $\neg(\alpha \to \beta), \neg\neg\alpha \to \neg\neg\beta \Rightarrow$, or alternatively

(A1) $\quad \neg(\alpha \to \beta) \Rightarrow \neg(\neg\neg\alpha \to \neg\neg\beta)$,

is provable in **LJ**, by the help of the *left-contraction rule* and the *weakening rules*. In fact, both sequents $\neg\alpha \Rightarrow \alpha \to \beta$ and $\beta \Rightarrow \alpha \to \beta$ are provable by using the right- and left-weakening rule, respectively. By taking the contraposition of either of them, both $\neg(\alpha \to \beta) \Rightarrow \neg\neg\alpha$ and $\neg(\alpha \to \beta) \Rightarrow \neg\beta$ are also provable. Then,

$$\dfrac{\dfrac{\neg(\alpha \to \beta) \Rightarrow \neg\neg\alpha \qquad \dfrac{\neg(\alpha \to \beta) \Rightarrow \neg\beta}{\neg\neg\beta, \neg(\alpha \to \beta) \Rightarrow}}{\neg(\alpha \to \beta), \neg(\alpha \to \beta), \neg\neg\alpha \to \neg\neg\beta \Rightarrow}}{\neg(\alpha \to \beta), \neg\neg\alpha \to \neg\neg\beta \Rightarrow}\ (\mathrm{left-c})$$

and thus by using this with the fact that $\Gamma \Rightarrow \neg\neg\alpha \to \neg\neg\beta$ is provable in **LJ**, we get the required result, i.e. $\neg(\alpha \to \beta), \Gamma \Rightarrow$ is provable in **LJ**.

For the case where I is the right rule for $\wedge$, it is necessary to show that

(A2) $\quad \neg(\alpha \wedge \beta) \Rightarrow \neg(\neg\neg\alpha \wedge \neg\neg\beta)$,

or equivalently, $\neg(\alpha \wedge \beta), \neg\neg\alpha \wedge \neg\neg\beta \Rightarrow$ is provable in **LJ**. In proving the sequent $\neg(\alpha \wedge \beta), \neg\neg\alpha \wedge \neg\neg\beta \Rightarrow$, both the *left-contraction rule* and the *left-weakening rule* are necessary. To see this, we note first that the sequent $\alpha, \beta \Rightarrow \alpha \wedge \beta$ is provable by using the left-weakening rule. From this, it follows that the sequent $\neg(\alpha \wedge \beta), \alpha, \beta \Rightarrow$ and therefore the sequent $\neg(\alpha \wedge \beta), \neg\neg\alpha, \neg\neg\beta \Rightarrow$ are provable. Hence, by continuing the proof in the following way we get our required result.

$$\dfrac{\dfrac{\dfrac{\neg(\alpha \wedge \beta), \neg\neg\alpha, \neg\neg\beta \Rightarrow}{\neg(\alpha \wedge \beta), \neg\neg\alpha \wedge \neg\neg\beta, \neg\neg\beta \Rightarrow}}{\neg(\alpha \wedge \beta), \neg\neg\alpha \wedge \neg\neg\beta, \neg\neg\alpha \wedge \neg\neg\beta \Rightarrow}}{\neg(\alpha \wedge \beta), \neg\neg\alpha \wedge \neg\neg\beta \Rightarrow} \text{ (left} - \text{c)}$$

2 Is Glivenko's result best possible?

The original Glivenko's result says that *Glivenko theorem* holds for intuitionistic logic (*relative to* classical logic **Cl**). Then it will be natural to explore the problem for which logics Glivenko theorem holds relative to **Cl**. It is trivial from Proposition 2 that Glivenko theorem holds also for every consistent extension of intuitionistic logic, i.e. every superintuitionistic logic except the inconsistent one. Can we find a logic weaker than intuitionistic logic for which Glivenko theorem holds? A breakthrough in this question was made by Cignoli and Torrens in (CT03), who showed that Glivenko theorem holds also for an extension **SBL** of Hájek's basic logic. Note that **SBL** is incomparable with intuitionistic logic.

Now let us consider the question for which substructural logic **L** in general Glivenko theorem holds (relative to **Cl**), i.e.

for any formula α, α is provable in **Cl** *if and only if $\neg\neg\alpha$ is provable in* **L**.

