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Author(s)	Kihara, Hitoshi; Ono, Hiroakira
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Description	

Interpolation properties, Beth definability properties and amalgamation properties for substructural logics

Hitoshi Kihara and Hiroakira Ono
School of Information Science
Japan Advanced Institute of Science and Technology
Asahidai, Nomi, Ishikawa, 923-1292, Japan
{h-kihara,ono}@jaist.ac.jp

Abstract. This paper develops a comprehensive study of various types of interpolation properties and Beth definability properties for substructural logics, and their algebraic characterizations through amalgamation properties and epimorphisms surjectivity. In general, substructural logics are algebraizable but lack many of the basic logical properties that modal and superintuitionistic logics enjoy (cf. [7]). In this case, careful examination is necessary to see how these logical and algebraic properties are related. To describe these relations exactly, many variants of interpolation properties and Beth definability properties, and also corresponding algebraic properties, are introduced. Because of their generality, the results reported here hold not only for substructural logics, but can be extended to a more general setting such as abstract algebraic logic [1], [6].

1 Introduction

This paper develops a comprehensive study of various types of interpolation properties and Beth definability properties for substructural logics and their algebraic characterizations. Much work has already been done in this direction for modal and superintuitionistic logics (see [7] for general information), for abstract algebraic logic (see [1] and [6] for general information and historical notes), and also in an abstract model-theoretic framework (see e.g. [4], [31]).

On the other hand, while the algebraization theorem holds for substructural logics, in general they lack many basic logical properties such as the (local) deduction theorem possessed by modal and superintuitionistic logics, and therefore we cannot expect that results for modal or superintuitionistic logics hold also for substructural logics in a similar way. In fact, we can give many examples where two equivalent conditions for modal or superintuitionistic logics are not equivalent for substructural logics. Thus, a careful and close examination is necessary to see precisely how logical and algebraic properties are related to each other. By introducing many variants of interpolation properties, Beth definability properties and algebraic properties, we obtain sharper results on the correspondence between these logical properties and algebraic properties. (See e.g. Figure 10 for relationships between interpolation properties and amalgamation properties, and Figure 12 for relationships between various Beth definability properties.)

From a close examination of the proofs of our results, we can see that only a few logical (algebraic) properties specific to substructural logics (**FL**-algebras or residuated lattices, respectively) are required. Therefore, we expect that these results can be stated in a more general setting, e.g. for abstract algebraic logic. We can safely say that our framework using substructural logics is quite adequate for the study of interpolation properties and definability properties, as it can connect the study of individual nonclassical logics in a natural way with abstract algebraic logic.

After giving preliminaries for substructural logics and residuated lattices in Section 2, in Section

3, we introduce the Craig interpolation property (CIP), deductive interpolation property (DIP) and their strong versions SCIP and SDIP, and establish some basic syntactic results. From a technical point of view, the Robinson property (RP) introduced there will play an essential role in Section 4, where it is shown that a logic has the (strong, super) RP if and only if the corresponding variety has the (strong, super) AP, respectively. On the other hand, a new algebraic notion, called the generalized amalgamation property (GAP), is introduced there to give an algebraic characterization of the CIP and the DIP, that are weaker than the superRP and the RP, respectively. Section 5 is devoted to an algebraic characterization of the SCIP and SDIP. For this purpose, the commutative homomorphism diagrams (CHD) property is introduced, which is obtained from the AP by replacing "embeddings" by "homomorphisms". Various types of the CHD are examined and their syntactic characterizations are given. In Section 6 variants of the Beth definability property (BDP) are introduced and their algebraic characterizations via variants of epimorphisms surjectivity are discussed. Relationships between the strong RP, the projective BDP and the BDP are clarified. Moreover, it is shown that the projective BDP can be expressed as a limit of a sequence of natural extensions of the BDP. Some remarks on further research, including open problems, are given in Section 7.

The present work was originally motivated by discussions of the authors with F. Montagna and D. Mundici on relationships between the CIP and the DIP. Some initial results were announced in the doctoral dissertation of the first author [15] and also in some international conferences. Some results have also appeared in Chapter 5 of [8] but in a weaker form. Also, algebraic characterizations of logical properties related to interpolation properties, like Halldén completeness and Maksimova's variable separation property are discussed in our paper [16]. We would like to express our thanks to T. Kowalski, F. Montagna and D. Mundici for their useful discussions in the early stages of our study, and J. Czelakowski, L. Maksimova and the anonymous referee for their helpful comments and suggestions on earlier drafts of this paper.

2 Residuated lattices and substructural logics

In the present paper, we assume a certain familiarity with definitions and basic results introduced in [9]. To make our paper self-contained, we will briefly describe some of these ideas below, referring to [8] for more details.

A *residuated lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ such that $\langle A, \wedge, \vee\rangle$ is a lattice, $\langle A, \cdot, 1\rangle$ is a monoid, and the monoid operation $\cdot$ is *residuated* with respect to the order by both the left- and right-division operations $\backslash, /$, i.e., for all $x, y, z \in A$,

$$x \cdot y \leq z \iff x \leq z/y \iff y \leq x\backslash z.$$

An $\mathbf{FL}$-*algebra* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ with a residuated lattice $\langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ and an arbitrary element 0 of A. An $\mathbf{FL}$-algebra is an $\mathbf{FL_e}$-*algebra* if the monoid operation $\cdot$ is commutative. An $\mathbf{FL}$-algebra is an $\mathbf{FL_{ew}}$-*algebra* if it is an $\mathbf{FL_e}$-algebra satisfying $0 \leq x \leq 1$ for each element x. It is easy to show that in any $\mathbf{FL}$-algebra the commutativity of the monoid operation is equivalent to the condition $x\backslash y = y/x$. In this case we sometimes denote $x\backslash y$ (and hence y/x also) by $x \to y$. It is easy to see that the class $\mathcal{FL}$ of $\mathbf{FL}$-algebras forms a variety. We denote the subvariety lattice of $\mathcal{FL}$ by $\mathbf{S}(\mathcal{FL})$.

We adopt the convention that the monoid operation has priority over the division operations, which have priority over the lattice operations. So, for example, we write $x/yz \wedge u\backslash v$ for $[x/(y \cdot z)] \wedge (u\backslash v)$.

The class of $\mathbf{FL}$-algebras provides algebraic semantics for the substructural logic $\mathbf{FL}$, called the full Lambek calculus. For the precise definition of the sequent calculus $\mathbf{FL}$, see [9]. By a substructural logic (over $\mathbf{FL}$), we mean an axiomatic extension of $\mathbf{FL}$. Here, a sequent calculus is an *axiomatic extension* of **FL** with axioms $\{\alpha_j : j \in J\}$ if it is obtained from **FL** by adding each sequent of the form $\Rightarrow \varphi$ as a new initial sequent, where φ is any *substitution instance* of some axiom α_j. When a substructural logic is obtained from **L** by adding axioms $\{\beta_k : k \in K\}$, it is denoted by $\mathbf{L} + \{\beta_k : k \in K\}$. As usual, we identify a given substructural logic **L** with the set of formulas provable in it.

The substructural logic $\mathbf{FL_e}$ ($\mathbf{FL_{ew}}$) is usually introduced as a sequent calculus obtained from **FL** by adding the exchange rule (the exchange rule, and left- and right-weakening rules, respectively). It can be easily seen that both of them are in fact axiomatic extensions of **FL**. Obviously, the set of all substructural logics over **FL** (as sets of formulas) forms a lattice **SL**.

For an arbitrary class $\mathcal{K}$ of **FL**-algebras, let $\mathsf{L}(\mathcal{K})$ be the set of formulas that are *valid* in all **FL**-algebras in $\mathcal{K}$. Then, we can show that $\mathsf{L}(\mathcal{K})$ is a substructural logic for any $\mathcal{K}$. When $\mathcal{K}$ consists of a single algebra $\mathbf{A}$, we denote $\mathsf{L}(\mathcal{K})$ by $\mathsf{L}(\mathbf{A})$. Conversely, for a given substructural logic **L**, let $\mathsf{V}(\mathbf{L})$ be the class of all **FL**-algebras in which every inequation $1 \leq \varphi$ holds for $\varphi \in \mathbf{L}$. Then $\mathsf{V}(\mathbf{L})$ belongs to $\mathbf{S}(\mathcal{FL})$. Moreover, we have the following.

PROPOSITION 1 *The maps* $\mathsf{L} : \mathbf{S}(\mathcal{FL}) \to \mathbf{SL}$ *and* $\mathsf{V} : \mathbf{SL} \to \mathbf{S}(\mathcal{FL})$ *are mutually inverse, dual lattice isomorphisms.*

For a set of formulas Γ and a formula ψ, we say that ψ is *deducible* from Γ in **FL** ($\Gamma \vdash_{\mathbf{FL}} \psi$, in symbols), when there is a proof of $\Rightarrow \psi$ in the calculus **FL** with new initial sequents of the form $\Rightarrow \gamma$ added for each $\gamma \in \Gamma$. Unlike the definition of an axiomatic extension, here we cannot use a sequent $\Rightarrow \delta$ as an initial sequent when δ is a substitution instance of a formula in Γ, except in the case where δ itself belongs to Γ.

The deducibility relation is naturally extended to each substructural logic **L** in the following way. For a set of formulas $\Gamma \cup \{\psi\}$, we write $\Gamma \vdash_{\mathbf{L}} \psi$ when $\Gamma \cup \mathbf{L} \vdash_{\mathbf{FL}} \psi$. Then we can show that the relation $\vdash_{\mathbf{L}}$ is a finitary, substitution invariant consequence relation (in the sense of abstract algebraic logic). See [9] for the details.

For formulas α, φ, the *left conjugate* $\lambda_\alpha(\varphi)$ and the *right conjugate* $\rho_\alpha(\varphi)$ of φ (with respect to α) are formulas $(\alpha\backslash\varphi\alpha) \wedge 1$ and $(\alpha\varphi/\alpha) \wedge 1$, respectively. An *iterated conjugate* γ of φ is a composition of left and right conjugates of the form $\delta_{\alpha_1}(\delta_{\alpha_2}(\cdots\delta_{\alpha_m}(\varphi)\cdots))$ for some formulas $\alpha_1, \ldots, \alpha_m$ (called *parameters*), where each δ_{α_i} is either a left or a right conjugate. The following result, called the *parameterized local deduction theorem*, is shown in [9]. Here, Π means a finite product of formulas by the fusion.

PROPOSITION 2 *If* $\Gamma \cup \Sigma \cup \{\psi\}$ *is a set of formulas and* **L** *is a logic over* **FL** *then*

$$\Gamma, \Sigma \vdash_{\mathbf{L}} \psi \textit{ iff } \Gamma \vdash_{\mathbf{L}} (\Pi_{i=1}^{n} \gamma_i(\varphi_i))\backslash\psi,$$

for some n*, some iterated conjugates* γ_i *of a formula* $\varphi_i \in \Sigma$ *for each* $i \leq n$*. In particular, if* **L** *is over* $\mathbf{FL_e}$ *then*

$$\Gamma, \Sigma \vdash_{\mathbf{L}} \psi \textit{ iff } \Gamma \vdash_{\mathbf{L}} (\Pi_{i=1}^{n} (\varphi_i \wedge 1)) \to \psi,$$

for some n *and some* $\varphi_i \in \Sigma$ *for each* $i \leq n$*. Moreover, if* **L** *is a logic over* $\mathbf{FL_{ew}}$ *then*

$$\Gamma, \Sigma \vdash_{\mathbf{L}} \psi \textit{ iff } \Gamma \vdash_{\mathbf{L}} (\Pi_{i=1}^{n} \varphi_i) \to \psi,$$

for some n *and some* $\varphi_i \in \Sigma$ *for each* $i \leq n$*.*

In the above, iterated conjugates, and therefore parameters do not appear in the second and the third results. Hence, they are called simply the *local deduction theorem.* They are derived from the first result using the fact that every formula φ follows from each left and right conjugate of φ with respect to 1, and also that φ implies both $(\alpha\backslash\varphi\alpha)$ and $(\alpha\varphi/\alpha)$ for any formula α in $\mathbf{FL_e}$.

The following proposition, called the *algebraization theorem*, is fundamental for considering relationships between logic and algebra, although we omit a detailed explanation here. Note that Proposition 1 follows from the following proposition. For further information, consult [9].

PROPOSITION 3 *For every substructural logic* $\mathbf{L}$ *over* $\mathbf{FL}$*, the deducibility relation* $\vdash_{\mathbf{L}}$ *is algebraizable with defining equation* $1 = x \wedge 1$ *and equivalence formula* $x\backslash y \wedge y\backslash x$*. An equivalent algebraic semantics for* $\vdash_{\mathbf{L}}$ *is the variety* $\mathsf{V}(\mathbf{L})$*.*

Let $\mathbf{A}$ be an $\mathbf{FL}$-algebra. Then, a subset F of A is a *deductive filter* or simply *filter*, if it satisfies the following;

1. $1 \leq x$ implies $x \in F$,
2. $x, x\backslash y \in F$ implies $y \in F$,
3. $x \in F$ implies $x \wedge 1 \in F$,
4. $x \in F$ implies $a\backslash xa, ax/a \in F$ for any a.

For a subset S of A, let $\mathrm{Fg}_{\mathbf{A}}(S)$ be the filter generated by S, i.e. the smallest filter containing S. Because of close resemblances between the deducibility and filter generation which comes from the algebraization theorem, we have the following lemma (cf. [9]). Here, algebraic analogues of *conjugates* are used. That is, for an $\mathbf{FL}$-algebra $\mathbf{A}$ and $a, x \in A$, the left conjugate $\lambda_a(x)$ of x with respect to a is the element $(a\backslash xa) \wedge 1$. Right conjugates and iterated conjugates are defined in a similar way.

LEMMA 4 *Let* $\mathbf{A}$ *be an* $\mathbf{FL}$*-algebra and* S *a subset of* A*. Then*

> $Fg_{\mathbf{A}}(S) = \{x \in A : \Pi_{i=1}^{n}\gamma_i(s_i) \leq x$ *for some* n*, for some* $s_i \in S$*, and some iterated conjugates* γ_i *with parameters from* $A\}$*.*

In particular, if $\mathbf{A}$ *is an* $\mathbf{FL_e}$*-algebra then*

$$Fg_{\mathbf{A}}(S) = \{x \in A : \Pi_{i=1}^{n}(s_i \wedge 1) \leq x \text{ for some } n \text{ and for some } s_i \in S\}.$$

Also, if $\mathbf{A}$ *is an* $\mathbf{FL_{ew}}$*-algebra then*

$$Fg_{\mathbf{A}}(S) = \{x \in A : \Pi_{i=1}^{n}s_i \leq x \text{ for some } n \text{ and for some } s_i \in S\}.$$

It is easy to show that all filters of an $\mathbf{FL}$-algebra $\mathbf{A}$ form a lattice denoted by $\mathbf{Fil}(\mathbf{A})$. Let $\mathbf{Con}(\mathbf{A})$ be the congruence lattice of $\mathbf{A}$. Then the following holds (see [9]).

LEMMA 5 *Let* $\mathbf{A}$ *be an* $\mathbf{FL}$*-algebra. Then, for* $F \in \mathbf{Fil}(\mathbf{A})$ *and* $\theta \in \mathbf{Con}(\mathbf{A})$*, the maps* $F \mapsto \Theta_F = \{(a,b) \in A^2 \mid a\backslash b \wedge b\backslash a \in F\}$ *and* $\theta \mapsto F_\theta = \{a \in A \mid (a \wedge 1, 1) \in \theta\}$ *are mutually inverse lattice isomorphisms between* $\mathbf{Fil}(\mathbf{A})$ *and* $\mathbf{Con}(\mathbf{A})$*.*

We say that an $\mathbf{FL}$-algebra $\mathbf{A}$ has the *congruence extension property* (CEP) if for any subalgebra $\mathbf{B}$ of $\mathbf{A}$ and $\theta \in \mathbf{Con}(\mathbf{B})$ there is a $\varphi \in \mathbf{Con}(\mathbf{A})$ such that $\theta = \varphi \cap B^2$. A class $\mathcal{K}$ of $\mathbf{FL}$-algebras has the CEP if every algebra in it has the CEP.

For any class of algebras $\mathcal{K}$ and a set of variables X, we denote the $\mathcal{K}$*-free algebra* by $\mathbf{Fm}_{\mathcal{K}}(X)$, or simply $\mathbf{Fm}(X)$ when the context is clear. Then the following holds.

LEMMA 6 *For each substructural logic* $\mathbf{L}$*, a set of variables* X *and a set of formulas* Γ*, define a binary relation* $\equiv_\Gamma$ *on* $\mathbf{Fm}_{\mathsf{V}(\mathbf{L})}(X)$ *by*

$$\alpha \equiv_\Gamma \beta \quad \textit{iff} \quad \Gamma \vdash_{\mathbf{L}} (\alpha\backslash\beta) \wedge (\beta\backslash\alpha).$$

Then, $\equiv_\Gamma$ *is a congruence relation on* $\mathbf{Fm}_{\mathsf{V}(\mathbf{L})}(X)$*.*

3 Interpolation properties

In this section, we discuss relationships between several types of interpolation properties for substructural logics.

3.1 Craig interpolation property and the deductive interpolation property

DEFINITION 1 *A substructural logic* $\mathbf{L}$ *over* $\mathbf{FL}$ *has the Craig interpolation property (CIP), if for all formulas* φ, ψ, *whenever* $\varphi\backslash\psi$ *is provable in* $\mathbf{L}$, *there exists a formula* δ *such that*

1. *both* $\varphi\backslash\delta$ *and* $\delta\backslash\psi$ *are provable in* $\mathbf{L}$,
2. $var(\delta) \subseteq var(\varphi) \cap var(\psi)$,

where $var(\gamma)$ *denotes the set of propositional variables in a formula* γ.

Now, we introduce an extension of the CIP.

DEFINITION 2 *A substructural logic* $\mathbf{L}$ *has the strong Craig interpolation property (SCIP), if for any set of formulas* $\Gamma \cup \Sigma \cup \{\varphi, \psi\}$, *if* $\Gamma, \Sigma \vdash_{\mathbf{L}} \varphi\backslash\psi$, *then there exists a formula* δ *such that*

1. $\Gamma \vdash_{\mathbf{L}} \varphi\backslash\delta$ *and* $\Sigma \vdash_{\mathbf{L}} \delta\backslash\psi$,
2. $var(\delta) \subseteq var(\Gamma \cup \{\varphi\}) \cap var(\Sigma \cup \{\psi\})$.

Since $\vdash_{\mathbf{L}}$ is finitary and conjunctive for each substructrual logic $\mathbf{L}$, the modifications of the SCIP obtained by stipulating that each of the sets Γ and Σ is finite or a single formula, are equivalent to the SCIP. This holds also for all of the interpolation properties discussed below.

By replacing the provability of implications in the definition of the CIP by the deducibility, we can introduce some other interpolation properties.

DEFINITION 3 *A substructural logic* $\mathbf{L}$ *has the deductive interpolation property (DIP), if for any set of formulas* $\Gamma \cup \{\psi\}$, *if* $\Gamma \vdash_{\mathbf{L}} \psi$, *then there exists a formula* δ *such that*

1. $\Gamma \vdash_{\mathbf{L}} \delta$ *and* $\delta \vdash_{\mathbf{L}} \psi$,
2. $var(\delta) \subseteq var(\Gamma) \cap var(\psi)$.

The DIP [1] is called also the *turnstile interpolation property* by Madarász [18], and the *interpolation property for deducibility* by Maksimova [25].

Next, we introduce an extension of the DIP.

DEFINITION 4 *A substructural logic* $\mathbf{L}$ *has the strong deductive interpolation property (SDIP), if for any set of formulas* $\Gamma \cup \Sigma \cup \{\psi\}$, *if* $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$, *then there exists a set of formulas* Δ *such that*

1. $\Gamma \vdash_{\mathbf{L}} \delta$ *for all* $\delta \in \Delta$, *and* $\Delta, \Sigma \vdash_{\mathbf{L}} \psi$,
2. $var(\Delta) \subseteq var(\Gamma) \cap var(\Sigma \cup \{\psi\})$.

The SDIP is called the *Maehara interpolation property* in [6] and GINT in [31]. Again, since $\vdash_{\mathbf{L}}$ is finitary and conjunctive, we can take a single formula δ for an interpolant, instead of a set Δ, in the definition of the SDIP, and thus δ is supposed to satisfy that

1. $\Gamma \vdash_{\mathbf{L}} \delta$ and $\delta, \Sigma \vdash_{\mathbf{L}} \psi$,

[1] Note that the DIP is called the CIP in [6].

2. $\mathrm{var}(\delta) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\Sigma \cup \{\psi\})$.

In [36], Wroński introduced the notion of the equational interpolation property of a class of algebras. In the present context, this is stated as follows. A subvariety $\mathcal{V}$ of $\mathcal{FL}$ has the *equational interpolation property* (eqIP), if for every set of equations $G \cup E \cup \{\varepsilon\}$, whenever $G, E \models_{\mathcal{V}} \varepsilon$, there exists a set of equations D such that

1. $G \models_{\mathcal{V}} \delta$ for all $\delta \in D$, and $D, E \models_{\mathcal{V}} \varepsilon$,
2. $\mathrm{var}(D) \subseteq \mathrm{var}(G) \cap \mathrm{var}(E \cup \{\varepsilon\})$.

This property with the extra assumption that E is empty is introduced and discussed also in [14] and [3]. By the algebraization theorem, the eqIP of a variety $\mathsf{V}(\mathbf{L})$ can be translated into the SDIP of a substructural logic $\mathbf{L}$. More precisely,

PROPOSITION 7 *A substructural logic* $\mathbf{L}$ *has the SDIP iff the variety* $\mathsf{V}(\mathbf{L})$ *has the eqIP.*

The following result shows some relationships between these interpolation properties.

LEMMA 8 *For every substructural logic, the following hold.*

(1) The SCIP implies the CIP.

(2) The SDIP implies the DIP.

(3) The SCIP implies the SDIP.

Proof. The implication (1) is clear since the CIP is a special case of the SCIP when Γ and Σ are empty sets. Similarly, the implication (2) can be shown. To show (3), suppose that $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$ holds. Then, $\Gamma, \Sigma \vdash_{\mathbf{L}} 1\backslash\psi$ also holds. Thus, by the SCIP, there exists a formula δ with $\mathrm{var}(\delta) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\Sigma \cap \psi)$ such that $\Gamma \vdash_{\mathbf{L}} 1\backslash\delta$ and $\Sigma \vdash_{\mathbf{L}} \delta\backslash\psi$. From them, both $\Gamma \vdash_{\mathbf{L}} \delta$ and $\delta, \Sigma \vdash_{\mathbf{L}} \psi$ follow. □

Thus we can summarize relations among these interpolation properties in Figure 1.

$$\begin{array}{ccc} \text{SCIP} & \Longrightarrow & \text{CIP} \\ \Downarrow & & \\ \text{SDIP} & \Longrightarrow & \text{DIP} \end{array}$$

Figure 1: Relationships among interpolation properties.