An answer is given in the paper (GO06b) by the present author with N. Galatos, in which algebraic methods are fully used. In the present paper we will take another way and show how proof-theoretic approach works well in solving the above problem. Though non-commutative cases can be treated in the same way, we restrict our attention in the following only to substructural logics over $\mathbf{FL_e}$, i.e. substructural logics which admit exchange rule, only for the brevity's sake. We assume here the familiarity with the sequent calculus $\mathbf{FL_e}$ for intuitionistic linear logic without exponentials (see e.g. (GJKO07)). Roughly speaking, it is the sequent calculus obtained from **LJ** by deleting both weakening rules and the left-contraction rule, and then adding the following rules for the connective $\cdot$, called *fusion*.

$$\frac{\Gamma, \alpha, \beta, \Sigma \Rightarrow \gamma}{\Gamma, \alpha \cdot \beta, \Sigma \Rightarrow \gamma} \text{ (left}\cdot\text{)} \qquad \frac{\Gamma \Rightarrow \alpha \quad \Sigma \Rightarrow \beta}{\Gamma, \Sigma \Rightarrow \alpha \cdot \beta} \text{ (right}\cdot\text{)}$$

For our present purpose, we will look up our proof-theoretic analysis given in the previous section, and examine whether and where, if any, our argument fails when we take $\mathbf{FL_e}$ instead of intuitionistic logic. Obviously, neither of sequents (**A1**) and (**A2**) are provable in $\mathbf{FL_e}$, since we need weakening rules and the left-contraction rule in order to derive them. Also, we meet some difficulties in the first case (1) in the previous section where I is either the right-weakening rule or the right-contraction rule (of **LK**), since $\mathbf{FL_e}$ has neither weakening rules nor the left-contraction rule. But, by a careful examination of the proof, we can see that our argument will still work well as long as we have the following restricted forms of these structural rules, instead of taking their full forms.

$$\frac{\alpha, \alpha, \Gamma \Rightarrow}{\alpha, \Gamma \Rightarrow} \qquad \frac{\Gamma \Rightarrow}{\alpha, \Gamma \Rightarrow}$$

Alternatively, these restricted rules can be expressed by the following sequents (**AC**) and (**AW**), respectively.

(**AC**) $\neg(\alpha \cdot \alpha) \Rightarrow \neg\alpha$,
(**AW**) $\neg\beta \Rightarrow \neg(\alpha \cdot \beta)$.

To sum up, we have the following theorem. Here $\mathbf{FL_e}^{\dagger}$ is the logic obtained from $\mathbf{FL_e}$ by adding all of (**A1**), (**A2**), (**AC**) and (**AW**) as axiom schemes (or, initial sequents). Note that each formula $\varphi \cdot \psi$ is understood as $\varphi \wedge \psi$ in classical logic.

Theorem 3 *Glivenko theorem holds for* $\mathbf{FL_e}^{\dagger}$ *relative to* **Cl**.

As a matter of fact, $\mathbf{FL_e}^{\dagger}$ is the weakest substructural logic over $\mathbf{FL_e}$ for which Glivenko theorem holds relative to **Cl**, as the next Theorem 5 says. (In the present paper, by a substructural logic over $\mathbf{FL_e}$ we mean an axiomatic extension of $\mathbf{FL_e}$, i.e. a sequent calculus obtained from $\mathbf{FL_e}$ by adding some axiom schemes as initial sequents. More precisely, we identify one logical calculus with another one, as long as the set of formulas provable in the first calculus coincides with that of the second. For, only the set of formulas provable in a given calculus does matter to Glivenko's theorem. So, in the present paper the word "*logics*" is used slightly in an ambiguous way, which denote sometimes calculi and sometimes sets of formulas, as long as no confusions may occur.) We can show the following.

Lemma 4 *For all formulas* φ *and* ψ, $\neg\neg(\varphi \to \psi) \Rightarrow \neg\psi \to \neg\varphi$ *is provable in* $\mathbf{FL_e}$.

Theorem 5 *For each (consistent) substructural logic* **L** *over* $\mathbf{FL_e}$, *Glivenko theorem holds for* **L** *relative to* **Cl** *if and only if* **L** *is an extension of* $\mathbf{FL_e}^{\dagger}$.

Proof. It is enough to assure that each of (**A1**), (**A2**), (**AC**) and (**AW**) is provable in **L**, whenever Glivenko theorem holds for **L**. For (**A1**), it is obvious that $(\neg\neg\alpha \to \neg\neg\beta) \to (\alpha \to \beta)$ is provable in **LK**, and hence by our assumption $\neg\neg((\neg\neg\alpha \to \neg\neg\beta) \to (\alpha \to \beta))$ is provable in **L**. Then, we have (**A1**) by using Lemma 4, since **L** contains $\mathbf{FL_e}$. Others can be shown similarly.