On the other hand, for any logic over $\mathbf{FL_e}$, we have the following.

LEMMA 9 *For every substructural logic over* $\mathbf{FL_e}$*, the following hold.*

(1) The SCIP and the CIP are equivalent.

(2) The SDIP and the DIP are equivalent.

Thus, the CIP implies the DIP.

Proof. To prove (1), it is sufficient to show that the CIP implies the SCIP. Let $\mathbf{L}$ be a substructural logic over $\mathbf{FL_e}$ with the CIP. To show the SCIP of $\mathbf{L}$, suppose that $\gamma, \sigma \vdash_{\mathbf{L}} \varphi \to \psi$ by taking single formulas γ and σ, instead of Γ and Σ. Then by the local deduction theorem a formula $((\gamma \wedge 1)^m \cdot \varphi) \to ((\sigma \wedge 1)^n \to \psi)$ is provable in $\mathbf{L}$ for some m and n. By the CIP, there exists a formula δ such that both $((\gamma \wedge 1)^m \cdot \varphi \to \delta)$ and $\delta \to ((\sigma \wedge 1)^m \to \psi)$ are provable in $\mathbf{L}$ and $\mathrm{var}(\delta) \subseteq \mathrm{var}((\gamma \wedge 1)^m \cdot \varphi) \cap \mathrm{var}((\sigma \wedge 1)^n \to \psi)$. Then, both $\gamma \vdash_{\mathbf{L}} \varphi \to \delta$ and $\sigma \vdash_{\mathbf{L}} \delta \to \psi$ hold also for this formula δ. Moreover, we can show that $\mathrm{var}(\delta) \subseteq \mathrm{var}(\gamma, \varphi) \cap \mathrm{var}(\sigma, \psi)$ holds. Thus, the SCIP holds for $\mathbf{L}$.

Next, we show that the DIP implies the SDIP. Suppose that a logic over $\mathbf{FL_e}$ has the DIP and that $\gamma, \sigma \vdash_{\mathbf{L}} \psi$ holds. By the local deduction theorem, we have $\gamma \vdash_{\mathbf{L}} (\sigma \wedge 1)^n \to \psi$ for some n. Then by the DIP, there exists a formula δ such that $\gamma \vdash_{\mathbf{L}} \delta$, $\delta \vdash_{\mathbf{L}} (\sigma \wedge 1)^n \to \psi$ and $\mathrm{var}(\delta) \subseteq \mathrm{var}(\gamma) \cap \mathrm{var}((\sigma \wedge 1)^n \to \psi) = \mathrm{var}(\gamma) \cap \mathrm{var}(\sigma, \psi)$. From the latter, $\delta, \sigma \vdash_{\mathbf{L}} \psi$ follows. Thus, the SDIP holds.

Combining these two equivalences with Lemma 8, we can show that the CIP implies the DIP. □

Thus, we have:

$$\boxed{\text{SCIP} \quad \Leftrightarrow \quad \text{CIP}}$$
$$\Downarrow$$
$$\boxed{\text{SDIP} \quad \Leftrightarrow \quad \text{DIP}}$$

Figure 2: Relationships among interpolation properties: commutative case.

The above figure tells us that the CIP implies always the SDIP for every substructural logic over $\mathbf{FL_e}$. But they are in general independent, as the following shows (see Lemma 5.49 in [8]).

PROPOSITION 10 *The properties CIP and SDIP are independent.*

3.2 Robinson property and Extension interpolation property

In [31], the second author introduced two properties for classes of algebras, ROB* and limited GINT, either of which is weaker than eqIP. In view of the algebraization theorem, they are translated into the following properties RP and ExIP for logics. Both of them are discussed also in [6], where the RP is called the *ordinary interpolation property.*

DEFINITION 5 *A substructural logic* $\mathbf{L}$ *has the Robinson property (RP) if for every set of formulas* $\Gamma \cup \Sigma \cup \{\psi\}$*,* $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$ *implies* $\Sigma \vdash_{\mathbf{L}} \psi$*, whenever* $\Gamma \vdash_{\mathbf{L}} \sigma$ *iff* $\Sigma \vdash_{\mathbf{L}} \sigma$ *for every formula* σ *such that* $var(\sigma) \subseteq var(\Gamma) \cap var(\Sigma \cup \{\psi\})$*.*

Now, we introduce two extensions of the RP.

DEFINITION 6 *(1) A substructural logic* $\mathbf{L}$ *has the super Robinson property (superRP), provided that for every set of formulas* $\Gamma \cup \Sigma \cup \{\varphi, \psi\}$ *such that* $\Gamma \vdash_{\mathbf{L}} \sigma$ *iff* $\Sigma \vdash_{\mathbf{L}} \sigma$ *for every formula* σ *with* $var(\sigma) \subseteq var(\Gamma \cup \{\varphi\}) \cap var(\Sigma \cup \{\psi\})$*, if* $\Gamma, \Sigma \vdash_{\mathbf{L}} \varphi \backslash \psi$ *then there exists a fomula* δ *such that*

(a) $\Gamma \vdash_{\mathbf{L}} \varphi \backslash \delta$ *and* $\Sigma \vdash_{\mathbf{L}} \delta \backslash \psi$*, and*

(b) $var(\delta) \subseteq var(\Gamma \cup \{\varphi\}) \cap var(\Sigma \cup \{\psi\})$*.*

(2) *A substructural logic* **L** *has the strong Robinson property (strongRP), provided that for every set of formulas* $\Gamma \cup \Sigma \cup \{\varphi, \psi\}$ *such that* $\Gamma \vdash_{\mathbf{L}} \sigma$ *iff* $\Sigma \vdash_{\mathbf{L}} \sigma$ *for every formula* σ *with* $var(\sigma) \subseteq var(\Gamma \cup \{\varphi\}) \cap var(\Sigma \cup \{\psi\})$, *if* $\Gamma, \Sigma \vdash_{\mathbf{L}} (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$ *then there exists a formula* δ *such that*

   (a) $\Gamma \vdash_{\mathbf{L}} (\varphi\backslash\delta) \wedge (\delta\backslash\varphi)$ *and* $\Sigma \vdash_{\mathbf{L}} (\psi\backslash\delta) \wedge (\delta\backslash\psi)$, *and*

   (b) $var(\delta) \subseteq var(\Gamma \cup \{\varphi\}) \cap var(\Sigma \cup \{\psi\})$.

Notice that the former extension was called the "strong" Robinson property in [8] and the latter is not defined so far. But, to introduce two extensions of the RP, we call the former, "super" RP instead. In fact, this way of naming these properties corresponds to relationships between the RP and the amalgamation properties shown later.

LEMMA 11 *For any substructural logic, the following hold.*

(1) *The superRP implies the strongRP, and the strongRP implies the RP.*

(2) *The SCIP implies the superRP, and the superRP implies the CIP.*

(3) *The SDIP implies the RP, and the RP implies the DIP.*

Proof. Suppose that a substructural logic **L** has the superRP, and $\Gamma, \Sigma \vdash_{\mathbf{L}} (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$ holds for a set of formulas $\Gamma \cup \Sigma \cup \{\varphi, \psi\}$ such that $\Gamma \vdash_{\mathbf{L}} \sigma$ iff $\Sigma \vdash_{\mathbf{L}} \sigma$ for every formula σ with $\mathrm{var}(\sigma) \subseteq \mathrm{var}(\Gamma \cup \{\varphi\}) \cap \mathrm{var}(\Sigma \cup \{\psi\})$. Then, we have that $\Gamma, \Sigma \vdash_{\mathbf{L}} \varphi\backslash\psi$ and $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi\backslash\varphi$ hold. By the superRP, there exist some formulas δ_1 and δ_2 such that

- $\Gamma \vdash_{\mathbf{L}} \varphi\backslash\delta_1$ and $\Sigma \vdash_{\mathbf{L}} \delta_1\backslash\psi$,
- $\Gamma \vdash_{\mathbf{L}} \delta_2\backslash\varphi$ and $\Sigma \vdash_{\mathbf{L}} \psi\backslash\delta_2$, and
- $\mathrm{var}(\delta_1), \mathrm{var}(\delta_2) \subseteq \mathrm{var}(\Gamma \cup \{\varphi\}) \cap (\Sigma \cup \{\psi\})$.

Thus, both $\Gamma \vdash_{\mathbf{L}} \delta_2\backslash\delta_1$ and $\Sigma \vdash_{\mathbf{L}} \delta_1\backslash\delta_2$ hold. Since $\mathrm{var}(\delta_1\backslash\delta_2) = \mathrm{var}(\delta_2\backslash\delta_1) \subseteq \mathrm{var}(\Gamma\cup\{\varphi\})\cap\mathrm{var}(\Sigma\cup\{\psi\})$, we have also that $\Gamma \vdash_{\mathbf{L}} \delta_1\backslash\delta_2$ and $\Sigma \vdash_{\mathbf{L}} \delta_2\backslash\delta_1$ hold by the assumption of $\Gamma \cup \Sigma \cup \{\varphi, \psi\}$. Therefore, $\Gamma \vdash_{\mathbf{L}} (\varphi\backslash\delta_1) \wedge (\delta_1\backslash\varphi)$ and $\Sigma \vdash_{\mathbf{L}} (\psi\backslash\delta_1) \wedge (\delta_1\backslash\psi)$, namely the strongRP holds. Next, suppose that a substructural logic **L** has the strongRP, and $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$ holds for a set of formulas $\Gamma \cup \Sigma \cup \{\psi\}$ such that $\Gamma \vdash_{\mathbf{L}} \sigma$ iff $\Sigma \vdash_{\mathbf{L}} \sigma$ for every formula σ with $\mathrm{var}(\sigma) \subseteq \mathrm{var}(\Gamma) \cup \mathrm{var}(\Sigma \cup \{\psi\})$. Then we have that $\Gamma, \Sigma \vdash_{\mathbf{L}} (1\backslash(\psi \wedge 1)) \wedge ((\psi \wedge 1)\backslash 1)$ holds. By the strongRP, there exists some formula δ_3 such that

1. $\Gamma \vdash_{\mathbf{L}} (1\backslash\delta_3) \wedge (\delta_3\backslash 1)$ and $\Sigma \vdash_{\mathbf{L}} ((\psi \wedge 1)\backslash\delta_3) \wedge (\delta_3\backslash(\psi \wedge 1))$, and
2. $\mathrm{var}(\delta_3) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\Sigma \cup \{\psi\})$.

Thus, $\Gamma \vdash_{\mathbf{L}} \delta_3$ and $\Sigma \vdash_{\mathbf{L}} \delta_3\backslash(\psi \wedge 1)$ hold. By the assumption of $\Gamma \cup \Sigma \cup \{\psi\}$, $\Sigma \vdash_{\mathbf{L}} \delta_3$ also holds, and hence $\Sigma \vdash_{\mathbf{L}} \psi$, namely the RP holds.

It is clear that the SCIP implies the superRP, and the CIP is nothing but the superRP where both Γ and Σ are empty sets.

We show next that a substructural logic **L** has the RP, assuming that it has the SDIP. Suppose that $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$ holds for a set of formulas $\Gamma\cup\Sigma\cup\{\psi\}$ such that $\Gamma \vdash_{\mathbf{L}} \sigma$ iff $\Sigma \vdash_{\mathbf{L}} \sigma$ for every formula σ with $\mathrm{var}(\sigma) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\Sigma \cup \{\psi\})$. Then by the SDIP there is some formula δ_4 such that $\Gamma \vdash_{\mathbf{L}} \delta_4$ and $\delta_4, \Sigma \vdash_{\mathbf{L}} \psi$, and $\mathrm{var}(\delta_4) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\Sigma \cup \{\psi\})$. By the assumption of $\Gamma \cup \Sigma \cup \{\psi\}$, $\Sigma \vdash_{\mathbf{L}} \delta_4$ holds, and hence $\Sigma \vdash_{\mathbf{L}} \psi$. Thus the RP holds. Finally, suppose that a substructural logic **L** has the RP, and $\Gamma \vdash_{\mathbf{L}} \psi$ holds. Define the set of formulas $\Gamma^\dagger$ by $\Gamma^\dagger = \{\gamma : \Gamma \vdash_{\mathbf{L}} \gamma \text{ and } \mathrm{var}(\gamma) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\psi)\}$. Then it is easy to see that $\Gamma \vdash_{\mathbf{L}} \sigma$ iff $\Gamma^\dagger \vdash_{\mathbf{L}} \sigma$ holds for every formula σ with $\mathrm{var}(\sigma) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\psi)$. Since $\mathrm{var}(\Gamma) \cap \mathrm{var}(\Gamma^\dagger \cup \{\psi\}) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\psi)$ and $\Gamma, \Gamma^\dagger \vdash_{\mathbf{L}} \psi$ hold, by the RP we have $\Gamma^{\dagger} \vdash_{\mathbf{L}} \psi$. Since $\vdash_{\mathbf{L}}$ is finitary and conjunctive, if we take a suitable finite conjunction δ_5 of $\Gamma^{\dagger}$, both $\Gamma \vdash_{\mathbf{L}} \delta_5$ and $\delta_5 \vdash_{\mathbf{L}} \psi$ hold, and moreover $\mathrm{var}(\delta_5) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\psi)$. Thus, the DIP holds.

□

The fact that the RP implies the DIP in Lemma 11 was originally shown by Czelakowski and Pigozzi (see Theorem 3.6 in [6]).

In Figure 3, we summarize the relationships between interpolation properties and Robinson properties discussed so far.

$$
\begin{array}{ccccc}
\text{SCIP} & \Rightarrow & \text{superRP} & \Rightarrow & \text{CIP} \\
 & & \Downarrow & & \\
\Downarrow & & \text{strongRP} & & \\
 & & \Downarrow & & \\
\text{SDIP} & \Rightarrow & \text{RP} & \Rightarrow & \text{DIP}
\end{array}
$$

Figure 3: Relationships between interpolation propeties and Robinson properties

The above lemma tells us that the SDIP implies the RP, and the RP implies the DIP. To see differences among them more explicitly, we introduce alternative definitions of the SDIP and the DIP. These definitions are also used later when we discuss their algebraic characterizations.

DEFINITION 7 *(1) A substructural logic* $\mathbf{L}$ *has the SDIP** *if for every set of formulas* $\Gamma \cup \Sigma \cup \{\psi\}$, $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$ *implies* $\Sigma \vdash_{\mathbf{L}} \psi$, *whenever* $\Gamma \vdash_{\mathbf{L}} \sigma$ *implies* $\Sigma \vdash_{\mathbf{L}} \sigma$ *for every formula* σ *with* $var(\sigma) \subseteq var(\Gamma) \cap var(\Sigma \cup \{\psi\})$.

(2) A substructural logic $\mathbf{L}$ *has the DIP** *if for every set of formulas* $\Gamma \cup \Sigma \cup \{\psi\}$, $\Gamma \vdash_{\mathbf{L}} \psi$ *implies* $\Sigma \vdash_{\mathbf{L}} \psi$, *whenever* $\Gamma \vdash_{\mathbf{L}} \sigma$ *iff* $\Sigma \vdash_{\mathbf{L}} \sigma$ *for every formula* σ *with* $var(\sigma) \subseteq var(\Gamma) \cap var(\Sigma \cup \{\psi\})$.

LEMMA 12 *(1) For each substructural logic, the SDIP holds iff the SDIP** *holds.*

*(2) For each substructural logic, the DIP holds iff the DIP** *holds.*

Proof. First suppose that the SDIP holds for a logic $\mathbf{L}$. We assume that $\Gamma \vdash_{\mathbf{L}} \sigma$ implies $\Sigma \vdash_{\mathbf{L}} \sigma$ for every formula σ with $\mathrm{var}(\sigma) \subseteq X$, and that $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$. Here, X is the set $\mathrm{var}(\Gamma) \cap \mathrm{var}(\Sigma \cup \{\psi\})$. By the SDIP, there exists a formula δ with $\mathrm{var}(\delta) \subseteq X$ such that $\Gamma \vdash_{\mathbf{L}} \delta$, and $\delta, \Sigma \vdash_{\mathbf{L}} \psi$. By our assumption, $\Sigma \vdash_{\mathbf{L}} \delta$. Thus, $\Sigma \vdash_{\mathbf{L}} \psi$. Hence the SDIP* holds. Similarly, we can show that the DIP implies the DIP*.

Conversely, suppose that the SDIP* holds for a logic $\mathbf{L}$, and that $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$. Define $\Gamma^{\dagger} = \{\gamma : \Gamma \vdash_{\mathbf{L}} \gamma \text{ and } \mathrm{var}(\gamma) \subseteq X\}$, where X is the set of variables defined as above. Clearly, for any formula σ with $\mathrm{var}(\sigma) \subseteq X$, $\Gamma \vdash_{\mathbf{L}} \sigma$ implies $\Gamma^{\dagger}, \Sigma \vdash_{\mathbf{L}} \sigma$. Now, $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$ implies always $\Gamma, \Gamma^{\dagger}, \Sigma \vdash_{\mathbf{L}} \psi$, from which $\Gamma^{\dagger}, \Sigma \vdash_{\mathbf{L}} \psi$ follows by using the SDIP*. Since $\vdash_{\mathbf{L}}$ is finitary and conjunctive, there exists a formula δ, which is a conjunction of finitely many formulas in $\Gamma^{\dagger}$, such that $\delta, \Sigma \vdash_{\mathbf{L}} \psi$. By the definition of $\Gamma^{\dagger}$ it is clear that $\Gamma \vdash_{\mathbf{L}} \delta$ and $\mathrm{var}(\delta) \subseteq X$. Thus, the SDIP holds.

To show that the DIP follows from the DIP*, we need to modify the above proof slightly. Suppose that the DIP* holds for a logic $\mathbf{L}$, and that $\Gamma \vdash_{\mathbf{L}} \psi$. Define $\Gamma^{\dagger}$ just in the same way as above, but taking the set $\mathrm{var}(\Gamma) \cap \mathrm{var}(\psi)$ for X. Then it is obvious that $\Gamma \vdash_{\mathbf{L}} \sigma$ iff $\Gamma^{\dagger} \vdash_{\mathbf{L}} \sigma$ for every formula σ with $\mathrm{var}(\sigma) \subseteq X$. Thus, by applying the DIP*, $\Gamma^{\dagger} \vdash_{\mathbf{L}} \psi$ follows from $\Gamma \vdash_{\mathbf{L}} \psi$. Then, by taking a suitable finite conjunction δ of $\Gamma^{\dagger}$, we can show that $\Gamma \vdash_{\mathbf{L}} \delta$ and $\delta \vdash_{\mathbf{L}} \psi$. As $\mathrm{var}(\delta) \subseteq X$, the DIP follows. Clearly, the SDIP* implies the RP, which in turn implies the DIP*.

□

The following is an immediate corollary of Lemma 9.

COROLLARY 13 *For every substructural logic over $\mathbf{FL_e}$, the following hold.*

(1) The SCIP, the superRP and the CIP are mutually equivalent.

(2) The SDIP, the RP and the DIP are mutually equivalent.

In fact, the above equivalences hold whenever a logic satisfies the *local deduction property* of the following form (see [18]):

> for any Γ, σ, ψ, if $\Gamma, \sigma \vdash_{\mathbf{L}} \psi$ then there exists a formula $\star(\sigma)$ with $\mathrm{var}(\star(\sigma)) \subseteq \mathrm{var}(\sigma)$ such that $\Gamma \vdash_{\mathbf{L}} \star(\sigma)\backslash\psi$ and $\sigma \vdash_{\mathbf{L}} \star(\sigma)$.

Note that the above form of the local deduction property always holds for any logic over $\mathbf{FL_e}$ since we can take $(\sigma \wedge 1)^n$ for some $n \in \omega$ as $\star(\sigma)$.

DEFINITION 8 *A substructural logic* $\mathbf{L}$ *has the extension interpolation property (ExIP),*[2] *if for every set of formulas* $\Gamma \cup \Sigma \cup \{\psi\}$*, if* $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$*, then there exists a formula* δ *such that*

1. $\Gamma \vdash_{\mathbf{L}} \delta$ *and* $\delta, \Sigma \vdash_{\mathbf{L}} \psi$*,*
2. *var*$(\delta) \subseteq$ *var*$(\Sigma \cup \{\psi\})$*.*

Notice that in case of the ExIP, $\mathrm{var}(\delta)$ might be unrelated to $\mathrm{var}(\Gamma)$.

The following shows relationships between the SDIP, the RP and the ExIP. (see also Theorem 3.6 in [6])

THEOREM 14 *For any substructural logic* $\mathbf{L}$*,* $\mathbf{L}$ *has the SDIP iff it has both the RP and the ExIP iff it has both the DIP and the ExIP.*

Proof. The ExIP follows immediately from the SDIP. By Lemma 11, the SDIP implies the RP, and the RP implies the DIP. Thus, it is sufficient to show that both the DIP and the ExIP imply the SDIP. Assume that $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$ holds. Then, by the ExIP, there exists some formula δ' such that

1. $\Gamma \vdash_{\mathbf{L}} \delta'$ and $\delta', \Sigma \vdash_{\mathbf{L}} \psi$, and
2. $\mathrm{var}(\delta') \subseteq \mathrm{var}(\Sigma \cup \{\psi\})$.

Now, by applying the DIP to $\Gamma \vdash_{\mathbf{L}} \delta'$, there exists some formula δ^* such that

3. $\Gamma \vdash_{\mathbf{L}} \delta^*$ and $\delta^* \vdash_{\mathbf{L}} \delta'$, and
4. $\mathrm{var}(\delta^*) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\delta') \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\Sigma \cup \{\psi\})$.

Thus, we have that $\Gamma \vdash_{\mathbf{L}} \delta^*$ and $\delta^*, \Sigma \vdash_{\mathbf{L}} \psi$. Therefore, the SDIP holds. □

LEMMA 15 *(1) For each substructural logic* $\mathbf{L}$*,* $\mathbf{L}$ *has the ExIP iff* $\mathsf{V}(\mathbf{L})$ *has the CEP.*

(2) For each substructural logic over $\mathbf{FL_e}$*, the ExIP holds always.*

Proof. Here we prove only (2). For (1), see [31], where the equivalence between the equational form of the ExIP and the CEP is shown.

Suppose that $\gamma, \sigma \vdash_{\mathbf{L}} \varphi$. By the local deduction theorem for $\mathbf{FL_e}$, $\gamma \vdash_{\mathbf{L}} (\sigma \wedge 1)^n \to \psi$ for some n. Let δ be $(\sigma \wedge 1)^n \to \psi$. Then, clearly both $\gamma \vdash_{\mathbf{L}} \delta$ and $\delta, \sigma \vdash_{\mathbf{L}} \psi$ hold. Moreover, $\mathrm{var}(\delta) \subseteq \mathrm{var}(\sigma \cup \{\psi\})$. Thus, the ExIP follows. □

---

[2] The notion of the ExIP is introduced in [6]. Note that in the definition of the limited GINT given in [31], it is required that $\mathrm{var}(\Sigma \cup \{\psi\}) \subseteq \mathrm{var}(\Gamma)$. But it is easy to see that this assumption can be omitted.