The above theorem says that $\mathbf{FL_e}^\dagger$ is the weakest logic among substructural logics (over $\mathbf{FL_e}$) for which Glivenko theorem holds relative to **Cl**. Thus, it is uniquely determined by **Cl**. So we denote it as $\mathbf{G}(\mathbf{Cl})$.

We have shown that the converse arrow of (**A1**) is provable in $\mathbf{FL_e}$. Similarly, the converse arrow of (**A2**) is also provable in $\mathbf{FL_e}$. Note that the following two sequents

(1) $\neg(\alpha \cdot \beta) \Rightarrow \neg(\neg\neg\alpha \cdot \neg\neg\beta)$
(2) $\neg(\alpha \vee \beta) \Rightarrow \neg(\neg\neg\alpha \vee \neg\neg\beta)$

and their converses are also provable in $\mathbf{FL_e}$. From these facts, we can derive the following algebraic consequence. Let us take any substructural logic which is an extension of $\mathbf{FL_e}$ satisfying both (**A1**) and (**A2**). Consider an arbitrary $\mathrm{FL_e}$-algebra $\mathbf{A}$ for this logic. Then, the following equation holds always in $\mathbf{A}$ (if we use the same symbols for both logical connectives and algebraic operations, by the abuse of language): for all $x, y \in A$

$$\neg(x \star y) = \neg(\neg\neg x \star \neg\neg y) \text{ for each } \star \in \{\rightarrow, \wedge, \cdot, \vee\}.$$

Define a binary relation $\simeq$ on $\mathbf{A}$ by $x \simeq y$ iff $\neg x = \neg y$. Then, by our assumption, $\simeq$ is shown to be a congruence relation on $\mathbf{A}$ which moreover satisfies $\neg\neg x \simeq x$. Thus, we can get the quotient algebra $(\mathbf{A}/\simeq)$ which is involutive. For further discussions on the present topic, see Section 4.4 of (GO06b).

3 Glivenko theorems for substructural logics

Until now, we consider only Glivenko theorems for a substructural logic **L** relative to **Cl**. It is interesting when Glivenko-type theorems hold in general for substructural logics. For substructural logics **L** and **K**, we say that *Glivenko theorem holds for* **L** *relative to* **K** if

for any formula α, α is provable in **K** *iff $\neg\neg\alpha$ is provable in* **L**.

We say that a logic **K** is *involutive* whenever $\neg\neg\alpha \rightarrow \alpha$ is provable in it for every formula α. Let us consider two examples of involutive substructural logics, $\mathbf{InFL_e}$ and $\mathbf{InFL_{ew}}$. They are obtained from $\mathbf{FL_e}$ and $\mathbf{FL_{ew}}$ ($\mathbf{FL_e}$ with weakening rules), respectively, by adding the axiom scheme of involution $\neg\neg\varphi \rightarrow \varphi$. Obviously, $\mathbf{InFL_e}$ is the weakest involutive logic which contains $\mathbf{FL_e}$. Alternatively, they are formulated by sequent calculi obtained from **LK** (but with rules for fusion) by deleting both contraction rules and weakening rules (i.e. **MALL**), and by deleting only contraction rules (i.e. Grishin's logic), respectively.

Theorem 6 *For each substructural logic* $\mathbf{L}$ *over* $\mathbf{FL_e}$*, Glivenko theorem holds for* $\mathbf{L}$ *relative to* $\mathbf{InFL_e}$ $(\mathbf{InFL_{ew}})$ *if and only if* $\mathbf{L}$ *is an extension of the calculus obtained from* $\mathbf{FL_e}$ *with axiom schemes* $(\mathbf{A1}), (\mathbf{A2})$ *(and* $(\mathbf{AW})$*) and is included by* $\mathbf{InFL_e}$ $(\mathbf{InFL_{ew}}$*, respectively).*

The above theorem is shown similarly to Theorem 5. We note that $\mathbf{L}$ must be included by $\mathbf{InFL_e}$ when Glivenko theorem holds for $\mathbf{L}$ relative to $\mathbf{InFL_e}$. In fact, if a formula β is provable in $\mathbf{L}$ then $\neg\neg\beta$ is also provable in it. By the assumption that Glivenko theorem holds for $\mathbf{L}$ relative to $\mathbf{InFL_e}$, β must be provable in $\mathbf{InFL_e}$.

By using the notation introduced in the previous section, we can express the calculus obtained from $\mathbf{FL_e}$ by adding both $(\mathbf{A1})$ and $(\mathbf{A2})$ as $\mathbf{G}(\mathbf{InFL_e})$. Our result can be extended easily to every involutive substructural logic over $\mathbf{FL_e}$ (and thus, it is over $\mathbf{InFL_e}$).

Theorem 7 *Suppose that a substructural logic* $\mathbf{K}$ *is axiomatized by axiom schemes* $\{\alpha_i : i \in I\}$ *over* $\mathbf{InFL_e}$*. Then, for each substructural logic* $\mathbf{L}$ *over* $\mathbf{FL_e}$*, Glivenko theorem holds for* $\mathbf{L}$ *relative to* $\mathbf{K}$ *if and only if* $\mathbf{L}$ *is an extension of the calculus obtained from* $\mathbf{G}(\mathbf{InFL_e})$ *with axiom schemes* $\{\neg\neg\alpha_i : i \in I\}$ *and is included by* $\mathbf{K}$*.*

We can prove this theorem almost in the same way as the proof of Theorem 6, but adding the following simple observation to it. That is, for each initial sequent of $\mathbf{K}$ of the form $\Rightarrow \alpha_i$ the corresponding sequent $\neg\alpha_i \Rightarrow$ is provable in $\mathbf{L}$. In fact, this is so since $\Rightarrow \neg\neg\alpha_i$ is an initial sequent of $\mathbf{L}$ by our assumption.

Let $\mathbf{G}(\mathbf{K})$ be the extension obtained from $\mathbf{G}(\mathbf{InFL_e})$ by adding axiom schemes $\{\neg\neg\alpha_i : i \in I\}$. As a consequence of the above Theorem 7, we have that when $\mathbf{K}$ is finitely axiomatized over $\mathbf{InFL_e}$ then so is $\mathbf{G}(\mathbf{K})$ over $\mathbf{G}(\mathbf{InFL_e})$ (and hence over $\mathbf{InFL_e}$). Here, we give some comments on the form of axiom schemes of $\mathbf{G}(\mathbf{K})$ in Theorem 7. It is obvious that $\mathbf{LK}$ is obtained from $\mathbf{InFL_e}$ by adding the following axiom schemes $(\mathbf{C})$ and $(\mathbf{W})$ for contraction and weakening:

$(\mathbf{C})$ $\quad \alpha \to (\alpha \cdot \alpha),$
$(\mathbf{W})$ $\quad (\alpha \cdot \beta) \to \beta.$

Thus by Theorem 7, $\mathbf{G}(\mathbf{Cl})$ (i.e. $\mathbf{G}(\mathbf{LK})$) must be axiomatized by the following axiom schemes over $\mathbf{G}(\mathbf{InFL_e})$:

$(\mathbf{AC}^\dagger)$ $\quad \neg\neg(\alpha \to (\alpha \cdot \alpha)),$
$(\mathbf{AW}^\dagger)$ $\quad \neg\neg((\alpha \cdot \beta) \to \beta).$

Then, what are relations between these schemes ($\mathbf{AC}^\dagger$) and ($\mathbf{AW}^\dagger$) and sequents (**AC**) and (**AW**) in Theorems 5? By Lemma 4 we know that the formers implies the latters, respectively. Also, the converse implication holds as long as (**A1**) is assumed, as shown in the following.

Lemma 8 *For all formulas φ and ψ, $\neg\varphi \to \neg\psi \Rightarrow \neg\neg(\psi \to \varphi)$ is provable in $\mathbf{FL_e}$ with* (**A1**)*.*

Proof. It is easy to derive the sequent $\neg(\neg\neg\varphi \to \neg\neg\psi), \neg\psi \to \neg\varphi \Rightarrow$ in $\mathbf{FL_e}$. By the cut rule applying to this with the sequent $\neg(\varphi \to \psi) \Rightarrow \neg(\neg\neg\varphi \to \neg\neg\psi)$ which is an instance of (**A1**), the sequent $\neg(\varphi \to \psi), \neg\psi \to \neg\varphi \Rightarrow$ is derived. Thus, we have the required result.