4 Algebraic characterization of the CIP and the DIP

4.1 Amalgamation property

DEFINITION 9 *A variety $\mathcal{V}$ has the amalgamation property (AP), if for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathcal{V}$ and for all embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, there exists an algebra $\mathbf{D}$ in $\mathcal{V}$ and embeddings $h : \mathbf{B} \to \mathbf{D}$, $k : \mathbf{C} \to \mathbf{D}$ such that $h \circ f = k \circ g$.*

The following result was originally shown by the second author in [31], with respect to equational calculi. Here, we give a detailed proof with respect to substructural logics since this argument is essential for the later algebraic characterizations.

THEOREM 16 *For each substructural logic $\mathbf{L}$, $\mathbf{L}$ has the RP iff $\mathsf{V}(\mathbf{L})$ has the AP.*

Proof. Suppose that $\mathbf{L}$ has the RP. We assume moreover that there exist embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$ for **FL**-algebras $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ in $\mathsf{V}(\mathbf{L})$. We introduce a set of variables X by $X = \{x_a : a \in A\}$, and a mapping $\eta'_{\mathbf{A}} : X \to \mathbf{A}$ by $\eta'_{\mathbf{A}}(x_a) = a$. Then, the mapping $\eta'_{\mathbf{A}}$ is uniquely extended to the mapping $\eta_{\mathbf{A}} : \mathbf{Fm}(X) \to \mathbf{A}$, which is in fact a surjective homomorphism. Next, we introduce sets of variables Y and Z by $Y = X \cup \{y_b : b \in B \setminus f(A)\}$, $Z = X \cup \{z_c : c \in C \setminus g(A)\}$ so that $Y \cap Z = X$, and mappings $\eta'_{\mathbf{B}} : Y \to \mathbf{B}$ and $\eta'_{\mathbf{C}} : Z \to \mathbf{C}$ by

$$\eta'_{\mathbf{B}}(y) = \begin{cases} f(a) & \text{if } y = x_a \text{ for some } x_a \in X \\ b & \text{if } y = y_b \text{ for some } b \in B \setminus f(A) \end{cases}$$

and

$$\eta'_{\mathbf{C}}(z) = \begin{cases} g(a) & \text{if } z = x_a \text{ for some } x_a \in X \\ c & \text{if } z = z_c \text{ for some } c \in C \setminus g(A). \end{cases}$$

Then $\eta'_{\mathbf{B}}$ and $\eta'_{\mathbf{C}}$ are extended to surjective homomorphisms $\eta_{\mathbf{B}} : \mathbf{Fm}(Y) \to \mathbf{B}$ and $\eta_{\mathbf{C}} : \mathbf{Fm}(Z) \to \mathbf{C}$, respectively, and they satisfy $\eta_{\mathbf{B}}(\alpha) = f(\eta_{\mathbf{A}}(\alpha))$ and $\eta_{\mathbf{C}}(\alpha) = g(\eta_{\mathbf{A}}(\alpha))$ for every $\alpha \in \mathbf{Fm}(X)$. Define sets of formulas $\Gamma_{\mathbf{B}}$ and $\Gamma_{\mathbf{C}}$ by

$$\Gamma_{\mathbf{B}} = \{\varphi \in \mathbf{Fm}(Y) : \eta_{\mathbf{B}}(\varphi) \geq 1_{\mathbf{B}}\} \text{ and } \Gamma_{\mathbf{C}} = \{\psi \in \mathbf{Fm}(Z) : \eta_{\mathbf{C}}(\psi) \geq 1_{\mathbf{C}}\},$$

respectively. We introduce a binary relation $\equiv$ on $\mathbf{Fm}(Y \cup Z)$ by

$$\beta \equiv \gamma \text{ iff } \Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash (\beta \backslash \gamma) \wedge (\gamma \backslash \beta),$$

where $\vdash$ denotes $\vdash_{\mathbf{L}}$. Then, by Lemma 6, $\equiv$ is a congruence relation on $\mathbf{Fm}(Y \cup Z)$, and hence the quotient algebra $\mathbf{Fm}(Y \cup Z)/\equiv$ is a member of $\mathsf{V}(\mathbf{L})$. Let us call this algebra, $\mathbf{D}$. We will show that this $\mathbf{D}$ is a required algebra satisfying the conditions for the AP.

To define the required embeddings $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$, we show that for each $\alpha \in \mathbf{Fm}(X)$, $\Gamma_{\mathbf{B}} \vdash \alpha$ iff $\Gamma_{\mathbf{C}} \vdash \alpha$. Suppose that $\Gamma_{\mathbf{B}} \vdash \alpha$. Then, $f(1_{\mathbf{A}}) = 1_{\mathbf{B}} \leq \eta_{\mathbf{B}}(\alpha) = f(\eta_{\mathbf{A}}(\alpha))$ using the definition of $\Gamma_{\mathbf{B}}$. Since f is injective, $1_{\mathbf{A}} \leq \eta_{\mathbf{A}}(\alpha)$ and hence $1_{\mathbf{C}} = g(1_{\mathbf{A}}) \leq g(\eta_{\mathbf{A}}(\alpha)) = \eta_{\mathbf{C}}(\alpha)$. Thus, $\Gamma_{\mathbf{C}} \vdash \alpha$. The converse implication can be shown in the same way using the fact that g is injective. Now define mappings $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ by

- $h(b) = (\varphi/\equiv)$ when $b = \eta_{\mathbf{B}}(\varphi)$ for a formula $\varphi \in \mathbf{Fm}(Y)$,
- $k(c) = (\psi/\equiv)$ when $c = \eta_{\mathbf{C}}(\psi)$ for a formula $\psi \in \mathbf{Fm}(Z)$.

We prove that both h and k are well-defined embeddings. To show the well-definedness of h, suppose that $\eta_{\mathbf{B}}(\varphi) = \eta_{\mathbf{B}}(\varphi')$ for $\varphi, \varphi' \in \mathbf{Fm}(Y)$. Then, $\Gamma_{\mathbf{B}} \vdash (\varphi \backslash \varphi') \wedge (\varphi' \backslash \varphi)$, and hence $\varphi \equiv \varphi'$. It is easy to see that h is a homomorphism. To show that h is injective, suppose that $h(b) = h(b')$, where $b = \eta_{\mathbf{B}}(\varphi)$ and $b' = \eta_{\mathbf{B}}(\varphi')$ for $\varphi, \varphi' \in \mathbf{Fm}(Y)$. Then, $\varphi \equiv \varphi'$, and thus $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash (\varphi \backslash \varphi') \wedge (\varphi' \backslash \varphi)$

by the definition of $\equiv$. From the RP, it follows that $\Gamma_{\mathbf{B}} \vdash (\varphi\backslash\varphi') \wedge (\varphi'\backslash\varphi)$, which implies that $b = \eta_{\mathbf{B}}(\varphi) = \eta_{\mathbf{B}}(\varphi') = b'$. Similarly, k is shown to be a well-defined embedding. Note that in this case it is necessary to interchange the role of $\Gamma_{\mathbf{B}}$ and $\Gamma_{\mathbf{C}}$ in applying the RP, as the RP is of the symmetric form.

It remains to see that $h \circ f = k \circ g$. Take an arbitrary element $a \in A$. Then there exists a formula $\alpha \in \mathbf{Fm}(X)$ such that $a = \eta_{\mathbf{A}}(\alpha)$. Then, $(h \circ f)(a) = h(f(\eta_{\mathbf{A}}(\alpha))) = h(\eta_{\mathbf{B}}(\alpha)) = (\alpha/\equiv)$. Similarly, $(k \circ g)(a) = (\alpha/\equiv)$. Thus, $h \circ f = k \circ g$.

We show next that the AP implies the RP. Let Γ and $\Sigma \cup \{\psi\}$ be sets of formulas, and let us denote sets of variables Y and Z by $Y = \mathrm{var}(\Gamma)$ and $Z = \mathrm{var}(\Sigma \cup \{\psi\})$. Moreover, we assume that

- $\Gamma \vdash \alpha$ iff $\Sigma \vdash \alpha$ for every formula $\alpha \in \mathbf{Fm}(X)$, where $X = Y \cap Z$,
- $\Gamma, \Sigma \vdash \psi$.

Define $\Delta = \{\alpha \in \mathbf{Fm}(X) : \Gamma \vdash \alpha\}$, which is obviously equal to $\{\alpha \in \mathbf{Fm}(X) : \Sigma \vdash \alpha\}$. The set Δ determines a binary relation $\equiv_\Delta$ on $\mathbf{Fm}(X)$ by

$$\alpha \equiv_\Delta \beta \text{ iff } \Delta \vdash (\alpha\backslash\beta) \wedge (\beta\backslash\alpha),$$

which is in fact a congruence relation. We denote the quotient algebra of $\mathbf{Fm}(X)$ determined by this $\equiv_\Delta$ as $\mathbf{Fm}(X)/\Delta$. Similarly, we can introduce quotient algebras $\mathbf{Fm}(Y)/\Gamma$ of $\mathbf{Fm}(Y)$, and $\mathbf{Fm}(Z)/\Sigma$ of $\mathbf{Fm}(Z)$, by taking sets of formulas Γ and Σ, respectively, in the place of Δ. It is clear that all of these algebras $\mathbf{Fm}(X)/\Delta$, $\mathbf{Fm}(Y)/\Gamma$ and $\mathbf{Fm}(Z)/\Sigma$ are members of $\mathsf{V}(\mathbf{L})$. We define mappings $f : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Y)/\Gamma$ and $g : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Z)/\Sigma$ by

$$f(\alpha/\equiv_\Delta) = \alpha/\equiv_\Gamma \text{ and } g(\alpha/\equiv_\Delta) = \alpha/\equiv_\Sigma.$$

They are shown to be well-defined embeddings, using the definition of Δ.

Since we assume that the AP holds for $\mathsf{V}(\mathbf{L})$, there exist an algebra $\mathbf{D} \in \mathsf{V}(\mathbf{L})$, and two embeddings $h : \mathbf{Fm}(Y)/\Gamma \to \mathbf{D}$ and $k : \mathbf{Fm}(Z)/\Sigma \to \mathbf{D}$ satisfying $h \circ f = k \circ g$. Now, consider a valuation w over $\mathbf{D}$ for formulas in $\mathbf{Fm}(Y \cup Z)$ defined as follows: For every $x \in Y \cup Z$,

$$w(x) = \begin{cases} h(x/\equiv_\Gamma) & \text{if } x \in Y \\ k(x/\equiv_\Sigma) & \text{if } x \in Z \end{cases}$$

The mapping w is well-defined, since if $x \in X$, $h(x/\equiv_\Gamma) = (h \circ f)(x/\equiv_\Delta) = (k \circ g)(x/\equiv_\Delta) = k(x/\equiv_\Sigma)$. As usual, w is extended to a mapping from $\mathbf{Fm}(Y \cup Z)$ to $\mathbf{D}$, which satisfies that

$$w(\gamma) = \begin{cases} h(\gamma/\equiv_\Gamma) & \text{if } \gamma \in \mathbf{Fm}(Y) \\ k(\gamma/\equiv_\Sigma) & \text{if } \gamma \in \mathbf{Fm}(Z) \end{cases}$$

For each formula $\gamma \in \Gamma$, $(\gamma/\equiv_\Gamma) \geq 1_{\mathbf{Fm}(Y)/\Gamma}$ and hence $w(\gamma) = h(\gamma/\equiv_\Gamma) \geq 1_{\mathbf{D}}$. Similarly, $w(\gamma) \geq 1_{\mathbf{D}}$ for each formula $\gamma \in \Sigma$. Thus, $\mathbf{D}, w \models \gamma$ for every $\gamma \in \Gamma \cup \Sigma$. Since $\Gamma, \Sigma \vdash \psi$ by our assumption and $\mathbf{D} \in \mathsf{V}(\mathbf{L})$, $\mathbf{D}, w \models \psi$ must hold. That is, $k(\psi/\equiv_\Sigma) \geq 1_{\mathbf{D}}$, which implies $\psi/\equiv_\Sigma \geq 1_{\mathbf{Fm}(Z)/\Sigma}$. This means that $\Sigma \vdash \psi$. This completes our proof. □

The following is an immediate consequence of Theorem 16, by considering Corollary 13.

COROLLARY 17 *For each substructural logic* $\mathbf{L}$ *over* $\mathbf{FL_e}$, $\mathbf{L}$ *has the DIP iff* $\mathsf{V}(\mathbf{L})$ *has the AP.*

DEFINITION 10 *A variety* $\mathcal{V}$ *has the super-amalgamation property (superAP), if whenever* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *are in* $\mathcal{V}$ *and* $f : \mathbf{A} \to \mathbf{B}$, $g : \mathbf{A} \to \mathbf{C}$ *are embeddings, then there exists an algebra* $\mathbf{D}$ *in* $\mathcal{V}$ *and embeddings* $h : \mathbf{B} \to \mathbf{D}$, $k : \mathbf{C} \to \mathbf{D}$ *such that*

1. $h \circ f = k \circ g$,

2. **(super)** *for all* $b \in B$ *and* $c \in C$*, if* $h(b) \leq k(c)$ $(k(c) \leq h(b))$ *then there exists* $a \in A$ *for which both* $h(b) \leq h \circ f(a)$ *and* $k \circ g(a) \leq k(c)$ $(k(c) \leq k \circ g(a)$ *and* $h \circ f(a) \leq h(b)$*, respectively).*

A variety $\mathcal{V}$ *has the strong-amalgamation property (strongAP), if whenever* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *are in* $\mathcal{V}$ *and* $f : \mathbf{A} \to \mathbf{B}$*,* $g : \mathbf{A} \to \mathbf{C}$ *are embeddings, then there exists an algebra* $\mathbf{D}$ *in* $\mathcal{V}$ *and embeddings* $h : \mathbf{B} \to \mathbf{D}$*,* $k : \mathbf{C} \to \mathbf{D}$ *such that*

1. $h \circ f = k \circ g$*,*

2. **(strong)** $h(B) \cap k(C) = h \circ f(A)$*.*

It is easily seen that the superAP implies the strongAP, and the strongAP implies the AP. In [24], it is shown that the converse implications do not hold in general for varieties of modal algebras. On the other hand, all of them become equivalent for varieties of Heyting algebras, as shown in [21].

Now, we extend Theorem 16 to characterization results of the superRP and the strongRP, respectively.

THEOREM 18 *For each substructural logic* $\mathbf{L}$*,* $\mathbf{L}$ *has the superRP iff* $\mathsf{V}(\mathbf{L})$ *has the superAP.*

Proof. We show our theorem using the proof of Theorem 16. Suppose first that $\mathbf{L}$ has the superRP. Since it has the RP, we can construct an algebra $\mathbf{D}$ and embeddings h and k as we have done in the proof of Theorem 16. For simplicity's sake, we use the same symbols as in the proof of Theorem 16 in the following. So, it remains to show that the last condition for the superAP holds. Let $b \in B$ and $c \in C$ such that $h(b) \leq k(c)$. We need to find an element $a \in A$ such that $h(b) \leq h \circ f(a)$ and $k \circ g(a) \leq k(c)$ hold. Then, for some formulas $\varphi \in \mathbf{Fm}(Y)$ and $\psi \in \mathbf{Fm}(Z)$, $b = \eta_{\mathbf{B}}(\varphi)$ and $c = \eta_{\mathbf{C}}(\psi)$. From the definitions of h, k and $\equiv$, we can see that the condition $h(b) \leq k(c)$ is equivalent to $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash \varphi \backslash \psi$. Since $\mathbf{L}$ has the superRP and the required condition for applying it to $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash \varphi \backslash \psi$ is satisfied, as shown in the proof of Theorem 16, there exists a formula $\delta \in \mathbf{Fm}(X)$ such that $\Gamma_{\mathbf{B}} \vdash \varphi \backslash \delta$ and $\Gamma_{\mathbf{C}} \vdash \delta \backslash \psi$. Let $a = \eta_{\mathbf{A}}(\delta)$. Then, we have $b = \eta_{\mathbf{B}}(\varphi) \leq \eta_{\mathbf{B}}(\delta) = f(\eta_{\mathbf{A}}(\delta)) = f(a)$ and $g(a) = g(\eta_{\mathbf{A}}(\delta)) = \eta_{\mathbf{C}}(\delta) \leq \eta_{\mathbf{C}}(\psi) = c$. Thus, the superAP holds.

Conversely, suppose that $\mathsf{V}(\mathbf{L})$ has the superAP. Let $\Gamma \cup \{\varphi\}$ and $\Sigma \cup \{\psi\}$ be sets of formulas, and let us denote sets of variables Y and Z by $Y = \mathrm{var}(\Gamma \cup \{\varphi\})$ and $Z = \mathrm{var}(\Sigma \cup \{\psi\})$. Moreover, we assume that

- $\Gamma \vdash \alpha$ iff $\Sigma \vdash \alpha$ for every formula $\alpha \in \mathbf{Fm}(X)$, where $X = Y \cap Z$,
- $\Gamma, \Sigma \vdash \varphi \backslash \psi$.

As we have shown in the proof of Theorem 16, the mappings $f : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Y)/\Gamma$ and $g : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Z)/\Sigma$ introduced there are embeddings. Thus, by the superAP, there exist an algebra $\mathbf{D}$ and embeddings $h : \mathbf{Fm}(Y)/\Gamma \to \mathbf{D}$ and $k : \mathbf{Fm}(Z)/\Sigma \to \mathbf{D}$ such that

1. $h \circ f = k \circ g$,

2. for all $b \in \mathbf{Fm}(Y)/\Gamma$ and $c \in \mathbf{Fm}(Z)/\Sigma$, if $h(b) \leq k(c)$ $(k(c) \leq h(b))$ then there exists $a \in \mathbf{Fm}(X)/\Delta$ such that $h(b) \leq h \circ f(a)$ and $k \circ g(a) \leq k(c)$ $(k(c) \leq k \circ g(a)$ and $h \circ f(a) \leq h(b)$, respectively) hold.

Then as before, we can take such a valuation w over $\mathbf{D}$ that satisfies that

$$w(\gamma) = \begin{cases} h(\gamma/\equiv_{\Gamma}) & \text{if } \gamma \in \mathbf{Fm}(Y) \\ k(\gamma/\equiv_{\Sigma}) & \text{if } \gamma \in \mathbf{Fm}(Z). \end{cases}$$

By our assumption, $\Gamma, \Sigma \vdash \varphi\backslash\psi$ holds. Then, $\mathbf{D}, w \models \varphi\backslash\psi$ holds, i.e., $w(\varphi) \leq w(\psi)$, or equivalently, $h(\varphi/\equiv_\Gamma) \leq k(\psi/\equiv_\Sigma)$. By the second condition of the superAP, there exists an element $d \in \mathbf{Fm}(X)/\Delta$ such that $h(\varphi/\equiv_\Gamma) \leq h \circ f(d)$ and $k \circ g(d) \leq k(\psi/\equiv_\Sigma)$. Then, there exists a formula $\delta \in \mathbf{Fm}(X)$ such that $d = (\delta/\equiv_\Delta)$, and by injectiveness of h and k, these imply $(\varphi/\equiv_\Gamma) \leq (\delta/\equiv_\Gamma)$ and $(\delta/\equiv_\Sigma) \leq (\psi/\equiv_\Sigma)$. This is equivalent to saying that $\Gamma \vdash \varphi\backslash\delta$ and $\Sigma \vdash \delta\backslash\psi$ for a formula $\delta \in \mathbf{Fm}(X)$. Therefore, the superRP holds. □

Thus, we have the following as a corollary.

COROLLARY 19 *For each substructural logic* $\mathbf{L}$ *over* $\mathbf{FL_e}$*,* $\mathbf{L}$ *has the CIP iff* $\mathsf{V}(\mathbf{L})$ *has the superAP.*

In [18], Madarász discussed how far this result can be extended and for which logics the equivalence holds.

THEOREM 20 *For each substructural logic* $\mathbf{L}$*,* $\mathbf{L}$ *has the strongRP iff* $\mathsf{V}(\mathbf{L})$ *has the strongAP.*

Proof. We show our theorem again by using the proof of Theorem 16. Suppose first that $\mathbf{L}$ has the strongRP. Since it has the RP, we can construct an algebra $\mathbf{D}$ and embeddings h and k as we have done in the proof of Theorem 16. So, it remains to show that the last condition $h(B) \cap k(C) = h \circ f(A)$ holds. Note that $h \circ f(A) \subseteq h(B) \cap k(C)$ holds since $f(A) \subseteq B$, $g(B) \subseteq C$ and $h \circ f = k \circ g$. Thus, it is sufficient to show that $h(B) \cap k(C) \subseteq h \circ f(A)$. Let $d \in h(B) \cap k(C)$, namely for some $b \in B$ and $c \in C$, $d = h(b) = k(c)$. Then, for some formulas $\varphi \in \mathbf{Fm}(Y)$ and $\psi \in \mathbf{Fm}(Z)$, $b = \eta_{\mathbf{B}}(\varphi)$ and $c = \eta_{\mathbf{C}}(\psi)$. From the definitions of h, k and $\equiv$, we can see that the condition $h(b) = k(c)$ is equivalent to $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$. Since the required condition for applying the strongRP is satisfied by $\Gamma_{\mathbf{B}}$ and $\Gamma_{\mathbf{C}}$, as shown in the proof of Theorem 16, there exists a formula $\delta \in \mathbf{Fm}(X)$ such that $\Gamma_{\mathbf{B}} \vdash (\varphi\backslash\delta) \wedge (\delta\backslash\psi)$ and $\Gamma_{\mathbf{C}} \vdash (\delta\backslash\psi) \wedge (\psi\backslash\delta)$. Let $a = \eta_{\mathbf{A}}(\delta)$. Then, we have $b = \eta_{\mathbf{B}}(\varphi) = \eta_{\mathbf{B}}(\delta) = f(\eta_{\mathbf{A}}(\delta)) = f(a)$, which implies $d = h(b) = h \circ f(a) \in h \circ f(A)$. Thus, the strongAP holds.

Conversely, suppose that $\mathsf{V}(\mathbf{L})$ has the strongAP. Let $\Gamma \cup \{\varphi\}$ and $\Sigma \cup \{\psi\}$ be sets of formulas, and let $Y = \mathrm{var}(\Gamma \cup \{\varphi\})$ and $Z = \mathrm{var}(\Sigma \cup \{\psi\})$. Moreover, we assume that

- $\Gamma \vdash \alpha$ iff $\Sigma \vdash \alpha$ for every formula $\alpha \in \mathbf{Fm}(X)$, where $X = Y \cap Z$,
- $\Gamma, \Sigma \vdash (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$.

Similarly to the proof of Theorem 16, for given embeddings $f : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Y)/\Gamma$ and $g : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Z)/\Sigma$, there exist an algebra $\mathbf{D}$ and embeddings $h : \mathbf{Fm}(Y)/\Gamma \to \mathbf{D}$ and $k : \mathbf{Fm}(Z)/\Sigma \to \mathbf{D}$ satisfying $h \circ f = k \circ g$ and $h(\mathbf{Fm}(Y)/\Gamma) \cap k(\mathbf{Fm}(Z)/\Sigma) = h \circ f(\mathbf{Fm}(X)/\Delta)$ by the strongAP of $\mathsf{V}(\mathbf{L})$. Then there exists such a valuation w over $\mathbf{D}$ that satisfies that

$$w(\gamma) = \begin{cases} h(\gamma/\equiv_\Gamma) & \text{if } \gamma \in \mathbf{Fm}(Y) \\ k(\gamma/\equiv_\Sigma) & \text{if } \gamma \in \mathbf{Fm}(Z). \end{cases}$$

By our assumption, $\Gamma, \Sigma \vdash (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$ holds. Then, $\mathbf{D}, w \models (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$ holds, i.e., $w(\varphi) = w(\psi)$, or equivalently, $h(\varphi/\equiv_\Gamma) = k(\psi/\equiv_\Sigma)$. By the second condition of the strongAP, we have $h(\varphi/\equiv_\Gamma) = k(\psi/\equiv_\Sigma) \in h \circ f(\mathbf{A})$. Hence there exists an element $d \in \mathbf{Fm}(X)/\Delta$ such that $h(\varphi/\equiv_\Gamma) = h \circ f(d) = k \circ g(d) = k(\psi/\equiv_\Sigma)$, using the fact that $h \circ f = k \circ g$. Since both h and k are injective, we have that $(\varphi/\equiv_\Gamma) = f(d)$ and $g(d) = (\psi/\equiv_\Sigma)$ hold. Let δ be a formula in $\mathbf{Fm}(X)$ such that $d = (\delta/\equiv_\Delta)$. Then, both $(\varphi/\equiv_\Gamma) = (\delta/\equiv_\Gamma)$ and $(\delta/\equiv_\Sigma) = (\psi/\equiv_\Sigma)$ hold. Therefore, $\Gamma \vdash (\varphi\backslash\delta) \wedge (\delta\backslash\varphi)$ and $\Sigma \vdash (\psi\backslash\delta) \wedge (\delta\backslash\psi)$ for a formula $\delta \in \mathbf{Fm}(X)$. Thus, the strongRP holds. □

4.2 Generalized amalgamation property

We give algebraic characterizations of both the CIP and the DIP. An algebraic characterization of the DIP by the *flat amalgamation property* is given and discussed in [14], [32], [3] and [6]. But we give here another kind of algebraic characterization.

The amalgamation property says that for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in a variety $\mathcal{V}$ and for all embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, there exists an algebra $\mathbf{D}$ in $\mathcal{V}$ and embeddings $h : \mathbf{B} \to \mathbf{D}$, $k : \mathbf{C} \to \mathbf{D}$ satisfying some conditions. Obviously, h and k give isomorphisms between $\mathbf{B}$ and a subalgebra $h(\mathbf{B})$ of $\mathbf{D}$, and between $\mathbf{C}$ and a subalgebra $k(\mathbf{C})$ of $\mathbf{D}$, respectively. Equivalently, this can be expressed in such a way that there exist subalgebras $\mathbf{D}_1$ and $\mathbf{D}_2$ of $\mathbf{D}$ and isomorphisms from $\mathbf{D}_1$ to $\mathbf{B}$ and from $\mathbf{D}_2$ to $\mathbf{C}$ (in fact, they are h^{-1} and k^{-1}, respectively) satisfying certain conditions. By replacing homomorphisms by isomorphisms, we have a definition of the generalized amalgamation property as follows.

DEFINITION 11 *A variety $\mathcal{V}$ has the generalized amalgamation property (GAP) if for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathcal{V}$ and for all embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, there exist an algebra $\mathbf{D}$ in $\mathcal{V}$, subalgebras $\mathbf{D}_1$ and $\mathbf{D}_2$ of $\mathbf{D}$, and surjective homomorphisms $i : \mathbf{D}_1 \to \mathbf{B}$ and $j : \mathbf{D}_2 \to \mathbf{C}$ such that*

1. *for all $a \in A$ there exists $d \in D_1 \cap D_2$ such that $f(a) = i(d)$ and $g(a) = j(d)$.*

When either of i or j is injective in the GAP, we call it the generalized amalgamation property with injections, and write it as the IGAP.

Note that the condition 1 in the GAP corresponds to the condition $h \circ f = k \circ g$ in the AP. More precisely, when both i and j are isomorphisms, we can easily show the equivalence of these two conditions, by taking $h = i^{-1}$ and $k = j^{-1}$. In this case, the GAP is nothing but the AP. Thus, the AP implies the IGAP, and the IGAP implies the GAP.

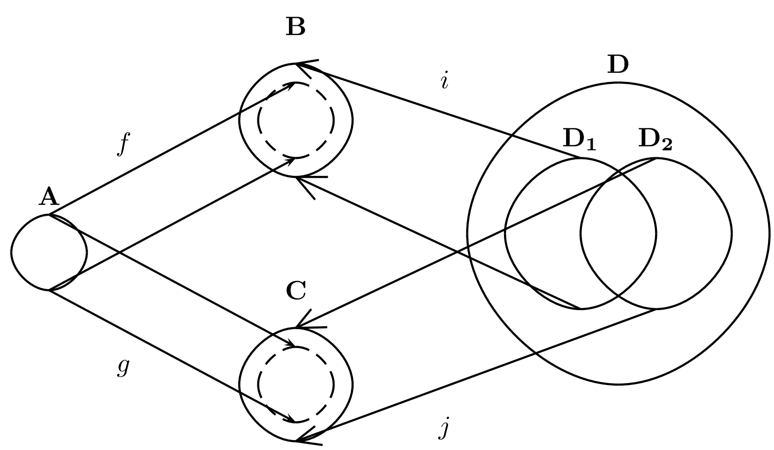

Figure 4: The generalized amalgamation property

In the previous subsection, we have introduced two strengthened forms of the AP, i.e., superAP and strongAP. In the same way, we introduce two strengthened forms of the GAP as follows.

DEFINITION 12 *A variety $\mathcal{V}$ has the super generalized amalgamation property (superGAP), if for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathcal{V}$ and for all embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, there exist an algebra $\mathbf{D}$*

in $\mathcal{V}$, *subalgebras* $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *of* $\mathbf{D}$, *and surjective homomorphisms* $i : \mathbf{D}_1 \to \mathbf{B}$ *and* $j : \mathbf{D}_2 \to \mathbf{C}$ *that satisfy the following:*

1. *for all* $a \in A$ *there exists* $d \in D_1 \cap D_2$ *such that* $f(a) = i(d)$ *and* $g(a) = j(d)$,

2. *for all* $d_1 \in D_1$, $d_2 \in D_2$ *such that* $d_1 \leq d_2$ $(d_2 \leq d_1)$, *there exists* $a \in A$ *such that* $i(d_1) \leq f(a)$ *and* $g(a) \leq j(d_2)$ $(j(d_2) \leq g(a)$ *and* $f(a) \leq i(d_1)$, *respectively).*

When either of i *or* j *is injective in the superGAP, we call it the super generalized amalgamation property with injections (superIGAP).*

A variety $\mathcal{V}$ *has the strong generalized amalgamation property (strongGAP), if for all* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *in* $\mathcal{V}$ *and for all embeddings* $f : \mathbf{A} \to \mathbf{B}$ *and* $g : \mathbf{A} \to \mathbf{C}$, *there exist an algebra* $\mathbf{D}$ *in* $\mathcal{V}$, *subalgebras* $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *of* $\mathbf{D}$, *and surjective homomorphisms* $i : \mathbf{D}_1 \to \mathbf{B}$ *and* $j : \mathbf{D}_2 \to \mathbf{C}$ *that satisfy the following:*

1. *for all* $a \in A$ *there exists* $d \in D_1 \cap D_2$ *such that* $f(a) = i(d)$ *and* $g(a) = j(d)$,

2. *for all* $d \in D_1 \cap D_2$, *there exists* $a \in A$ *such that* $i(d) = f(a)$ *and* $j(d) = g(a)$.

When either of i *or* j *is injective in the strongGAP, we call it the strong generalized amalgamation property with injections (strongIGAP).*

Note that the second condition in the superGAP corresponds to the condition **(super)** in the superAP, namely

2. **(super)** for all $b \in B$ and $c \in C$, if $h(b) \leq k(c)$ $(k(c) \leq h(b))$ then there exists $a \in A$ for which both $h(b) \leq h \circ f(a)$ and $k \circ g(a) \leq k(c)$ $(k(c) \leq k \circ g(a)$ and $h \circ f(a) \leq h(b)$, respectively).

Similarly, the second condition in the strongGAP corresponds to the condition **(strong)** in the strongAP, namely $h(B) \cap k(C) = h \circ f(A)$. Moreover, the superGAP implies the strongGAP. For, if $d \in D_1 \cap D_2$ then, by the condition 2 of the superGAP, there exist some $a_1, a_2 \in A$ such that the following two conditions follows;

1. $i(d) \leq f(a_1)$ and $g(a_1) \leq j(d)$, and

2. $j(d) \leq g(a_2)$ and $f(a_2) \leq i(d)$.

Thus, $f(a_2) \leq f(a_1)$ and $g(a_1) \leq g(a_2)$ hold. Since f and g are injective, we have that $a_1 = a_2$. Therefore, the strongGAP holds.

We can summarize relations among the APs and the GAPs in Figure 5.

$$
\begin{array}{ccccc}
\text{superAP} & \Rightarrow & \text{superIGAP} & \Rightarrow & \text{superGAP} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{strongAP} & \Rightarrow & \text{strongIGAP} & \Rightarrow & \text{strongGAP} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{AP} & \Rightarrow & \text{IGAP} & \Rightarrow & \text{GAP}
\end{array}
$$

Figure 5: Relationships between the APs and the GAPs

4.3 Algebraic characterizations of the DIP and the CIP

Now, we give algebraic characterizations of the DIP and the CIP, respectively.

THEOREM 21 *For each substructural logic* $\mathbf{L}$*,* $\mathbf{L}$ *has the DIP iff* $\mathsf{V}(\mathbf{L})$ *has the IGAP.*

Proof. By Lemma 12, it is enough to show that $\mathbf{L}$ has the DIP* iff $\mathsf{V}(\mathbf{L})$ has the IGAP. Once again, the proof goes in the similar way to the proof of Theorem 16. So, we point out only places where modifications are necessary. First, we show that $\mathsf{V}(\mathbf{L})$ has the IGAP, by assuming the DIP* of $\mathbf{L}$. Define $\Gamma_{\mathbf{B}}$ and $\Gamma_{\mathbf{C}}$ in the same way as before. But, differently from the proof of Theorem 16, we define a binary relation $\equiv$ on $\mathbf{Fm}(Y \cup Z)$ by

$$\beta \equiv \gamma \text{ iff } \Gamma_{\mathbf{B}} \vdash (\beta\backslash\gamma) \wedge (\gamma\backslash\beta).$$

Now, let $\mathbf{D}$ be the quotient algebra $\mathbf{Fm}(Y \cup Z)/\equiv$ which is in $\mathsf{V}(\mathbf{L})$. Also, let $\mathbf{D}_1$ and $\mathbf{D}_2$ be $\mathbf{Fm}(Y)/\equiv$ and $\mathbf{Fm}(Z)/\equiv$, respectively. Clearly they are subalgebras of $\mathbf{D}$. Now define mappings $i : \mathbf{D}_1 \to \mathbf{B}$ and $j : \mathbf{D}_2 \to \mathbf{C}$ by

- $i(\varphi/\equiv) = \eta_{\mathbf{B}}(\varphi)$ for a formula $\varphi \in \mathbf{Fm}(Y)$,
- $j(\psi/\equiv) = \eta_{\mathbf{C}}(\psi)$ for a formula $\psi \in \mathbf{Fm}(Z)$.

For all formulas $\varphi, \varphi' \in \mathbf{Fm}(Y)$,

$$\varphi \equiv \varphi' \text{ iff } \Gamma_{\mathbf{B}} \vdash (\varphi\backslash\varphi') \wedge (\varphi'\backslash\varphi) \text{ iff } \eta_{\mathbf{B}}(\varphi) = \eta_{\mathbf{B}}(\varphi').$$

Thus, i is a well-defined isomorphism. On the other hand, since for each $\alpha \in \mathbf{Fm}(X)$, $\Gamma_{\mathbf{B}} \vdash \alpha$ iff $\Gamma_{\mathbf{C}} \vdash \alpha$, and $\Gamma_{\mathbf{B}} \vdash \beta$ implies $\Gamma_{\mathbf{C}} \vdash \beta$ for any $\beta \in \mathbf{Fm}(Z)$, by the DIP*. Therefore, for all formulas $\psi, \psi' \in \mathbf{Fm}(Z)$,

$$\psi \equiv \psi' \text{ iff } \Gamma_{\mathbf{B}} \vdash (\psi\backslash\psi') \wedge (\psi'\backslash\psi), \text{ which implies } \Gamma_{\mathbf{C}} \vdash (\psi\backslash\psi') \wedge (\psi'\backslash\psi), \text{ iff } \eta_{\mathbf{C}}(\psi) = \eta_{\mathbf{C}}(\psi').$$

Thus, j is well-defined and is a surjective homomorphism. Moreover, for each $a \in A$ there exists a formula $\alpha \in \mathbf{Fm}(X)$ such that $\eta_{\mathbf{A}}(\alpha) = a$. Take $(\alpha/\equiv)$ for d in the definition of the GAP. Then it is easy to see that this d satisfies the required conditions for the GAP. Thus the IGAP holds.

Conversely, suppose that $\mathsf{V}(\mathbf{L})$ has the IGAP. Let Γ and $\Sigma \cup \{\psi\}$ be sets of formulas, and let $Y = \mathrm{var}(\Gamma)$ and $Z = \mathrm{var}(\Sigma \cup \{\psi\})$. Moreover, we assume that

- $\Gamma \vdash \alpha$ iff $\Sigma \vdash \alpha$ for every formula $\alpha \in \mathbf{Fm}(X)$, where $X = Y \cap Z$,
- $\Gamma \vdash \psi$.

The proof goes also similarly to the proof of Theorem 16. We introduce algebras $\mathbf{Fm}(X)/\Delta$, $\mathbf{Fm}(Y)/\Gamma$ and $\mathbf{Fm}(Z)/\Sigma$, and then define mappings $f : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Y)/\Gamma$ and $g : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Z)/\Sigma$ by

$$f(\alpha/\equiv_{\Delta}) = \alpha/\equiv_{\Gamma} \text{ and } g(\alpha/\equiv_{\Delta}) = \alpha/\equiv_{\Sigma}.$$

As before, f and g are shown to be well-defined embeddings. By using the IGAP of $\mathsf{V}(\mathbf{L})$, we have that there exist an algebra $\mathbf{D} \in \mathsf{V}(\mathbf{L})$, subalgebras $\mathbf{D}_1$ and $\mathbf{D}_2$ of $\mathbf{D}$, and an isomorphism $i : \mathbf{D}_1 \to \mathbf{Fm}(Y)/\Gamma$ and a surjective homomorphism $j : \mathbf{D}_2 \to \mathbf{Fm}(Z)/\Sigma$ such that for all $a \in \mathbf{Fm}(X)/\Delta$ there exists $d \in D_1 \cap D_2$ such that $f(a) = i(d)$ and $g(a) = j(d)$. Note that such an element d determines uniquely since i is injective. Now, we define a valuation w over $\mathbf{D}$ for formulas in $\mathbf{Fm}(Y \cup Z)$ so as to satisfy the following: For every $x \in Y \cup Z$,

$$w(x) = \begin{cases} i^{-1}(x/\equiv_{\Gamma}) & \text{if } x \in Y \\ \text{an element in } j^{-1}(x/\equiv_{\Sigma}) & \text{if } x \in Z \setminus Y. \end{cases}$$

Then, $(i \circ w)(x) = (x/\equiv_\Gamma)$ for $x \in Y$. Moreover, we can show that $(j \circ w)(x) = (x/\equiv_\Sigma)$ for $x \in Z$. For, if $x \in Z \setminus Y$ then this is trivial by the definition of w. Otherwise, $x \in Z \cap Y \subseteq Y$. By the definition of w, we have $w(x) = i^{-1}(x/\equiv_\Gamma)$, namely $f(x/\equiv_\Delta) = (x/\equiv_\Gamma) = i(w(x))$ holds. On the other hand, since $x/\equiv_\Delta \in \mathbf{Fm}(X)/\Delta$, by the condition of the IGAP, there exists some $d \in D_1 \cap D_2$ such that $f(x/\equiv_\Delta) = i(d)$ and $g(x/\equiv_\Delta) = j(d)$. Thus, $i(d) = f(x/\equiv_\Delta) = i(w(x))$ holds. Since i is injective, we have $d = w(x)$, and hence $g(x/\equiv_\Delta) = j(w(x))$. Therefore, $(j \circ w)(x) = (x/\equiv_\Sigma)$ holds for $x \in Z$. Now, the mapping w is extended to a homomorphism from $\mathbf{Fm}(Y \cup Z)$ to $\mathbf{D}$, which satisfies that:

- $(i \circ w)(\gamma) = (\gamma/\equiv_\Gamma)$ for $\gamma \in \mathbf{Fm}(Y)$,
- $(j \circ w)(\beta) = (\beta/\equiv_\Sigma)$ for $\beta \in \mathbf{Fm}(Z)$.

For each formula $\gamma \in \Gamma$, $(i \circ w)(\gamma) = (\gamma/\equiv_\Gamma) \geq 1_{\mathbf{Fm}(Y)/\Gamma}$ and hence $w(\gamma) \geq 1_\mathbf{D}$ by the injectivity of i. Thus, $\mathbf{D}, w \models \gamma$ for every $\gamma \in \Gamma$. So, our assumption $\Gamma \vdash \psi$ implies $\mathbf{D}, w \models \psi$. Thus, $(j \circ w)(\psi) = (\psi/\equiv_\Sigma) \geq 1_{\mathbf{Fm}(Z)/\Sigma}$, which implies $\Sigma \vdash \psi$. This completes our proof. □

THEOREM 22 *For each substructural logic* $\mathbf{L}$, $\mathbf{L}$ *has the CIP iff* $\mathsf{V}(\mathbf{L})$ *has the superGAP.*

Proof. The proof goes similarly to the proofs of Theorems 18 and 21. First, we show that the superGAP holds, by assuming the CIP. This time we define a binary relation $\equiv$ on $\mathbf{Fm}(Y \cup Z)$ simply by

$$\beta \equiv \gamma \text{ iff } \vdash (\beta\backslash\gamma) \wedge (\gamma\backslash\beta).$$

Then, let $\mathbf{D}$ be the quotient algebra $\mathbf{Fm}(Y \cup Z)/\equiv$ which is in $\mathsf{V}(\mathbf{L})$, and let $\mathbf{D}_1$ and $\mathbf{D}_2$ be $\mathbf{Fm}(Y)/\equiv$ and $\mathbf{Fm}(Z)/\equiv$, respectively. We define mappings $i : \mathbf{D}_1 \to \mathbf{B}$ and $j : \mathbf{D}_2 \to \mathbf{C}$ by

- $i(\varphi/\equiv) = \eta_\mathbf{B}(\varphi)$ for a formula $\varphi \in \mathbf{Fm}(Y)$,
- $j(\psi/\equiv) = \eta_\mathbf{C}(\psi)$ for a formula $\psi \in \mathbf{Fm}(Z)$.

First, we show that both i and j are well-defined and are surjective homomorphisms. For all formulas $\varphi, \varphi' \in \mathbf{Fm}(Y)$,

$$\varphi \equiv \varphi' \text{ iff } \vdash (\varphi\backslash\varphi') \wedge (\varphi\backslash\varphi'), \text{ which implies } \Gamma_\mathbf{B} \vdash (\varphi\backslash\varphi') \wedge (\varphi'\backslash\varphi), \text{ iff } \eta_\mathbf{B}(\varphi) = \eta_\mathbf{B}(\varphi').$$

Thus, i is a well-defined surjective homomorphism. Similarly, j is also a well-defined surjective homomorphism. It is easily shown that they satisfy the first condition of the superGAP, in a similar way to the proof of Theorem 21. To show the second condition, suppose that $d_1 \in D_1$, $d_2 \in D_2$ such that $d_1 \leq d_2$. Then, there exist formulas $\varphi \in \mathbf{Fm}(Y)$ and $\psi \in \mathbf{Fm}(Z)$ such that $d_1 = (\varphi/\equiv), d_2 = (\psi/\equiv)$ and $\vdash \varphi\backslash\psi$. By the CIP, there exists a formula $\delta \in \mathbf{Fm}(X)$ such that both $\vdash \varphi\backslash\delta$ and $\vdash \delta\backslash\psi$ hold. Thus, $(\varphi/\equiv) \leq (\delta/\equiv)$ holds in $\mathbf{D}_1$ and $(\delta/\equiv) \leq (\psi/\equiv)$ holds in $\mathbf{D}_2$. Let $a = \eta_\mathbf{A}(\delta)$. Then, we have $i(d_1) = i(\varphi/\equiv) \leq i(\delta/\equiv) = \eta_\mathbf{B}(\delta) = f(\eta_\mathbf{A}(\delta)) = f(a)$, and similarly $g(a) \leq j(d_2)$. Thus, the superGAP holds.

Conversely, suppose that the superGAP holds for $\mathsf{V}(\mathbf{L})$, and $\vdash_\mathbf{L} \varphi\backslash\psi$ holds. Let's define sets of variables X, Y and Z by

$$Y = \mathrm{var}(\varphi), \quad Z = \mathrm{var}(\psi) \quad \text{and} \quad X = Y \cap Z.$$

By using the same congruence relation $\equiv$ as in the above, define $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ by $\mathbf{Fm}(X)/\equiv$, $\mathbf{Fm}(Y)/\equiv$ and $\mathbf{Fm}(Z)/\equiv$, respectively. Then there exist inclusion maps $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$. By the superGAP, there exist an algebra $\mathbf{D}$ in $\mathsf{V}(\mathbf{L})$, subalgebras $\mathbf{D}_1$ and $\mathbf{D}_2$ of $\mathbf{D}$, and surjective homomorphisms $i : \mathbf{D}_1 \to \mathbf{B}$ and $j : \mathbf{D}_2 \to \mathbf{C}$ that satisfy the following:

1. for all $a \in A$ there exists $d \in D_1 \cap D_2$ such that $f(a) = i(d)$ and $g(a) = j(d)$,

2. for all $d_1 \in D_1$, $d_2 \in D_2$ such that $d_1 \leq d_2$ ($d_2 \leq d_1$), there exists $a \in A$ such that $i(d_1) \leq f(a)$ and $g(a) \leq j(d_2)$ ($j(d_2) \leq g(a)$ and $f(a) \leq i(d_1)$, respectively).