Our final remark is on the involutiveness. As mentioned before, a logic **K** is said to be involutive when $\neg\neg\alpha \to \alpha$ is provable in it for every formula α. According to the definitions in Section 4 of (GO06b), **K** is *weakly involutive* when $\neg\neg\alpha \vdash_{\mathbf{K}} \alpha$ for every formula α, and is *Glivenko involutive* when for every formula α if $\neg\neg\alpha$ is provable in **K** then α is provable in it. Here, the deducibility relation $\Pi \vdash_{\mathbf{K}} \varphi$ (with a set of formulas Π) means that the sequent $\Rightarrow \varphi$ can be provable in the system obtained from **K** by adding $\Rightarrow \delta$ as new initial sequents for each $\delta \in \Pi$. Note that Theorem 7 is shown under the assumption that **K** is involutive. But, by a close examination of the proof of Theorem 5, it holds still under a weaker assumption that **K** is Glivenko involutive. That is, *Theorem 7 holds even if* **K** *is a Glivenko involutive substructural logic over* $\mathbf{FL_e}$. (See Corollary 4.7 of (GO06b).)

For substructural logics **L** and **K**, we say that *deductive Glivenko theorem* holds for **L** relative to **K** if

for any set $\Pi \cup \{\alpha\}$ of formulas, $\Pi \vdash_{\mathbf{K}} \alpha$ if and only if $\Pi \vdash_{\mathbf{L}} \neg\neg\alpha$.

Then, by modifying slightly the proof of Theorem 7, we have the following. (See also Corollary 4.7 of (GO06b).)

Theorem 9 *Suppose that* **K** *is a weakly involutive substructural logic over* $\mathbf{FL_e}$*. Then, for each substructural logic* **L** *over* $\mathbf{FL_e}$*, deductive Glivenko theorem holds for* **L** *relative to* **K** *if and only if* **L** *is an extension of* $\mathbf{G}(\mathbf{K})$ *and is included by* **K***.*

In fact, to show this it is enough to consider moreover the case where a sequent of the form $\Rightarrow \gamma$ for $\gamma \in \Pi$ appears as an initial sequent in the *derivation* of $\Pi \vdash_{\mathbf{K}} \alpha$. Similarly to the proof of Theorem 7, we can get a derivation of $\neg\neg\Pi \vdash_{\mathbf{L}} \neg\neg\alpha$. But each sequent of the form $\Rightarrow \neg\neg\gamma$ follows from $\Rightarrow \gamma$. Thus, $\Pi \vdash_{\mathbf{L}} \neg\neg\alpha$ holds. The assumption that **K** is a weakly involutive is necessary when we show that deductive Glivenko theorem holds for **L** relative to **K**, if **L** is an extension of $\mathbf{G}(\mathbf{K})$ and is included by **K**. In fact, if $\Pi \vdash_{\mathbf{L}} \neg\neg\alpha$ then

$\Pi \vdash_{\mathbf{K}} \neg\neg\alpha$, as **K** includes **L**. Then, since **K** is a weakly involutive, we have $\Pi \vdash_{\mathbf{K}} \alpha$.

We say that *equational Glivenko theorem* holds for **L** relative to **K** if

> *for all formulas α and β, $\alpha \to \beta$ is provable in* **K** *if and only if $\neg\beta \to \neg\alpha$ is provable in* **L**.

Note that the above condition can be replaced by the following:

> *for all formulas α and β, $\alpha \leftrightarrow \beta$ is provable in* **K** *if and only if $\neg\beta \leftrightarrow \neg\alpha$ is provable in* **L**.

We remark that our proof of Theorem 5 is essentially a proof of equational Glivenko theorem relative to **Cl**, if we make a minor change of the last part of the proof, i.e. using the definition of equational Glivenko theorem and not applying Lemma 4. We can show the following similarly to Theorem 7, using Lemmas 4 and 8.

Theorem 10 *Suppose that* **K** *is an involutive substructural logic over* $\mathbf{FL_e}$*. Then, for each substructural logic* **L** *over* $\mathbf{FL_e}$*, equational Glivenko theorem holds for* **L** *relative to* **K** *if and only if* **L** *is an extension of* $\mathbf{G}(\mathbf{K})$ *and is included by* **K**.

When we show that equational Glivenko theorem holds for **L** relative to **K** if **L** is an extension of $\mathbf{G}(\mathbf{K})$ and is included by **K**, the assumption that **K** is involutive is used in the following way. Suppose that $\neg\beta \to \neg\alpha$ is provable in **L**. Since **K** includes **L**, $\neg\neg\alpha \to \neg\neg\beta$ and hence $\alpha \to \neg\neg\beta$ are provable in **K**. Since **K** is involutive, $\neg\neg\beta \to \beta$ is also provable in it, and hence $\alpha \to \beta$ is provable in **K**.

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