Now, we define a valuation w over $\mathbf{D}$ for formulas in $\mathbf{Fm}(Y \cup Z)$. By the above condition 1, for each variable $x \in X$, there exists $d \in D_1 \cap D_2$ such that $f(x/\equiv) = i(d)$ and $g(x/\equiv) = j(d)$. Let d_x be one of such elements d. Now, define w as follows: For every $x \in Y \cup Z$,

$$w(x) = \begin{cases} d_x & \text{if } x \in X \\ \text{an element in } i^{-1}(x/\equiv) & \text{if } x \in Y \setminus X \\ \text{an element in } j^{-1}(x/\equiv) & \text{if } x \in Z \setminus X. \end{cases}$$

Thus, we can show that $(i \circ w)(x) = (x/\equiv)$ for $x \in Y$ and $(j \circ w)(x) = (x/\equiv)$ for $x \in Z$. As usual, the mapping w is extended to a mapping from $\mathbf{Fm}(Y \cup Z)$ to $\mathbf{D}$, for which the following holds.

- $(i \circ w)(\gamma) = (\gamma/\equiv)$ for $\gamma \in \mathbf{Fm}(Y)$,
- $(j \circ w)(\beta) = (\beta/\equiv)$ for $\beta \in \mathbf{Fm}(Z)$.

Since $\vdash \varphi\backslash\psi$ holds by our assumption, we have $w(\varphi\backslash\psi) \geq 1_{\mathbf{D}}$. Thus, $w(\varphi) \leq w(\psi)$. Since $w(\varphi) \in D_1$ and $w(\psi) \in D_2$, by the second condition of the superGAP, there exists $a \in A$ such that $i(w(\varphi)) \leq f(a) = a$ and $a = g(a) \leq j(w(\psi))$ (recall that both f and g are inclusion maps). Let $a = (\delta/\equiv)$ for $\delta \in \mathbf{Fm}(X)$. Then, $(\varphi/\equiv) \leq (\delta/\equiv)$ and $(\delta/\equiv) \leq (\psi/\equiv)$. That is, $\vdash \varphi\backslash\delta$ and $\vdash \delta\backslash\psi$. Thus, the CIP holds. □

In the rest of this subsection, we discuss what constitutes a logical propety corresponding to each of the other GAPs. Interestingly enough, the GAP always holds. More precisely, the following holds.

THEOREM 23 *For each substructural logic* $\mathbf{L}$*,* $\mathsf{V}(\mathbf{L})$ *has the GAP. In other words, every subvariety of* $\mathbf{FL}$*-algebras has the GAP.*

Proof. The proof goes in the same way as the first half of the proof of Theorem 22. Notice that in the proof of the GAP of $\mathsf{V}(\mathbf{L})$, any logical property of $\mathbf{L}$ is not used at all. Thus, $\mathsf{V}(\mathbf{L})$ has the GAP for each substructural logic $\mathbf{L}$. □

THEOREM 24 *For each substructural logic* $\mathbf{L}$*, the following are equivalent.*

(1) For all formulas φ, ψ*, if* $\vdash_{\mathbf{L}} (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$ *then there exists some formula* δ *such that*

(a) $\vdash_{\mathbf{L}} (\varphi\backslash\delta) \wedge (\delta\backslash\varphi)$ *and* $\vdash_{\mathbf{L}} (\psi\backslash\delta) \wedge (\delta\backslash\psi)$*, and*

(b) $var(\delta) \subseteq var(\varphi) \cap var(\psi)$*.*

(2) $\mathsf{V}(\mathbf{L})$ *has the strongGAP.*

Proof. The proof goes quite similarly to the proof of Theorem 22. First, we show that $\mathsf{V}(\mathbf{L})$ has the strongGAP, by assuming the condition (1). Take $\mathbf{D}$, $\mathbf{D}_1$, $\mathbf{D}_2$, i and j as before. As shown before, they satisfy the first conditon of the strongGAP. To show the second condition, let $d \in D_1 \cap D_2$. Then, there exist formulas $\varphi \in \mathbf{Fm}(Y)$ and $\psi \in \mathbf{Fm}(Z)$ such that $d = (\varphi/\equiv) = (\psi/\equiv)$ and $\vdash (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$. By our assumption (1), there exists a formula $\delta \in \mathbf{Fm}(X)$ such that both $\vdash (\varphi\backslash\delta) \wedge (\delta\backslash\varphi)$ and $\vdash (\psi\backslash\delta) \wedge (\delta\backslash\psi)$ hold. Thus, $(\varphi/\equiv) = (\delta/\equiv)$ holds in $\mathbf{D_1}$ and $(\psi/\equiv) = (\delta/\equiv)$ holds in $\mathbf{D_2}$. Let $a = \eta_{\mathbf{A}}(\delta)$. Then, we have $i(d) = i(\varphi/\equiv) = i(\delta/\equiv) = \eta_{\mathbf{B}}(\delta) = f(\eta_{\mathbf{A}}(\delta)) = f(a)$, and similary $j(d) = g(a)$. Thus, the strongGAP holds.

Conversely, suppose that the strongGAP holds for $\mathsf{V}(\mathbf{L})$, and $\vdash_{\mathbf{L}} (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$ holds. Let us take $\mathbf{A}, \mathbf{B}, \mathbf{C}, f$ and g in the same way as the proof of Theorem 22. By the strongGAP, there exist an algebra $\mathbf{D}$ in $\mathsf{V}(\mathbf{L})$, subalgebras $\mathbf{D}_1$ and $\mathbf{D}_2$ of $\mathbf{D}$, and surjective homomorphisms $i : \mathbf{D}_1 \to \mathbf{B}$ and $j : \mathbf{D}_2 \to \mathbf{C}$ that satisfy the following:

1. for all $a \in A$ there exists $d \in D_1 \cap D_2$ such that $f(a) = i(d)$ and $g(a) = j(d)$,

2. for all $d \in D_1 \cap D_2$, there exists $a \in A$ such that $i(d) = f(a)$ and $j(d) = g(a)$.

Define a valuation w over $\mathbf{D}$ for formulas in $\mathbf{Fm}(Y \cup Z)$ in the same way as the proof of Theorem 22. Since $\vdash (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$ holds by our assumption, we have $w((\varphi\backslash\psi) \wedge (\psi\backslash\varphi)) \geq 1_{\mathbf{D}}$. Thus, $w(\varphi) = w(\psi)$. Since $w(\varphi) \in D_1$ and $w(\psi) \in D_2$, by the second condition of the strongGAP, there exists $a \in A$ such that $i(w(\varphi)) = f(a) = a$ and $j(w(\psi)) = g(a) = a$. Let $a = (\delta/\equiv)$ for $\delta \in \mathbf{Fm}(X)$. Then $(\varphi/\equiv) = (\delta/\equiv)$ and $(\psi/\equiv) = (\delta/\equiv)$. That is, $\vdash (\varphi\backslash\delta) \wedge (\delta\backslash\varphi)$ and $\vdash (\psi\backslash\delta) \wedge (\delta\backslash\psi)$. Thus, the condition (1) holds. □

THEOREM 25 *For each substructural logic* $\mathbf{L}$*, the following are equivalent.*

(1) For any set of formulas $\Gamma \cup \Sigma \cup \{\varphi, \psi\}$ *such that* $\Gamma \vdash_{\mathbf{L}} \sigma$ *iff* $\Sigma \vdash_{\mathbf{L}} \sigma$ *for every formula* σ *with* $var(\sigma) \subseteq var(\Gamma \cup \{\varphi\}) \cap var(\Sigma \cup \{\psi\})$*, if* $\Gamma \vdash_{\mathbf{L}} \varphi\backslash\psi$ $(\Gamma \vdash_{\mathbf{L}} \psi\backslash\varphi)$ *then there exists some formula* δ *such that*

   (a) $\Gamma \vdash_{\mathbf{L}} \varphi\backslash\delta$ *and* $\Sigma \vdash_{\mathbf{L}} \delta\backslash\psi$ $(\Gamma \vdash_{\mathbf{L}} \delta\backslash\varphi$ *and* $\Sigma \vdash_{\mathbf{L}} \psi\backslash\delta$*, respectively), and*

   (b) $var(\delta) \subseteq var(\Gamma \cup \{\varphi\}) \cap var(\Sigma \cup \{\psi\})$*.*

(2) $\mathsf{V}(\mathbf{L})$ *has the superIGAP.*

Proof. First, we show that $\mathsf{V}(\mathbf{L})$ has the superIGAP, by assuming the condition (1). Notice that the condition (1) implies the DIP*. For, if $\Gamma \vdash_{\mathbf{L}} \psi$ holds for a set of formulas $\Gamma \cup \Sigma \cup \{\psi\}$ which satisfies the assumption of the DIP*, then $\Gamma \vdash_{\mathbf{L}} 1\backslash\psi$ also holds. By the condition (1), there exists a formula δ with $\mathrm{var}(\delta) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\Sigma \cup \{\psi\})$ such that both $\Gamma \vdash_{\mathbf{L}} 1\backslash\delta$ and $\Sigma \vdash_{\mathbf{L}} \delta\backslash\psi$, which imply $\Gamma \vdash_{\mathbf{L}} \delta$ and $\delta, \Sigma \vdash_{\mathbf{L}} \psi$. By the definition of $\Gamma \cup \Sigma \cup \{\psi\}$, $\Gamma \vdash_{\mathbf{L}} \delta$ implies $\Sigma \vdash_{\mathbf{L}} \delta$. Thus, we have $\Sigma \vdash_{\mathbf{L}} \psi$.

Now, for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathsf{V}(\mathbf{L})$ and embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, define $\mathbf{D}, \mathbf{D}_1, \mathbf{D}_2, i$ and j in the same way as in the proof of Theorem 21. Then, they satisfy the first condition of the superIGAP. To show the second condition of the superIGAP, suppose that $d_1 \leq d_2$ $(d_2 \leq d_1)$ holds for $d_1 \in D_1$ and $d_2 \in D_2$. Then, there exist some formulas $\varphi \in \mathbf{Fm}(Y)$ and $\psi \in \mathbf{Fm}(Z)$ such that $d_1 = (\varphi/\equiv), d_2 = (\psi/\equiv)$ and $\Gamma_{\mathbf{B}} \vdash \varphi\backslash\psi$ $(\Gamma_{\mathbf{B}} \vdash \psi\backslash\varphi$, respectively). By our assumption of the logical property (1), there is a formula δ with $\mathrm{var}(\delta) \subseteq \mathrm{var}(\Gamma_{\mathbf{B}} \cup \{\varphi\}) \cap \mathrm{var}(\Gamma_{\mathbf{C}} \cup \{\psi\}) \subseteq Y \cap Z = X$ such that $\Gamma_{\mathbf{B}} \vdash \varphi\backslash\delta$ and $\Gamma_{\mathbf{C}} \vdash \delta\backslash\psi$ $(\Gamma_{\mathbf{B}} \vdash \delta\backslash\varphi$ and $\Gamma_{\mathbf{C}} \vdash \psi\backslash\delta$, respectively). Let $a = \eta_{\mathbf{A}}(\delta) \in A$. Then, $i(d_1) = i(\varphi/\equiv) = \eta_{\mathbf{B}}(\varphi) \leq \eta_{\mathbf{B}}(\delta) = f(\eta_{\mathbf{A}}(\delta)) = f(a)$ and $g(a) = g(\eta_{\mathbf{A}}(\delta)) = \eta_{\mathbf{C}}(\delta) \leq \eta_{\mathbf{C}}(\psi) = j(\psi/\equiv) = j(d_2)$ $(f(a) \leq i(d_1)$ and $j(d_2) \leq g(a)$, respectively).

Conversely, suppose that the superIGAP holds for $\mathsf{V}(\mathbf{L})$. Let $\Gamma \cup \Sigma \cup \{\varphi, \psi\}$ be a set of formulas, and let $Y = \mathrm{var}(\Gamma \cup \{\varphi\})$ and $Z = \mathrm{var}(\Sigma \cup \{\psi\})$. Moreover, we assume that

- $\Gamma \vdash \sigma$ iff $\Sigma \vdash \sigma$ for every formula $\sigma \in \mathbf{Fm}(X)$, where $X = Y \cap Z$,
- $\Gamma \vdash \varphi\backslash\psi$ $(\Gamma \vdash \psi\backslash\varphi)$.

Let us take $\mathbf{Fm}(X)/\Delta, \mathbf{Fm}(Y)/\Gamma, \mathbf{Fm}(Z)/\Sigma, f$ and g in the same way as in the proof of Theorem 21. By the superIGAP, there exist an algebra $\mathbf{D}$ in $\mathsf{V}(\mathbf{L})$, subalgebras $\mathbf{D}_1$ and $\mathbf{D}_2$ of $\mathbf{D}$, and an isomorphism $i : \mathbf{D}_1 \to \mathbf{Fm}(Y)/\Gamma$ and a surjective homomorphism $j : \mathbf{D}_2 \to \mathbf{Fm}(Z)/\Sigma$ such that

1. for all $a \in \mathbf{Fm}(X)/\Delta$ there exists $d \in D_1 \cap D_2$ such that $f(a) = i(d)$ and $g(a) = j(d)$,

2. for all $d_1 \in D_1,\ d_2 \in D_2$ such that $d_1 \leq d_2$ $(d_2 \leq d_1)$, there exists $a \in \mathbf{Fm}(X)/\Delta$ such that $i(d_1) \leq f(a)$ and $g(a) \leq j(d_2)$ $(j(d_2) \leq g(a)$ and $f(a) \leq i(d_1)$, respectively).

Define a valuation w over $\mathbf{D}$ for formulas in $\mathbf{Fm}(Y \cup Z)$ in the same way as in the proof of Theorem 21. As shown before, for each formula $\gamma \in \Gamma$, we have $\mathbf{D}, w \models \gamma$. So, our assumpiton $\Gamma \vdash \varphi\backslash\psi$ ($\Gamma \vdash \psi\backslash\varphi$) implies $w(\varphi) \leq w(\psi)$ ($w(\psi) \leq w(\varphi)$, respectively). Since $w(\varphi) \in D_1$ and $w(\psi) \in D_2$, by the second condition of the superIGAP, there exists some $\delta/\equiv_\Delta \in \mathbf{Fm}(X)/\Delta$ such that $(\varphi/\equiv_\Gamma) = (i \circ w)(\varphi) \leq f(\delta/\equiv_\Delta) = (\delta/\equiv_\Gamma)$ and $(\delta/\equiv_\Sigma) = g(\delta/\equiv_\Delta) \leq (j \circ w)(\psi) = (\psi/\equiv_\Sigma)$ ($(\psi/\equiv_\Sigma) \leq (\delta/\equiv_\Sigma)$ and $(\delta/\equiv_\Gamma) \leq (\varphi/\equiv_\Gamma)$, respectively). That is, $\Gamma \vdash_\mathbf{L} \varphi\backslash\delta$ and $\Sigma \vdash_\mathbf{L} \delta\backslash\psi$ ($\Gamma \vdash_\mathbf{L} \delta\backslash\varphi$ and $\Sigma \vdash_\mathbf{L} \psi\backslash\delta$, respectively). Thus, the condition (1) holds. □

THEOREM 26 *For each substructural logic* $\mathbf{L}$*, the following are equivalent.*

(1) For any set of formulas $\Gamma \cup \Sigma \cup \{\varphi, \psi\}$ *such that* $\Gamma \vdash_\mathbf{L} \sigma$ *iff* $\Sigma \vdash_\mathbf{L} \sigma$ *for every formula* σ *with* $var(\sigma) \subseteq var(\Gamma \cup \{\varphi\}) \cap var(\Sigma \cup \{\psi\})$*, if* $\Gamma \vdash_\mathbf{L} (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$ *then there exists some formula* δ *such that*

(a) $\Gamma \vdash_\mathbf{L} (\varphi\backslash\delta) \wedge (\delta\backslash\varphi)$ *and* $\Sigma \vdash_\mathbf{L} (\psi\backslash\delta) \wedge (\delta\backslash\psi)$*, and*

(b) $var(\delta) \subseteq var(\Gamma \cup \{\varphi\}) \cap var(\Sigma \cup \{\psi\})$*.*

(2) $\mathsf{V}(\mathbf{L})$ *has the strongIGAP.*

Proof. First, we show that $\mathsf{V}(\mathbf{L})$ has the strongIGAP, by assuming the condition (1). As with the first condition in Theorem 25, the condition (1) also implies the DIP*. For, if $\Gamma \vdash_\mathbf{L} \psi$ holds for a set of formulas $\Gamma \cup \Sigma \cup \{\psi\}$ which satisfies the assumption of the DIP*, then we have $\Gamma \vdash_\mathbf{L} (1\backslash(\psi \wedge 1)) \wedge ((\psi \wedge 1)\backslash 1)$. By (1), there exists a formula δ with $\mathrm{var}(\delta) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\Sigma \cup \{\psi\})$ such that both $\Gamma \vdash_\mathbf{L} (1\backslash\delta) \wedge (\delta\backslash 1)$ and $\Sigma \vdash_\mathbf{L} ((\psi \wedge 1)\backslash\delta) \wedge (\delta\backslash(\psi \wedge 1))$, which imply $\Gamma \vdash_\mathbf{L} \delta$ and $\delta, \Sigma, \vdash_\mathbf{L} \psi$. Thus, by the definition of $\Gamma \cup \Sigma \cup \{\psi\}$, we have $\Sigma \vdash_\mathbf{L} \psi$. In the same way as in the proof of Theorem 21, for any embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, define $\mathbf{D}, \mathbf{D}_1, \mathbf{D}_2, i$ and j. Of course, they satisfy the first condition of the strongIGAP. To show the second condition of the strongIGAP, let $d \in D_1 \cap D_2$. Then, there exist some formulas $\varphi \in \mathbf{Fm}(Y)$ and $\psi \in \mathbf{Fm}(Z)$ such that $d = (\varphi/\equiv) = (\psi/\equiv)$ and $\Gamma_\mathbf{B} \vdash (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$. By our assumption (1), there is a formula δ with $\mathrm{var}(\delta) \subseteq X$ such that $\Gamma_\mathbf{B} \vdash (\varphi\backslash\delta) \wedge (\delta\backslash\varphi)$ and $\Gamma_\mathbf{C} \vdash (\psi\backslash\delta) \wedge (\delta\backslash\psi)$. Let $a = \eta_\mathbf{A}(\delta) \in A$. Then, $i(d) = i(\varphi/\equiv) = \eta_\mathbf{B}(\varphi) = \eta_\mathbf{B}(\delta) = f(\eta_\mathbf{A}(\delta)) = f(a)$ and $g(a) = g(\eta_\mathbf{A}(\delta)) = \eta_\mathbf{C}(\delta) = \eta_\mathbf{C}(\psi) = j(\psi/\equiv) = j(d)$.

Conversely, suppose that the strongIGAP holds for $\mathsf{V}(\mathbf{L})$. Let $\Gamma \cup \Sigma \cup \{\varphi, \psi\}$ be a set of formulas, and let $Y = \mathrm{var}(\Gamma \cup \{\varphi\})$ and $Z = \mathrm{var}(\Sigma \cup \{\psi\})$. Moreover, we assume that

- $\Gamma \vdash \sigma$ iff $\Sigma \vdash \sigma$ for every formula $\sigma \in \mathbf{Fm}(X)$, where $X = Y \cap Z$,
- $\Gamma \vdash (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$.

In the same way as in the proof of Theorem 21, let us take $\mathbf{Fm}(X)/\Delta, \mathbf{Fm}(Y)/\Gamma, \mathbf{Fm}(Z)/\Sigma, f$ and g. By the strongIGAP, there exist an algebra $\mathbf{D}$ in $\mathsf{V}(\mathbf{L})$, subalgebras $\mathbf{D}_1$ and $\mathbf{D}_2$ of $\mathbf{D}$, and an isomorphism $i : \mathbf{D}_1 \to \mathbf{Fm}(Y)/\Gamma$ and a surjective homomorphism $j : \mathbf{D}_2 \to \mathbf{Fm}(Z)/\Sigma$ such that

1. for all $a \in \mathbf{Fm}(X)/\Delta$ there exists $d \in D_1 \cap D_2$ such that $f(a) = i(d)$ and $g(a) = j(d)$,
2. for all $d \in D_1 \cap D_2$, there exists $a \in \mathbf{Fm}(X)/\Delta$ such that $i(d) = f(a)$ and $j(d) = g(a)$.

Define a valuation w over $\mathbf{D}$ for formulas in $\mathbf{Fm}(Y \cup Z)$ in the same way as in the proof of Theorem 21. As shown before, for each formula $\gamma \in \Gamma$, we have $\mathbf{D}, w \models \gamma$. So, our assumption $\Gamma \vdash (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$ implies $w(\varphi) = w(\psi)$. Since $w(\varphi) \in D_1$ and $w(\psi) \in D_2$, by the second condition of the strongIGAP, there exists some $\delta \in \mathbf{Fm}(X)$ such that $(\varphi/\equiv_\Gamma) = (i \circ w)(\varphi) = f(\delta/\equiv_\Delta) = (\delta/\equiv_\Gamma)$ and $(\psi/\equiv_\Sigma) = (j \circ w)(\psi) = g(\delta/\equiv_\Delta) = (\delta/\equiv_\Sigma)$. That is, $\Gamma \vdash_\mathbf{L} (\varphi\backslash\delta) \wedge (\delta\backslash\varphi)$ and $\Sigma \vdash_\mathbf{L} (\psi\backslash\delta) \wedge (\delta\backslash\psi)$. Thus, the condition (1) holds. □

We summarize the algebraic characterizations of the DIP and the CIP, and relationships between GAPs in Figure 6.

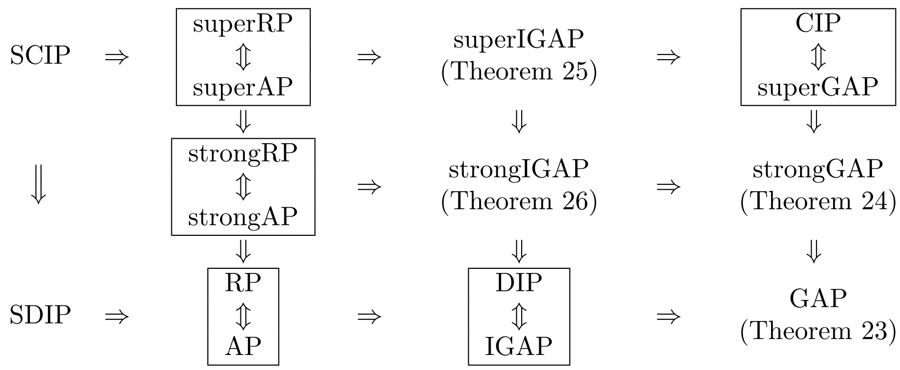

Figure 6: Algebraic characterizations of the DIP and the CIP

5 Algebraic characterizations of the SCIP and the SDIP

We give next algebraic characterizations of both the SCIP and the SDIP. An algebraic characterization of the SDIP by transferable injections is essentially given by Wroński in [36]. Here, we give another characterization of the SDIP.

5.1 Commutative homomorphisms diagrams

DEFINITION 13 *A variety $\mathcal{V}$ has the transferable injections property (TI), if for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathcal{V}$ and for any embedding $f : \mathbf{A} \to \mathbf{B}$ and any homomorphism $g : \mathbf{A} \to \mathbf{C}$, there exists an algebra $\mathbf{D}$ in $\mathcal{V}$, a homomorphism $h : \mathbf{B} \to \mathbf{D}$, and an embedding $k : \mathbf{C} \to \mathbf{D}$ such that $h \circ f = k \circ g$.*

Notice that the difference between the TI and the AP is that the mappings g and h are *homomorphisms* in the case of the TI.

The following result is well-known. In fact, Bacsich showed in [2] that the TI is equivalent to the AP with the CEP (congruence extension property). For the completeness of the present paper, we give here a proof.

LEMMA 27 *For any variety $\mathcal{V}$, if $\mathcal{V}$ has the TI then it has the AP.*

Proof. Suppose that $\mathcal{V}$ has the TI, and let $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$ be embeddings for $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{V}$. Then, by the TI, there exist $\mathbf{D}_1, \mathbf{D}_2 \in \mathcal{V}$, and homomorphisms $h_1 : \mathbf{B} \to \mathbf{D}_1$ and $k_2 : \mathbf{C} \to \mathbf{D}_2$ and embeddings $k_1 : \mathbf{C} \to \mathbf{D}_1$ and $h_2 : \mathbf{B} \to \mathbf{D}_2$ such that $h_1 \circ f = k_1 \circ g$ and $h_2 \circ f = k_2 \circ g$. Let $\mathbf{D} = \mathbf{D}_1 \times \mathbf{D}_2$ and define $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ by $h(b) = (h_1(b), h_2(b))$ and $k(c) = (k_1(c), k_2(c))$. Then, it is easy to see that both h and k are embeddings satisfying $h \circ f = k \circ g$. Thus, $\mathcal{V}$ has the AP. □

Now, we can introduce the most general algebraic property of this kind by replacing all embeddings by homomorphisms in the definition of the TI.

DEFINITION 14 *A variety $\mathcal{V}$ has the commutative homomorphisms diagrams property (or simply, $\mathcal{V}$ has the CHD), if for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathcal{V}$ and for all homomorphisms $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, there exist an algebra $\mathbf{D}$ in $\mathcal{V}$, and homomorphisms $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ such that $h \circ f = k \circ g$, i.e. every commutaive homomorphisms diagram is completed in $\mathcal{V}$.*

We can regard both the AP and the TI as special cases of the CHD. Besides the AP and the TI, we can introduce several algebraic properties which can be regarded as special cases of the CHD, by replacing some of the homomorphisms by embeddings in the definition. It is easy to see that relationships between these algebraic properties can be shown in the following Figure 7. Here, "$\rightarrow$" and "$\hookrightarrow$" denote a homomorphism and an embedding, respectively, and the arrow $\Rightarrow$ means the implication between two properties.

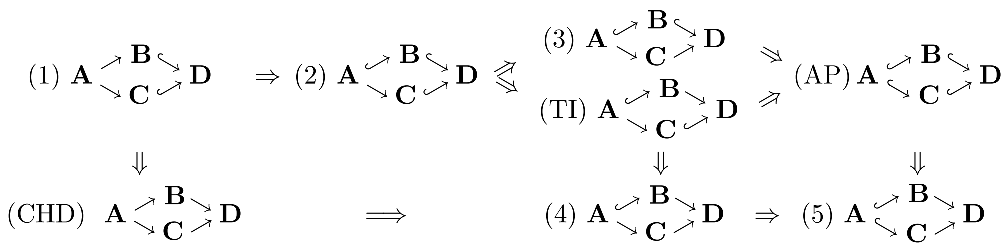

Figure 7: Relationships between the AP, the TI and the CHD

Then the following holds.

LEMMA 28 *The algebraic properties (1), (2) and (3) in Figure 7 are mutually equivalent. In fact, the only trivial variety satisfies them.*

Proof. It is sufficient to show that the only trivial variety satisfies the algebraic property (3). Suppose that a non-trivial variety $\mathcal{V}$ has the algebaric property (3). Then, there exists a non-trivial algebra $\mathbf{A} \in \mathcal{V}$. Take $\mathbf{A}$ for $\mathbf{B}$ and the trivial algebra consisting of the single element $1_{\mathbf{C}}$ for $\mathbf{C}$, respectively, and define $f : \mathbf{A} \rightarrow \mathbf{B}$ and $g : \mathbf{A} \rightarrow \mathbf{C}$ by $f(a) = a$ and $g(a) = 1_{\mathbf{C}}$ for each $a \in A$. Clearly, $\mathbf{B}, \mathbf{C} \in \mathcal{V}$, f is an embedding and g is a homomorphism. By the algebraic property (3), there exist an algebra $\mathbf{D} \in \mathcal{V}$, an embedding $h : \mathbf{B} \rightarrow \mathbf{D}$ and a homomorphism $k : \mathbf{C} \rightarrow \mathbf{D}$ such that $h \circ f = k \circ g$. Since $\mathbf{A}$ is a non-trivial algebra, there exist some $a_1, a_2 \in A$ such that $a_1 \neq a_2$. Then, $h \circ f(a_1) \neq h \circ f(a_2)$ holds. But, by the condition $h \circ f = k \circ g$, we have $h \circ f(a_1) = k \circ g(a_1) = k(1_{\mathbf{C}}) = k \circ g(a_2) = h \circ f(a_2)$. This is a contradiction. Thus, the only trivial variety satisfies (3). $\square$

LEMMA 29 *The CHD, the algebraic properties (4) and (5) in Figure 7 are mutually equivalent. In fact, all varieties satisfy them.*[3]

Proof. It is enough to show that every variety satisfies the CHD. In fact, this is shown simply by taking the one-element algebra for $\mathbf{D}$, when algebras $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{V}$ and homomorphisms $f : \mathbf{A} \rightarrow \mathbf{B}$ and $g : \mathbf{A} \rightarrow \mathbf{C}$ are given for an arbitrary variety $\mathcal{V}$. To make a comparison with algebraic characterizations of related properties introduced later, we give an alternative proof of our lemma below.

Let $\mathcal{V}$ be a variety, and both $f : \mathbf{A} \rightarrow \mathbf{B}$ and $g : \mathbf{A} \rightarrow \mathbf{C}$ be homomorphisms for $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{V}$. In the same way as in the proof of Theorem 16, define sets of variables X, Y, Z and sets of formulas $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}}$. Again, we introduce a binary relation $\equiv$ on $\mathbf{Fm}(Y \cup Z)$ by

$$\beta \equiv \gamma \text{ iff } \Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathsf{L}(\mathcal{V})} (\beta \backslash \gamma) \wedge (\gamma \backslash \beta),$$

[3]Note that RAP introduced by Maksimova is the property (5) but with the condition that $h \circ f$ and hence also $k \circ g$ are isomorphisms (see e.g. [28]).

where $\mathsf{L}(\mathcal{V})$ denotes the substructural logic determined by $\mathcal{V}$. Then, it is easily seen that $\equiv$ is a congruence relation on $\mathbf{Fm}(Y \cup Z)$ and the quotient algebra $\mathbf{Fm}(Y \cup Z)/\equiv$ is a member of $\mathcal{V}$. Define mappings $h : \mathbf{B} \to \mathbf{Fm}(Y \cup Z)/\equiv$ and $k : \mathbf{C} \to \mathbf{Fm}(Y \cup Z)/\equiv$ by

- $h(b) = (\varphi/\equiv)$ when $b = \eta_{\mathbf{B}}(\varphi)$ for a formula $\varphi \in \mathbf{Fm}(Y)$,
- $k(c) = (\psi/\equiv)$ when $c = \eta_{\mathbf{C}}(\psi)$ for a formula $\psi \in \mathbf{Fm}(Z)$.

First, we prove that both h and k are well-defined homomorphisms. To show the well-definedness of h, let $b = b'$, namely there exist some $\varphi, \varphi' \in \mathbf{Fm}(Y)$ such that $b = \eta_{\mathbf{B}}(\varphi) = \eta_{\mathbf{B}}(\varphi') = b'$. Then, $\Gamma_{\mathbf{B}} \vdash_{\mathsf{L}(\mathcal{L})} (\varphi\backslash\varphi') \wedge (\varphi'\backslash\varphi)$, and hence $h(b) = (\varphi/\equiv) = (\varphi'/\equiv) = h(b')$. It is easy to see that h is a homomorphism. Similarly, k is also a well-defined homomorphism. It remains to show that $h \circ f = k \circ g$. Take an arbitrary element $a \in A$. Then there exists a formula $\alpha \in \mathbf{Fm}(X)$ such that $a = \eta_{\mathbf{A}}(\alpha)$. Then, $(h \circ f)(a) = h(f(\eta_{\mathbf{A}}(\alpha))) = h(\eta_{\mathbf{B}}(\alpha)) = (\alpha/\equiv) = k(\eta_{\mathbf{C}}(\alpha)) = k(g(\eta_{\mathbf{A}}(\alpha))) = (k \circ g)(a)$. Thus, every variety has the CHD, by taking this $\mathbf{Fm}(Y \cup Z)/\equiv$ for $\mathbf{D}$. □

The above two lemmas say that there are only four cases of the CHD for varieties.

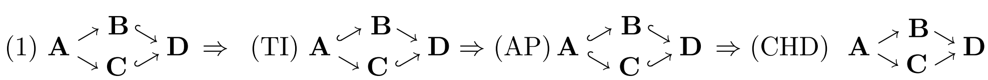

Figure 8: The CHDs for varieties

Now, we strengthen the CHD.

DEFINITION 15 *A variety $\mathcal{V}$ has the commutative homomorphisms diagrams with the condition (α) (CHD(α)), if for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathcal{V}$ and for all homomorphisms $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, there exist an algebra $\mathbf{D}$ in $\mathcal{V}$ and homomorphisms $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ such that*

1. *$h \circ f = k \circ g$,*
2. *(α) for all $b \in B$ and $a \in A$, if $h(b) = k \circ g(a)$ then there exists some $a' \in A$ such that $b = f(a')$ and $g(a) = g(a')$,*

   for all $c \in C$ and $a \in A$, if $k(c) = h \circ f(a)$ then there exists some $a' \in A$ such that $c = g(a')$ and $f(a) = f(a')$.

A variety $\mathcal{V}$ has the commutative homomorphisms diagrams with the condition (β) (CHD(β)), if for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathcal{V}$ and for all homomorphisms $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, there exist an algebra $\mathbf{D}$ in $\mathcal{V}$ and homomorphisms $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ such that

1. *$h \circ f = k \circ g$,*
2. *(β) for all $b \in B$ and $c \in C$, if $1_{\mathbf{D}} \leq h(b)$ ($1_{\mathbf{D}} \leq k(c)$) then there exists some $a \in A$ such that $f(a) \leq b$ and $1_{\mathbf{C}} \leq g(a)$ ($g(a) \leq c$ and $1_{\mathbf{B}} \leq f(a)$, respectively).*

A variety $\mathcal{V}$ has the commutative homomorphisms diagrams with the condition (γ) (CHD(γ)), if for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathcal{V}$ and for all homomorphisms $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, there exist an algebra $\mathbf{D}$ in $\mathcal{V}$ and homomorphisms $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ such that

1. *$h \circ f = k \circ g$,*

2. *(γ) for all $b \in B$ and $c \in C$, if $1_{\mathbf{D}} \leq h(b)$ ($1_{\mathbf{D}} \leq k(c)$) then there exists some $a \in A$ such that $b \in Fg_{\mathbf{B}}(f(a))$ and $1_{\mathbf{C}} \leq g(a)$ ($c \in Fg_{\mathbf{C}}(g(a))$ and $1_{\mathbf{B}} \leq f(a)$, respectivley),*

where $Fg_{\mathbf{E}}(x)$ denotes the deductive filter generated by x in an algebra $\mathbf{E}$.

It is not hard to see that the condition (α) is stronger than (β). For, if $1_{\mathbf{D}} \leq h(b)$ holds for $b \in B$ then $h(b \wedge 1_{\mathbf{B}}) = 1_{\mathbf{D}} = k \circ g(1_{\mathbf{A}})$ also holds. Thus, by the condition (α), there exists some $a \in A$ such that $b \wedge 1_{\mathbf{B}} = f(a)$ and $g(1_{\mathbf{A}}) = g(a)$, which imply $f(a) \leq b$ and $1_{\mathbf{C}} \leq g(a)$. Also, the condition (β) is stronger than (γ) since $x \leq y$ implies $y \in \mathrm{Fg}(x)$.

On the other hand, the CHD(γ) always implies the AP. For, if both f and g are embeddings then, by the condition of (γ), for each $b \in B$ and $c \in C$ with $1_{\mathbf{D}} \leq h(b)$ and $1_{\mathbf{D}} \leq k(c)$, there exist $a_1, a_2 \in A$ such that $b \in \mathrm{Fg}_{\mathbf{B}}(f(a_1))$ and $1_{\mathbf{C}} \leq g(a_1)$, and $c \in \mathrm{Fg}_{\mathbf{C}}(g(a_2))$ and $1_{\mathbf{B}} \leq f(a_2)$. Since g is injective, $1_{\mathbf{C}} \leq g(a_1)$ implies $1_{\mathbf{A}} \leq a_1$, and hence $1_{\mathbf{B}} \leq f(a_1)$ holds also. As $b \in \mathrm{Fg}_{\mathbf{B}}(f(a_1))$, $1_{\mathbf{B}} \leq b$. Thus, h is an embedding. Similarly, from $1_{\mathbf{B}} \leq f(a_2)$, $1_{\mathbf{C}} \leq c$ follows. Thus, k is also an embedding.

Consequently, the injectivity of k (and h) follows from that of f (and g, respectively) if the condition (γ) (a forteriori, (α) or (β)) is supposed.

Corresponding to either the superAP or strongAP, we can introduce a stronger form of the CHD(α), CHD(β) and CHD(γ).

DEFINITION 16 *A variety $\mathcal{V}$ has the super commutative homomorphisms diagrams with (α) (superCHD(α)), if for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathcal{V}$ and for all homomorphisms $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, there exist an algebra $\mathbf{D}$ in $\mathcal{V}$ and homomorphims $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ such that*

1. *$h \circ f = k \circ g$*
2. *(α) for all $b \in B$ and $a \in A$, if $h(b) = k \circ g(a)$ then there exists some $a' \in A$ such that $b = f(a')$ and $g(a) = g(a')$,*

   for all $c \in C$ and $a \in A$, if $k(c) = h \circ f(a)$ then there exists some $a' \in A$ such that $c = g(a')$ and $f(a) = f(a')$.
3. **(super)** *for all $b \in B$ and $c \in C$ if $h(b) \leq k(c)$ ($k(c) \leq h(b)$) then there exists $a \in A$ for which both $h(b) \leq h \circ f(a)$ and $k \circ g(a) \leq k(c)$ ($k(c) \leq k \circ g(a)$ and $h \circ f(a) \leq h(b)$, respectively) hold.*

A variety $\mathcal{V}$ has the strong commutative homomorphisms diagrams with (α) (strongCHD(α)), if for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathcal{V}$ and for all homomorphisms $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$, there exist an algebra $\mathbf{D}$ in $\mathcal{V}$ and homomorphisms $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ such that

1. *$h \circ f = k \circ g$*
2. *(α) for all $b \in B$ and $a \in A$, if $h(b) = k \circ g(a)$ then there exists some $a' \in A$ such that $b = f(a')$ and $g(a) = g(a')$,*

   for all $c \in C$ and $a \in A$, if $k(c) = h \circ f(a)$ then there exists some $a' \in A$ such that $c = g(a')$ and $f(a) = f(a')$.
3. **(strong)** *$h(B) \cap k(C) = h \circ f(A)$.*

Similarly, we define the super commutaive homomorphisms diagrams with (β) or (γ) (shortly, superCHD(β) or superCHD(γ)) and strong commutative homomorphisms diagrams with (β) or (γ) (shortly, strongCHD(β) or strongCHD(γ)), respectively.

We summarize the relationships between the APs, the CHD(α)s, the CHD(β)s and the CHD(γ)s in Figure 9.

$$\begin{array}{ccccccc}
\text{superCHD}(\alpha) & \Rightarrow & \text{superCHD}(\beta) & \Rightarrow & \text{superCHD}(\gamma) & \Rightarrow & \text{superAP} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{strongCHD}(\alpha) & \Rightarrow & \text{strongCHD}(\beta) & \Rightarrow & \text{strongCHD}(\gamma) & \Rightarrow & \text{strongAP} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{CHD}(\alpha) & \Rightarrow & \text{CHD}(\beta) & \Rightarrow & \text{CHD}(\gamma) & \Rightarrow & \text{AP}
\end{array}$$

Figure 9: Relationships between the APs, the CHD(α)s, the CHD(β)s and the CHD(γ)s

5.2 Strong deductive and strong Craig interpolation proprety

First, we give an algebraic characterization of the SDIP.

THEOREM 30 *For each substructural logic* $\mathbf{L}$, $\mathbf{L}$ *has the SDIP iff* $\mathsf{V}(\mathbf{L})$ *has the CHD(γ).*

Proof. The proof proceeds similarly to the proof of Theorem 16. Suppose first that $\mathbf{L}$ has the SDIP, and let $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$ be homomorphisms for $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathsf{V}(\mathbf{L})$. Define $\mathbf{D} = \mathbf{Fm}(Y \cup Z)/\equiv$, and mappings $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ in the same way. Then, we can show that both h and k are well-defined homomorphisms and $h \circ f = k \circ g$ holds (see the proof of Lemma 29). To show that the condition (γ) holds, suppose that $1_{\mathbf{D}} \le h(b)$ for $b \in B$. Then, there exists some formula $\varphi \in \mathbf{Fm}(Y)$ such that $b = \eta_{\mathbf{B}}(\varphi)$. By the definitions of h and $\equiv$, $1_{\mathbf{D}} \le h(b)$ implies $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} \varphi$. Thus, by the SDIP, there exists some formula δ with $\mathrm{var}(\delta) \subseteq \mathrm{var}(\Gamma_{\mathbf{B}} \cup \{\varphi\}) \cap \mathrm{var}(\Gamma_{\mathbf{C}}) \subseteq Y \cap Z = X$ such that $\delta, \Gamma_{\mathbf{B}} \vdash_{\mathbf{L}} \varphi$ and $\Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} \delta$ hold. Let $a = \eta_{\mathbf{A}}(\delta) \in A$. Then, $\Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} \delta$ implies $1_{\mathbf{C}} \le \eta_{\mathbf{C}}(\delta) = g(\eta_{\mathbf{A}}(\delta)) = g(a)$. We show that $\delta, \Gamma_{\mathbf{B}} \vdash_{\mathbf{L}} \varphi$ implies $b \in \mathrm{Fg}_{\mathbf{B}}(f(a))$. Construct the filter $\mathrm{Fg}_{\mathbf{B}}(f(a))$ of $\mathbf{B}$ generated by $f(a)$, and determine the quotient algebra $\mathbf{B}/\mathrm{Fg}_{\mathbf{B}}(f(a))$ of $\mathbf{B}$. Since $\eta_{\mathbf{B}}(\delta) = f(\eta_{\mathbf{A}}(\delta)) = f(a) \in \mathrm{Fg}_{\mathbf{B}}(f(a))$, we have $1_{\mathbf{B}/\mathrm{Fg}_{\mathbf{B}}(f(a))} \le \eta_{\mathbf{B}}(\delta)/\mathrm{Fg}_{\mathbf{B}}(f(a))$. Define a valuation u on $\mathbf{B}/\mathrm{Fg}_{\mathbf{B}}(f(a))$ for formulas in $\mathbf{Fm}(Y)$ by $u(\alpha) = \eta_{\mathbf{B}}(\alpha)/\mathrm{Fg}_{\mathbf{B}}(f(a))$. Then, $\delta, \Gamma_{\mathbf{B}} \vdash_{\mathbf{L}} \varphi$ implies $1_{\mathbf{B}/\mathrm{Fg}_{\mathbf{B}}(f(a))} \le u(\varphi) = \eta_{\mathbf{B}}(\varphi)/\mathrm{Fg}_{\mathbf{B}}(f(a))$, and hence $b = \eta_{\mathbf{B}}(\varphi) \in \mathrm{Fg}_{\mathbf{B}}(f(a))$. Similarly, if $1_{\mathbf{D}} \le k(c)$ holds for $c \in C$ then there exists some $a' \in A$ such that $c \in \mathrm{Fg}_{\mathbf{C}}(g(a'))$ and $1_{\mathbf{B}} \le f(a')$ hold. Thus, $\mathsf{V}(\mathbf{L})$ has the CHD(γ).

Conversely, suppose that $\mathsf{V}(\mathbf{L})$ has the CHD(γ). Moreover, we assume that $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$ holds for a set of formulas $\Gamma \cup \Sigma \cup \{\psi\}$, and let $Y = \mathrm{var}(\Gamma)$, $Z = \mathrm{var}(\Sigma \cup \{\psi\})$ and $X = Y \cap Z$. In the same way as the proof of Theorem 16, we introduce quotient algebras $\mathbf{Fm}(Y)/\Gamma$ and $\mathbf{Fm}(Z)/\Sigma$. But, differently to the proof of Theorem 16, we define a set of fromulas Δ by

$$\Delta = \{\alpha \in \mathbf{Fm}(X) : \vdash_{\mathbf{L}} \alpha\}.$$

Then, a binary relation $\equiv_\Delta$ on $\mathbf{Fm}(X)$ determined by Δ is defined by

$$\alpha \equiv_\Delta \beta \text{ iff } \Delta \vdash_{\mathbf{L}} (\alpha\backslash\beta) \wedge (\beta\backslash\alpha) \text{ iff } \vdash_{\mathbf{L}} (\alpha\backslash\beta) \wedge (\beta\backslash\alpha),$$

which is a congruence relation. We denote the quotient algebra of $\mathbf{Fm}(X)$ determined by this $\equiv_\Delta$ as $\mathbf{Fm}(X)/\Delta$. Again, we define mappings $f : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Y)/\Gamma$ and $g : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Z)/\Sigma$ in the same way as before by

$$f(\alpha/\equiv_\Delta) = \alpha/\equiv_\Gamma \text{ and } g(\alpha/\equiv_\Delta) = \alpha/\equiv_\Sigma .$$

It is easy to see that both f and g are well-defined homomorphisms. By the CHD(γ), there exist an algebra $\mathbf{D}$ in $\mathsf{V}(\mathbf{L})$ and homomorphisms $h : \mathbf{Fm}(Y)/\Gamma \to \mathbf{D}$ and $k : \mathbf{Fm}(Z)/\Sigma \to \mathbf{D}$ such that $h \circ f = k \circ g$ and the condition (γ) hold. In the same way as the proof of Theorem 16, we construct a valuation w over $\mathbf{D}$ for formulas in $\mathbf{Fm}(Y \cup Z)$. By our assumption $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$, we have $\mathbf{D}, w \models \psi$, namely $1_{\mathbf{D}} \le k(\psi/\equiv_\Sigma)$ holds. Since $\psi/\equiv_\Sigma \in \mathbf{Fm}(Z)/\Sigma$, by the condition ($\gamma$), there exists some $\delta/\equiv_\Delta \in \mathbf{Fm}(X)/\Delta$ such that $\psi/\equiv_\Sigma \in \mathrm{Fg}_{\mathbf{Fm}(Z)/\Sigma}(g(\delta/\equiv_\Delta)) = \mathrm{Fg}_{\mathbf{Fm}(Z)/\Sigma}(\delta/\equiv_\Sigma)$ and $1_{\mathbf{Fm}(Y)/\Gamma} \le f(\delta/\equiv_\Delta) = (\delta/\equiv_\Gamma)$. Clearly, the latter implies $\Gamma \vdash_{\mathbf{L}} \delta$. We show that the former implies $\delta, \Sigma \vdash_{\mathbf{L}} \psi$. By $\psi/\equiv_\Sigma \in \mathrm{Fg}_{\mathbf{Fm}(Z)/\Sigma}(\delta/\equiv_\Sigma)$, there exist some $n \in \omega$ and iterated conjugates γ_i with $1 \le i \le n$ on $\mathbf{Fm}(Z)$ such that $1_{\mathbf{Fm}(Z)/\equiv_\Sigma} \le ((\Pi_{i=1}^n \gamma_i(\delta)\backslash\psi)/\equiv_\Sigma)$. Thus, we have $\Sigma \vdash_{\mathbf{L}} \Pi_{i=1}^n \gamma_i(\delta)\backslash\psi$, namely, $\delta, \Sigma \vdash_{\mathbf{L}} \psi$ holds. Therefore, the SDIP holds on $\mathbf{L}$. □

In [36], Wroński showed that the TI is equivalent to the eqIP with respect to equational theories. As shown in Proposition 7, the eqIP is equivalent to the SDIP. Thus, we have the following.

COROLLARY 31 *For any variety $\mathcal{V}$, $\mathcal{V}$ has the CHD(γ) if and only if it has the TI.*

Before showing an algebraic characterization of the SCIP, we give a logical property corresponding to the CHD(β).

DEFINITION 17 *A substructural logic $\mathbf{L}$ has the implicational strong deductive interpolation property (ISDIP), if for any set of formulas $\Gamma \cup \Sigma \cup \{\psi\}$, if $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$ then there exists a formula δ such that*

1. *$\Gamma \vdash_{\mathbf{L}} \delta$ and $\Sigma \vdash_{\mathbf{L}} \delta\backslash\psi$, and*
2. *$var(\delta) \subseteq var(\Gamma) \cap var(\Sigma \cup \{\psi\})$.*

Clearly, the ISDIP implies the SDIP since $\Sigma \vdash_{\mathbf{L}} \delta\backslash\psi$ implies $\delta, \Sigma \vdash_{\mathbf{L}} \psi$. On the other hand, it is easily seen that the SCIP implies the ISDIP. Then, we have the following.

THEOREM 32 *For each substructural logic $\mathbf{L}$, $\mathbf{L}$ has the ISDIP iff $\mathsf{V}(\mathbf{L})$ has the CHD(β).*

Proof. As in Theorem 30, the proof proceeds similarly to the proof of Theorem 16. Suppose first that $\mathbf{L}$ has the ISDIP, and let $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$ be homomorphisms for $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathsf{V}(\mathbf{L})$. Define $\mathbf{D} = \mathbf{Fm}(Y \cup Z)/\equiv$, and mappings $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ in the same way as before. Then, they satisfy the conditions of the CHD. To show that the condition (β) holds, suppose that $1_{\mathbf{D}} \le h(b)$ for $b \in B$. Then, there exists some formula $\varphi \in \mathbf{Fm}(Y)$ such that $b = \eta_{\mathbf{B}}(\varphi)$. By the definitions of h and $\equiv$, $1_{\mathbf{D}} \le h(b)$ implies $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} \varphi$. Thus, by the ISDIP, there exists some formula δ with $\mathrm{var}(\delta) \subseteq \mathrm{var}(\Gamma_{\mathbf{B}} \cup \{\varphi\}) \cap \mathrm{var}(\Gamma_{\mathbf{C}}) \subseteq X$ such that $\Gamma_{\mathbf{B}} \vdash_{\mathbf{L}} \delta\backslash\varphi$ and $\Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} \delta$ hold. Let $a = \eta_{\mathbf{A}}(\delta) \in A$. Then, we have that $f(a) = f(\eta_{\mathbf{A}}(\delta)) = \eta_{\mathbf{B}}(\delta) \le \eta_{\mathbf{B}}(\varphi) = b$ and $1_{\mathbf{C}} \le \eta_{\mathbf{C}}(\delta) = g(\eta_{\mathbf{A}}(\delta)) = g(a)$. Similarly, if $1_{\mathbf{D}} \le k(c)$ holds for $c \in C$ then there exists some $a' \in A$ such that $g(a') \le c$ and $1_{\mathbf{B}} \le f(a')$ hold. Thus, $\mathsf{V}(\mathbf{L})$ has the CHD(β).

Conversely, suppose that $\mathsf{V}(\mathbf{L})$ has the CHD(β). Moreover, we assume that $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$ holds for a set of formulas $\Gamma \cup \Sigma \cup \{\psi\}$, and let $Y = \mathrm{var}(\Gamma)$, $Z = \mathrm{var}(\Sigma \cup \{\psi\})$ and $X = Y \cap Z$. In the same way as the proof of Theorem 30, we introduce quotient algebras $\mathbf{Fm}(X)/\Delta$, $\mathbf{Fm}(Y)/\Gamma$ and $\mathbf{Fm}(Z)/\Sigma$, and homomorphisms $f : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Y)/\Gamma$ and $g : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Z)/\Sigma$. By the CHD($\beta$), there exist an algebra $\mathbf{D}$ in $\mathsf{V}(\mathbf{L})$ and homomorphisms $h : \mathbf{Fm}(Y)/\Gamma \to \mathbf{D}$ and $k : \mathbf{Fm}(Z)/\Sigma \to \mathbf{D}$ such that $h \circ f = k \circ g$ and the condition (β) hold. Again, we construct a valuation w over $\mathbf{D}$ for formulas in $\mathbf{Fm}(Y \cup Z)$ in the same way as the proof of Theorem 16. By our assumption $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$, we have $\mathbf{D}, w \models \psi$, namely $1_{\mathbf{D}} \le k(\psi/\equiv_\Sigma)$ holds. Since $\psi/\equiv_\Sigma \in \mathbf{Fm}(Z)/\Sigma$, by the condition ($\beta$), there exists some $\delta/\equiv_\Delta \in \mathbf{Fm}(X)/\Delta$ such that $(\delta/\equiv_\Sigma) = g(\delta/\equiv_\Delta) \le (\psi/\equiv_\Sigma)$ and $1_{\mathbf{Fm}(Y)/\Gamma} \le f(\delta/\equiv_\Delta) = (\delta/\equiv_\Gamma)$. Thus, both $\Gamma \vdash_{\mathbf{L}} \delta$ and $\Sigma \vdash_{\mathbf{L}} \delta\backslash\psi$ hold. Therefore, $\mathbf{L}$ has the ISDIP. □

Next, we give an algebraic property which is equivalent to the superCHD(β).

LEMMA 33 *For any variety $\mathcal{V}$, the following are equivalent.*

(1) $\mathcal{V}$ has the superCHD(β).

(2) $\mathcal{V}$ has the CHD, and also it satisfies the following:

(*) for all $b \in B$ and $c \in C$, if $h(b) \leq k(c)$ ($k(c) \leq h(b)$) then there exists some $a \in A$ such that $b \leq f(a)$ and $g(a) \leq c$ ($c \leq g(a)$ and $f(a) \leq b$, respectively).

Proof. Suppose that a variety $\mathcal{V}$ has the superCHD(β). Clearly, it has the CHD. Assume that $h(b) \leq k(c)$ holds for $b \in B$ and $c \in C$. Then, by the third condition of the superCHD(β), there exists some $a_1 \in A$ such that both $h(b) \leq h \circ f(a_1)$ and $k \circ g(a_1) \leq k(c)$ hold, and hence $1_{\mathbf{D}} \leq h(b \backslash f(a_1))$ and $1_{\mathbf{D}} \leq k(g(a_1) \backslash c)$ hold. Since $b \backslash f(a_1) \in B$ and $g(a_1) \backslash c \in C$, by the condition ($\beta$), there exist some $a_2, a_3 \in A$ such that

$$\begin{cases} f(a_2) \leq b \backslash f(a_1) \\ \text{and} \\ 1_{\mathbf{C}} \leq g(a_2) \end{cases} \quad \text{and} \quad \begin{cases} g(a_3) \leq g(a_1) \backslash c \\ \text{and} \\ 1_{\mathbf{B}} \leq f(a_3), \end{cases}$$

and hence

$$\begin{cases} b \cdot f(a_2) \leq f(a_1) \\ \text{and} \\ 1_{\mathbf{C}} \leq g(a_2) \end{cases} \quad \text{and} \quad \begin{cases} g(a_1) \cdot g(a_3) \leq c \\ \text{and} \\ 1_{\mathbf{B}} \leq f(a_3) \end{cases}$$

hold. Let $a = a_2 \backslash (a_1 \cdot a_3) \in A$. Then, by $b \cdot f(a_2) \leq f(a_1)$ and $1_{\mathbf{B}} \leq f(a_3)$, we have $b \cdot f(a_2) \leq f(a_1) \cdot f(a_3)$, and hence $b \leq f(a_2) \backslash (f(a_1) \cdot f(a_3)) = f(a)$. Also, by $1_{\mathbf{C}} \leq g(a_2)$ and $g(a_1) \cdot g(a_3) \leq c$, we have $g(a) = g(a_2) \backslash (g(a_1) \cdot g(a_3)) \leq 1_{\mathbf{C}} \backslash (g(a_1) \cdot g(a_3)) = g(a_1) \cdot g(a_3) \leq c$. Similarly, we can show the converse inequality. Therefore, the condition (2) holds.

Conversely, we assume that the condition (2) holds. Since $b \leq f(a)$ and $g(a) \leq c$ imply $h(b) \leq h \circ f(a)$ and $k \circ g(a) \leq k(c)$, the third condition of the superCHD(β) hold. Thus, it is enough to show that the condition (β) holds. Suppose that $1_{\mathbf{D}} \leq h(b)$ holds for some $b \in B$. Then, $k(1_{\mathbf{C}}) = 1_{\mathbf{D}} \leq h(b)$ also holds. By our assumption (2), there exists some $a \in A$ such that $1_{\mathbf{C}} \leq g(a)$ and $f(a) \leq b$ hold. Hence, $\mathcal{V}$ has the superCHD(β). □

Note that in the case of the AP or where both h and k are injective, the condition (*) in the above Lemma 33 is equivalent to the condition **(super)** of the superAP. But, to show the same equivalence in the case of CHD, we need also the condition (β).

DEFINITION 18 *A substructural logic $\mathbf{L}$ has the nonseparable strong Craig interpolation property (nonsepSCIP), if for any set of formulas $\Gamma \cup \Sigma \cup \{\varphi, \psi\}$, if $\Gamma, \Sigma \vdash_{\mathbf{L}} \varphi \backslash \psi$ then there exists a formula δ such that*

1. *$\Gamma, \Sigma \vdash_{\mathbf{L}} \varphi \backslash \delta$ and $\Gamma, \Sigma \vdash_{\mathbf{L}} \delta \backslash \psi$,*
2. *$var(\delta) \subseteq var(\Gamma \cup \{\varphi\}) \cap var(\Sigma \cup \{\psi\})$.*

It is easily seen that the SCIP implies the nonsepSCIP since $\Gamma \vdash_{\mathbf{L}} \varphi \backslash \delta$ and $\Sigma \vdash_{\mathbf{L}} \delta \backslash \psi$ imply $\Gamma, \Sigma \vdash_{\mathbf{L}} \varphi \backslash \delta$ and $\Gamma, \Sigma \vdash_{\mathbf{L}} \delta \backslash \psi$, respectively.

Now, we give an algebraic characterization of the SCIP.

THEOREM 34 *For each substructural logic $\mathbf{L}$, the following are equivalent.*

(1) $\mathbf{L}$ has the SCIP.

(2) $\mathbf{L}$ has both the ISDIP and the nonsepSCIP.

(3) $\mathsf{V}(\mathbf{L})$ has the superCHD(β).

Proof. (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) Suppose that $\mathbf{L}$ satisfies the condition (2), and let $f : \mathbf{A} \to \mathbf{B}$ and $\mathbf{A} \to \mathbf{C}$ be homomorphsims for $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathsf{V}(\mathbf{L})$. Since $\mathbf{L}$ has the ISDIP, we can construct an algebra $\mathbf{D}$ and homomorphisms $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ satisfying the CHD(β) in the same way as the proof of Theorem 32. Thus, it is sufficient to show that they satisfy also the third condition of the superCHD(β), namely,

> for all $b \in B$ and $c \in C$, if $h(b) \leq k(c)$ ($k(c) \leq h(b)$) then there exists $a \in A$ such that $h(b) \leq h \circ f(a)$ and $k \circ g(a) \leq k(c)$ ($k(c) \leq k \circ g(a)$ and $h \circ f(a) \leq h(b)$, respectively).

Suppose that $h(b) \leq k(c)$ holds for $b \in B$ and $c \in C$. Then, there exist some formulas $\varphi \in \mathbf{Fm}(Y)$ and $\psi \in \mathbf{Fm}(Z)$ such that $b = \eta_{\mathbf{B}}(\varphi)$ and $c = \eta_{\mathbf{C}}(\psi)$. By the definitions of h, k and $\equiv$, we can see that $h(b) \leq k(c)$ is equivalent to $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} \varphi\backslash\psi$. Then, by the nonsepSCIP, there exists some formula δ with $\mathrm{var}(\delta) \subseteq \mathrm{var}(\Gamma_{\mathbf{B}} \cup \{\varphi\}) \cap \mathrm{var}(\Gamma_{\mathbf{C}} \cup \{\psi\}) \subseteq Y \cap Z = X$ such that both $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} \varphi\backslash\delta$ and $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} \delta\backslash\psi$ hold. Let $a = \eta_{\mathbf{A}}(\delta) \in A$. Then, $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} \varphi\backslash\delta$ implies $h(b) = h(\eta_{\mathbf{B}}(\varphi)) = (\varphi/\equiv) \leq (\delta/\equiv) = h(\eta_{\mathbf{B}}(\delta)) = h \circ f(\eta_{\mathbf{A}}(\delta)) = h \circ f(a)$. In the same way, $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} \delta\backslash\psi$ implies $k \circ g(a) \leq k(c)$. Similarly, if $k(c') \leq h(b')$ holds for $b' \in B$ and $c' \in C$ then there exists some $a' \in A$ such that $k(c') \leq k \circ g(a')$ and $h \circ f(a') \leq h(b')$ hold. Therefore, $\mathsf{V}(\mathbf{L})$ has the superCHD(β).

(3) $\Rightarrow$ (1) Suppose that $\mathsf{V}(\mathbf{L})$ has the superCHD(β). Moreover, we assume that $\Gamma, \Sigma \vdash_{\mathbf{L}} \varphi\backslash\psi$ holds for a set of formulas $\Gamma\cup\Sigma\cup\{\varphi, \psi\}$, and let $Y = \mathrm{var}(\Gamma\cup\{\varphi\})$, $Z = \mathrm{var}(\Sigma\cup\{\psi\})$ and $X = Y\cap Z$. In the same way as the proof of Theorem 30, we introduce quotient algebras $\mathbf{Fm}(X)/\Delta, \mathbf{Fm}(Y)/\Gamma$ and $\mathbf{Fm}(Z)/\Sigma$, and homomorphisms $f : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Y)/\Gamma$ and $g : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Z)/\Sigma$. By the superCHD($\beta$) and Lemma 33, there exist an algebra $\mathbf{D}$ in $\mathsf{V}(\mathbf{L})$ and homomorphisms $h : \mathbf{Fm}(Y)/\Gamma \to \mathbf{D}$ and $k : \mathbf{Fm}(Z)/\Sigma \to \mathbf{D}$ such that $h \circ f = k \circ g$ and the following hold;

(*) for all $b \in \mathbf{Fm}(Y)/\Gamma$ and $c \in \mathbf{Fm}(Z)/\Sigma$, if $h(b) \leq k(c)$ ($k(c) \leq h(b)$) then there exists $a \in \mathbf{Fm}(X)/\Delta$ such that $b \leq f(a)$ and $g(a) \leq c$ ($c \leq g(a)$ and $f(a) \leq b$, respectively).

We construct a valuation w over $\mathbf{D}$ for formulas from $\mathbf{Fm}(Y \cup Z)$ in the same way as the proof of Theorem 16. Then, our assumption $\Gamma, \Sigma \vdash_{\mathbf{L}} \varphi\backslash\psi$ implies $\mathbf{D}, w \models \varphi\backslash\psi$, and hence $h(\varphi/\equiv_{\Gamma}) \leq k(\psi/\equiv_{\Sigma})$ holds. Since $\varphi/\equiv_{\Gamma}\in \mathbf{Fm}(Y)/\Gamma$ and $\psi/\equiv_{\Sigma}\in \mathbf{Fm}(Z)/\Sigma$, by the above condition (*), there exists some $\delta/\equiv_{\Delta}\in \mathbf{Fm}(X)/\Delta$ such that both $(\varphi/\equiv_{\Gamma}) \leq f(\delta/\equiv_{\Delta}) = (\delta/\equiv_{\Gamma})$ and $(\delta/\equiv_{\Sigma}) = g(\delta/\equiv_{\Delta}) \leq (\psi/\equiv_{\Sigma})$ hold. Thus, both $\Gamma \vdash_{\mathbf{L}} \varphi\backslash\delta$ and $\Sigma \vdash_{\mathbf{L}} \delta\backslash\psi$ hold. Therefore, $\mathbf{L}$ has the SCIP. □

5.3 Syntactic characterization of other CHDs

In the rest of this section, we give logical properties which correspond to other CHDs.

DEFINITION 19 *A substructural logic* $\mathbf{L}$ *has the equivalential strong deductive interpolation property (EqSDIP), if for any set of formulas* $\Gamma \cup \Sigma \cup \{\psi\}$ *and a formula* σ *with* $var(\sigma) \subseteq var(\Gamma) \cap var(\Sigma \cup \{\psi\})$, *if* $\Gamma, \Sigma \vdash_{\mathbf{L}} (\sigma\backslash\psi) \wedge (\psi\backslash\sigma)$ *then there exists some formula* δ *such that*

1. $\Gamma \vdash_{\mathbf{L}} (\sigma\backslash\delta) \wedge (\delta\backslash\sigma)$ *and* $\Sigma \vdash_{\mathbf{L}} (\psi\backslash\delta) \wedge (\delta\backslash\psi)$, *and*
2. $var(\delta) \subseteq var(\Gamma) \cap var(\Sigma \cup \{\psi\})$.

A substructural logic $\mathbf{L}$ *has the equivalential strong Craig interpolation property (EqSCIP), if for any set of formulas* $\Gamma \cup \Sigma \cup \{\varphi, \psi\}$, *if* $\Gamma, \Sigma \vdash_{\mathbf{L}} (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$ *then there exists some formula* δ *such that*

1. $\Gamma \vdash_{\mathbf{L}} (\varphi\backslash\delta) \wedge (\delta\backslash\varphi)$ *and* $\Sigma \vdash_{\mathbf{L}} (\psi\backslash\delta) \wedge (\delta\backslash\psi)$, *and*
2. $var(\delta) \subseteq var(\Gamma \cup \{\varphi\}) \cap var(\Sigma \cup \{\psi\})$.

A substructural logic $\mathbf{L}$ *has the nonseparable equivalential strong Craig interpolation property (nonsepEqSCIP), if for any set of formulas* $\Gamma \cup \Sigma \cup \{\varphi, \psi\}$*, if* $\Gamma, \Sigma \vdash_{\mathbf{L}} (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$ *then there exists some formula* δ *such that*

1. $\Gamma, \Sigma \vdash_{\mathbf{L}} (\varphi\backslash\delta) \wedge (\delta\backslash\varphi)$ *and* $\Gamma, \Sigma \vdash_{\mathbf{L}} (\psi\backslash\delta) \wedge (\delta\backslash\psi)$*, and*

2. $var(\delta) \subseteq var(\Gamma \cup \{\varphi\}) \cap var(\Sigma \cup \{\psi\})$.

Clearly, the EqSCIP implies both the EqSDIP and the nonsepEqSCIP. Also, the EqSDIP implies the ISDIP. For, if $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$ holds for a set of formulas $\Gamma \cup \Sigma \cup \{\psi\}$ then $\Gamma, \Sigma \vdash_{\mathbf{L}} (1\backslash(\psi \wedge 1)) \wedge ((\psi \wedge 1)\backslash 1)$ also holds. Since $\mathrm{var}(1) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\Sigma \cup \{\psi\})$, by the EqSDIP, there exists some δ with $\mathrm{var}(\delta) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\Sigma \cup \{\psi\})$ such that $\Gamma \vdash_{\mathbf{L}} (1\backslash\delta) \wedge (\delta\backslash 1)$ and $\Sigma \vdash_{\mathbf{L}} ((\psi \wedge 1)\backslash\delta) \wedge (\delta\backslash(\psi \wedge 1))$, which imply $\Gamma \vdash_{\mathbf{L}} \delta$ and $\Sigma \vdash_{\mathbf{L}} \delta\backslash\psi$.

THEOREM 35 *For each substructural logic* $\mathbf{L}$*,* $\mathbf{L}$ *has the EqSDIP iff* $\mathsf{V}(\mathbf{L})$ *has the CHD(*α*).*

Proof. As in Theorem 30, the proof proceeds similarly to the proof of Theorem 16. Suppose first that $\mathbf{L}$ has the EqSDIP, and let $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$ be homomorphisms for $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathsf{V}(\mathbf{L})$. Define $\mathbf{D} = \mathbf{Fm}(Y \cup Z)/\equiv$, and mappings $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ in the same way as before. Then, they satisfy the conditions of the CHD. To show that the condition (α) holds, suppose that $h(b) = k \circ g(a)$ for $b \in B$ and $a \in A$. Then, there exist some formulas $\varphi \in \mathbf{Fm}(Y)$ and $\sigma \in \mathbf{Fm}(X)$ such that $b = \eta_{\mathbf{B}}(\varphi)$ and $a = \eta_{\mathbf{A}}(\sigma)$. By the definitions of h, k, g and $\equiv$, $h(b) = k \circ g(a)$ implies $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} (\sigma\backslash\varphi) \wedge (\varphi\backslash\sigma)$. Thus, by the EqSDIP, there exists some formula δ with $\mathrm{var}(\delta) \subseteq \mathrm{var}(\Gamma_{\mathbf{B}} \cup \{\varphi\}) \cap \mathrm{var}(\Gamma_{\mathbf{C}}) \subseteq X$ such that $\Gamma_{\mathbf{B}} \vdash_{\mathbf{L}} (\varphi\backslash\delta) \wedge (\delta\backslash\varphi)$ and $\Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} (\sigma\backslash\delta) \wedge (\delta\backslash\sigma)$ hold. Let $a' = \eta_{\mathbf{A}}(\delta) \in A$. Then, we have that $b = \eta_{\mathbf{B}}(\varphi) = \eta_{\mathbf{B}}(\delta) = f(\eta_{\mathbf{A}}(\delta)) = f(a')$ and $g(a) = g(\eta_{\mathbf{A}}(\sigma)) = \eta_{\mathbf{C}}(\sigma) = \eta_{\mathbf{C}}(\delta) = g(\eta_{\mathbf{A}}(\delta)) = g(a')$. Similarly, if $k(c) = h \circ f(a)$ holds for $c \in C$ and $a \in A$ then there exists some $a' \in A$ such that $c = g(a')$ and $f(a) = f(a')$ hold. Thus, $\mathsf{V}(\mathbf{L})$ has the CHD(α).

Conversely, suppose that $\mathsf{V}(\mathbf{L})$ has the CHD(α). Moreover, we assume that $\Gamma, \Sigma \vdash_{\mathbf{L}} (\sigma\backslash\psi) \wedge (\psi\backslash\sigma)$ holds for a set of formulas $\Gamma \cup \Sigma \cup \{\psi\}$ and a formula σ with $\mathrm{var}(\sigma) \subseteq \mathrm{var}(\Gamma) \cap \mathrm{var}(\Sigma \cup \{\psi\})$. Let $Y = \mathrm{var}(\Gamma), Z = \mathrm{var}(\Sigma \cup \{\psi\})$ and $X = Y \cap Z$. In the same way as the proof of Theorem 30, we introduce quotient algebras $\mathbf{Fm}(X)/\Delta, \mathbf{Fm}(Y)/\Gamma$ and $\mathbf{Fm}(Z)/\Sigma$, and homomorphisms $f : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Y)/\Gamma$ and $g : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Z)/\Sigma$. By the CHD($\alpha$), there exist an algebra $\mathbf{D}$ in $\mathsf{V}(\mathbf{L})$ and homomorphisms $h : \mathbf{Fm}(Y)/\Gamma \to \mathbf{D}$ and $k : \mathbf{Fm}(Z)/\Sigma \to \mathbf{D}$ such that $h \circ f = k \circ g$ and the condition (α) hold. Again, we construct a valuation w over $\mathbf{D}$ for formulas in $\mathbf{Fm}(Y \cup Z)$ in the same way as the proof of Theorem 16. By our assumption $\Gamma, \Sigma \vdash_{\mathbf{L}} (\sigma\backslash\psi) \wedge (\psi\backslash\sigma)$, we have $\mathbf{D}, w \models (\sigma\backslash\psi) \wedge (\psi\backslash\sigma)$, namely $k(\psi/\equiv_{\Sigma}) = h \circ f(\sigma/\equiv_{\Delta})$ holds. Since $\psi/\equiv_{\Sigma} \in \mathbf{Fm}(Z)/\Sigma$ and $\sigma/\equiv_{\Delta} \in \mathbf{Fm}(X)/\Delta$, by the condition (α), there exists some $\delta/\equiv_{\Delta} \in \mathbf{Fm}(X)/\Delta$ such that $(\psi/\equiv_{\Sigma}) = g(\delta/\equiv_{\Delta}) = (\delta/\equiv_{\Sigma})$ and $(\sigma/\equiv_{\Gamma}) = f(\sigma/\equiv_{\Delta}) = f(\delta/\equiv_{\Delta}) = (\delta/\equiv_{\Gamma})$. Thus, both $\Gamma \vdash_{\mathbf{L}} (\sigma\backslash\delta) \wedge (\delta\backslash\sigma)$ and $\Sigma \vdash_{\mathbf{L}} (\psi\backslash\delta) \wedge (\delta\backslash\psi)$ hold. Therefore, $\mathbf{L}$ has the EqSDIP. □

Before giving an algebraic characterization of the EqSCIP, we give an algebraic property which is equivalent to the strongCHD(α).

LEMMA 36 *For any variety* $\mathcal{V}$*, the following are equivalent.*

(1) $\mathcal{V}$ *has the strongCHD(*α*).*

(2) $\mathcal{V}$ *has the CHD, and also it satisfies the following:*

(**) for all $b \in B$ *and* $c \in C$*, if* $h(b) = k(c)$ *then there exists some* $a \in A$ *such that* $b = f(a)$ *and* $c = g(a)$*.*

Proof. Suppose that a variety $\mathcal{V}$ has the strongCHD(α). Clearly, it has the CHD. Assume that $h(b) = k(c)$ holds for $b \in B$ and $c \in C$. Then, by the first and third conditions of the strongCHD(α), there exists some $a_1 \in A$ such that both $h(b) = h \circ f(a_1) = k \circ g(a_1)$ and $k(c) = k \circ g(a_1)$ hold. Now, by applying the condition (α) to $h(b) = k \circ g(a_1)$, there exists some $a_2 \in A$ such that $b = f(a_2)$ and $g(a_1) = g(a_2)$ hold. Then, we have $k(c) = k \circ g(a_1) = k \circ g(a_2) = h \circ f(a_2)$. Again, by applying ($\alpha$) to $k(c) = h \circ f(a_2)$, there exists some $a' \in A$ such that $f(a_2) = f(a')$ and $c = g(a')$ hold. Thus, $b = f(a')$ and $c = g(a')$, namely, the condition (**) holds.

The converse direction is shown easily. □

Note that in the case of the AP or where both h and k are injective, the condition (**) in the above Lemma 36 is equivalent to the condition **(strong)** of the strongAP. But, as with the case of the condition **(super)** (see Lemma 33), we need also the condition (α) to show the same equivalence in the case of CHD.

THEOREM 37 *For each substructural logic* $\mathbf{L}$*, the following are equivalent.*

(1) $\mathbf{L}$ *has the EqSCIP.*

(2) $\mathbf{L}$ *has both the EqSDIP and the nonsepEqSCIP.*

(3) $\mathsf{V}(\mathbf{L})$ *has the strongCHD(α).*

Proof. (1) ⇒ (2) Obvious.

(2) ⇒ (3) Suppose first that $\mathbf{L}$ has the condition (2), and let $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$ be homomorphisms for $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathsf{V}(\mathbf{L})$. Since $\mathbf{L}$ has the EqSDIP, we can construct an algebra $\mathbf{D}$ in $\mathsf{V}(\mathbf{L})$ and homomorphisms $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ satisfying the CHD(α) in the same way as the proof of Theorem 35. Thus, it is sufficient to show that they satisfy also the third condition of the strongCHD(α), namely, $h(B) \cap k(C) = h \circ f(A)$. Clearly, $h \circ f(A) \subseteq h(B) \cap k(C)$ holds since $f(A) \subseteq B$, $g(A) \subseteq C$ and $h \circ f = k \circ g$. We show that $h(B) \cap k(C) \subseteq h \circ f(A)$. Let $d \in h(B) \cap k(C)$, namely for some $b \in B$ and $c \in C$, $d = h(b) = k(c)$. Then, there exist some formulas $\varphi \in \mathbf{Fm}(Y)$ and $\psi \in \mathbf{Fm}(Z)$ such that $b = \eta_{\mathbf{B}}(\varphi)$ and $c = \eta_{\mathbf{C}}(\psi)$. From the definitions of h, k and $\equiv$, we can see that the condition $h(b) = k(c)$ is equivalent to $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$. By the nonsepEqSCIP, there is a formula δ with $\mathrm{var}(\delta) \subseteq X$ such that $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} (\varphi\backslash\delta) \wedge (\delta\backslash\varphi)$ and $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} (\psi\backslash\delta) \wedge (\delta\backslash\psi)$. Let $a = \eta_{\mathbf{A}}(\delta) \in A$. Then, $d = h(b) = h(\eta_{\mathbf{B}}(\varphi)) = (\varphi/\equiv) = (\delta/\equiv) = h(\eta_{\mathbf{B}}(\delta)) = h \circ f(\eta_{\mathbf{A}}(\delta)) = h \circ f(a)$, which implies $d \in h \circ f(\mathbf{A})$. Thus, $\mathsf{V}(\mathbf{L})$ has the strongCHD(α).

(3) ⇒ (1) Suppose that $\mathsf{V}(\mathbf{L})$ has the strong*CHD. Moreover, we assume that $\Gamma, \Sigma \vdash_{\mathbf{L}} (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$ holds for a set of formulas $\Gamma \cup \Sigma \cup \{\varphi, \psi\}$, and let $Y = \mathrm{var}(\Gamma \cup \{\varphi\})$ $Z = \mathrm{var}(\Sigma \cup \{\psi\})$ and $X = Y \cap Z$. In the same way as the proof of Theorem 30, we introduce quotient algebras $\mathbf{Fm}(X)/\Delta, \mathbf{Fm}(Y)/\Gamma$ and $\mathbf{Fm}(Z)/\Sigma$, and homomorphisms $f : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Y)/\Gamma$ and $g : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Z)/\Sigma$. By the strongCHD($\alpha$) and Lemma 36 there exist an algebra $\mathbf{D}$ in $\mathsf{V}(\mathbf{L})$ and homomorphsims $h : \mathbf{Fm}(Y)/\Gamma \to \mathbf{D}$ and $k : \mathbf{Fm}(Z)/\Sigma \to \mathbf{D}$ such that $h \circ f = k \circ g$ and the following hold;

(**) for all $b \in \mathbf{Fm}(Y)/\Gamma$ and $c \in \mathbf{Fm}(Z)/\Sigma$, if $h(b) = k(c)$ then there exists some $a \in \mathbf{Fm}(X)/\Delta$ such that $b = f(a)$ and $c = g(a)$.

We construct a valuation w over $\mathbf{D}$ for formulas from $\mathbf{Fm}(Y \cup Z)$ in the same way as the proof of Theorem 16. Then, our assumption $\Gamma, \Sigma \vdash_{\mathbf{L}} (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$ implies $\mathbf{D}, w \models (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$, and hence $h(\varphi/\equiv_\Gamma) = k(\psi/\equiv_\Sigma)$ holds. Since $\varphi/\equiv_\Gamma \in \mathbf{Fm}(Y)/\Gamma$ and $\psi/\equiv_\Sigma \in \mathbf{Fm}(Z)/\Sigma$, by the above condition (**), there exists some $\delta/\equiv_\Delta \in \mathbf{Fm}(X)/\Delta$ such that $(\varphi/\equiv_\Gamma) = f(\delta/\equiv_\Delta) = (\delta/\equiv_\Gamma)$ and $(\psi/\equiv_\Sigma) = g(\delta/\equiv_\Delta) = (\delta/\equiv_\Sigma)$, which imply $\Gamma \vdash_{\mathbf{L}} (\varphi\backslash\delta) \wedge (\delta\backslash\varphi)$ and $\Sigma \vdash_{\mathbf{L}} (\psi\backslash\delta) \wedge (\delta\backslash\psi)$. Therefore, $\mathbf{L}$ satisfies the EqSCIP. □

THEOREM 38 *For each substructural logic* $\mathbf{L}$*, the following are equivalent.*

(1) $\mathbf{L}$ *has both the SCIP and the EqSCIP.*

(2) $\mathbf{L}$ *has both the EqSDIP and the nonsepSCIP.*

(3) $\mathsf{V}(\mathbf{L})$ *has the superCHD(α).*

Proof. (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) Suppose first that $\mathbf{L}$ has the condition (2), and let $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$ be homomorphisms for $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathsf{V}(\mathbf{L})$. Since $\mathbf{L}$ has the EqSDIP, we can construct an algebra $\mathbf{D}$ in $\mathsf{V}(\mathbf{L})$ and homomorphisms $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ satisfying the CHD(α) in the same way as the proof of Theorem 35. Thus, it is sufficient to show that they satisfy also the third condition of the superCHD(α), namely,

> for all $b \in B$ and $c \in C$, if $h(b) \leq k(c)$ ($k(c) \leq h(b)$) then there exists $a \in A$ such that $h(b) \leq h \circ f(a)$ and $k \circ g(a) \leq k(c)$ ($k(c) \leq k \circ g(a)$ and $h \circ f(a) \leq h(b)$, respectively).

Suppose that $h(b) \leq k(c)$ holds for $b \in B$ and $c \in C$. Then, there exist some formulas $\varphi \in \mathbf{Fm}(Y)$ and $\psi \in \mathbf{Fm}(Z)$ such that $b = \eta_{\mathbf{B}}(\varphi)$ and $c = \eta_{\mathbf{C}}(\psi)$. By the definitions of h, k and $\equiv$, we can see that $h(b) \leq k(c)$ is equivalent to $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} \varphi\backslash\psi$. Then, by the nonsepSCIP, there exists some formula δ with $\mathrm{var}(\delta) \subseteq \mathrm{var}(\Gamma_{\mathbf{B}} \cup \{\varphi\}) \cap \mathrm{var}(\Gamma_{\mathbf{C}} \cup \{\psi\}) \subseteq Y \cap Z = X$ such that both $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} \varphi\backslash\delta$ and $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} \delta\backslash\psi$ hold. Let $a = \eta_{\mathbf{A}}(\delta) \in A$. Then, $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} \varphi\backslash\delta$ implies $h(b) = h(\eta_{\mathbf{B}}(\varphi)) = (\varphi/\equiv) \leq (\delta/\equiv) = h(\eta_{\mathbf{B}}(\delta)) = h \circ f(\eta_{\mathbf{A}}(\delta)) = h \circ f(a)$. In the same way, $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} \delta\backslash\psi$ implies $k \circ g(a) \leq k(c)$. Similarly, if $k(c') \leq h(b')$ holds for $b' \in B$ and $c' \in C$ then there exists some $a' \in A$ such that $k(c') \leq k \circ g(a')$ and $h \circ f(a') \leq h(b')$ hold. Therefore, $\mathsf{V}(\mathbf{L})$ has the superCHD(α).

(3) $\Rightarrow$ (1) Suppose that $\mathsf{V}(\mathbf{L})$ has the superCHD(α). Then, it is easily shown that the superCHD(α) implies both the superCHD(β) and the strongCHD(α). Thus, by Theorem 34 and 37, $\mathbf{L}$ has both the SCIP and the EqSCIP. $\square$

THEOREM 39 *For each substructural logic* $\mathbf{L}$*,* $\mathbf{L}$ *has both the ISDIP and the nonsepEqSCIP iff* $\mathsf{V}(\mathbf{L})$ *has the strongCHD(β).*

Proof. Suppose that $\mathbf{L}$ has both the ISDIP and the nonsepEqSCIP, and let $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$ be homomorphisms for $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathsf{V}(\mathbf{L})$. Then, by the ISDIP, we can construct an algebra $\mathbf{D}$ and homomorphisms $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ satisfying the CHD(β) as we have done in the proof of Theorem 32. Thus, it is sufficent to show that they satisfy also the third condition of the strongCHD(β), namely, $h(B) \cap k(C) = h \circ f(A)$. Clearly, $h \circ f(A) \subseteq h(B) \cap k(C)$ holds since $f(A) \subseteq B$, $g(A) \subseteq C$ and $h \circ f = k \circ g$. We show that $h(B) \cap k(C) \subseteq h \circ f(A)$. Let $d \in h(B) \cap k(C)$, namely for some $b \in B$ and $c \in C$, $d = h(b) = k(c)$. Then, there exist some formulas $\varphi \in \mathbf{Fm}(Y)$ and $\psi \in \mathbf{Fm}(Z)$ such that $b = \eta_{\mathbf{B}}(\varphi)$ and $c = \eta_{\mathbf{C}}(\psi)$. From the definitions of h, k and $\equiv$, we can see that the condition $h(b) = k(c)$ is equivalent to $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$. By the nonsepEqSCIP, there is a formula δ with $\mathrm{var}(\delta) \subseteq X$ such that $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} (\varphi\backslash\delta) \wedge (\delta\backslash\varphi)$ and $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} (\psi\backslash\delta) \wedge (\delta\backslash\psi)$. Let $a = \eta_{\mathbf{A}}(\delta) \in A$. Then, $d = h(b) = h(\eta_{\mathbf{B}}(\varphi)) = (\varphi/\equiv) = (\delta/\equiv) = h(\eta_{\mathbf{B}}(\delta)) = h \circ f(\eta_{\mathbf{A}}(\delta)) = h \circ f(a)$, which implies $d \in h \circ f(\mathbf{A})$. Thus, $\mathsf{V}(\mathbf{L})$ has the strongCHD(β).

Conversely, suppose that $\mathsf{V}(\mathbf{L})$ has the strongCHD(β). Clearly, by Theorem 32, $\mathbf{L}$ has the ISDIP. We assume that $\Gamma, \Sigma \vdash_{\mathbf{L}} (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$ holds for a set of formulas $\Gamma \cup \Sigma \cup \{\varphi, \psi\}$ and let $Y = \mathrm{var}(\Gamma \cup \{\varphi\})$, $Z = \mathrm{var}(\Sigma \cup \{\psi\})$ and $X = Y \cap Z$. In the same way as the proof of Theorem 30, we introduce quotient algebras $\mathbf{Fm}(X)/\Delta, \mathbf{Fm}(Y)/\Gamma$ and $\mathbf{Fm}(Z)/\Sigma$, and homomorphisms $f : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Y)/\Gamma$ and $g : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Z)/\Sigma$. By the strongCHD($\beta$), there exist an algebra $\mathbf{D}$ in $\mathsf{V}(\mathbf{L})$ and homomorphisms $h : \mathbf{Fm}(Y)/\Gamma \to \mathbf{D}$ and $k : \mathbf{Fm}(Z)/\Sigma \to \mathbf{D}$ such that all of three conditions $h \circ f = k \circ g$, (β) and $h(\mathbf{Fm}(Y)/\Gamma) \cap k(\mathbf{Fm}(Z)/\Sigma) = h \circ f(\mathbf{Fm}(X)/\Delta)$

hold. We construct a valuation w over $\mathbf{D}$ for formulas from $\mathbf{Fm}(Y \cup Z)$ in the same way as the proof of Theorem 16. Then, our assumption $\Gamma, \Sigma \vdash_{\mathbf{L}} (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$ implies $\mathbf{D}, w \models (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$, and hence $h(\varphi/\equiv_\Gamma) = k(\psi/\equiv_\Sigma)$ holds. Since $\varphi/\equiv_\Gamma \in \mathbf{Fm}(Y)/\Gamma$ and $\psi/\equiv_\Sigma \in \mathbf{Fm}(Z)/\Sigma$, the conditions $h \circ f = k \circ g$ and $h(\mathbf{Fm}(Y)/\Gamma) \cap k(\mathbf{Fm}(Z)/\Sigma) = h \circ f(\mathbf{Fm}(X)/\Delta)$ imply that there exists some $\delta/\equiv_\Delta \in \mathbf{Fm}(X)/\Delta$ such that $h(\varphi/\equiv_\Gamma) = h \circ f(\delta/\equiv_\Delta) = h(\delta/\equiv_\Gamma)$ and $k(\psi/\equiv_\Sigma) = k \circ g(\delta/\equiv_\Delta) = k(\delta/\equiv_\Sigma)$. We show that $h(\varphi/\equiv_\Gamma) = h(\delta/\equiv_\Gamma)$ implies $\Gamma, \Sigma \vdash_{\mathbf{L}} (\varphi\backslash\delta) \wedge (\delta\backslash\varphi)$. So, suppose that $h(\varphi/\equiv_\Gamma) = h(\delta/\equiv_\Gamma)$. Then we have $1_{\mathbf{D}} \leq h(((\varphi\backslash\delta) \wedge (\delta\backslash\varphi))/\equiv_\Gamma)$. Since $((\varphi\backslash\delta) \wedge (\delta\backslash\varphi))/\equiv_\Gamma \in \mathbf{Fm}(Y)/\Gamma$, by the condition (β) there exists some $\delta'/\equiv_\Delta \in \mathbf{Fm}(X)/\Delta$ such that

$$\begin{cases} (\delta'/\equiv_\Gamma) = f(\delta'/\equiv_\Delta) \leq (((\varphi\backslash\delta) \wedge (\delta\backslash\varphi))/\equiv_\Gamma) \\ \text{and} \\ 1_{\mathbf{Fm}(Z)/\Sigma} \leq g(\delta'/\equiv_\Delta) = (\delta'/\equiv_\Sigma). \end{cases}$$

Hence $\Gamma \vdash_{\mathbf{L}} \delta'\backslash((\varphi\backslash\delta) \wedge (\delta\backslash\varphi))$ and $\Sigma \vdash_{\mathbf{L}} \delta'$ follow from them, respectively. Therefore, $\Gamma, \Sigma \vdash_{\mathbf{L}} (\varphi\backslash\delta) \wedge (\delta\backslash\varphi)$ holds. Similarly, we can show that $k(\psi/\equiv_\Sigma) = k(\delta/\equiv_\Sigma)$ implies $\Gamma, \Sigma \vdash_{\mathbf{L}} (\psi\backslash\delta) \wedge (\delta\backslash\psi)$. Thus, $\mathbf{L}$ has both the ISDIP and the nonsepEqSCIP. □

THEOREM 40 *For each substructural logic* $\mathbf{L}$*,* $\mathbf{L}$ *has both the SDIP and the nonsepSCIP iff* $\mathsf{V}(\mathbf{L})$ *has the superCHD(*γ*).*

Proof. Suppose that $\mathbf{L}$ has both the SDIP and the nonsepSCIP, and let $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$ be homomorphisms for $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathsf{V}(\mathbf{L})$. By the SDIP, we can construct an algebra $\mathbf{D}$ and homomorphisms $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ satisfying the CHD(γ) as we have done in the proof fo Theorem 30. It suffices to show that they satisfy also the third condition of the superCHD(γ), namely,

> for all $b \in B$ and $c \in C$, if $h(b) \leq k(c)$ $(k(c) \leq h(b))$ then there exists $a \in A$ such that $h(b) \leq h \circ f(a)$ and $k \circ g(a) \leq k(c)$ ($k(c) \leq k \circ g(a)$ and $h \circ f(a) \leq h(b)$, respectively).

Suppose that $h(b) \leq k(c)$ holds for $b \in B$ and $c \in C$. Then, there exist some formulas $\varphi \in \mathbf{Fm}(Y)$ and $\psi \in \mathbf{Fm}(Z)$ such that $b = \eta_{\mathbf{B}}(\varphi)$ and $c = \eta_{\mathbf{C}}(\psi)$. From the definition of h, k and $\equiv$, we can see that $h(b) \leq k(c)$ is equivalent to $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} \varphi\backslash\psi$. Then, by the nonsepSCIP, there exists some formula δ with $\mathrm{var}(\delta) \subseteq \mathrm{var}(\Gamma_{\mathbf{B}} \cup \{\varphi\}) \cap \mathrm{var}(\Gamma_{\mathbf{C}} \cup \{\psi\}) \subseteq Y \cap Z = X$ such that both $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} \varphi\backslash\delta$ and $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} \delta\backslash\psi$ hold. Let $a = \eta_{\mathbf{A}}(\delta) \in A$. Then, $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} \varphi\backslash\delta$ implies $h(b) = h(\eta_{\mathbf{B}}(\varphi)) = (\varphi/\equiv) \leq (\delta/\equiv) = h(\eta_{\mathbf{B}}(\delta)) = h \circ f(\eta_{\mathbf{A}}(\delta)) = h \circ f(a)$. In the same way, $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} \delta\backslash\psi$ implies $k \circ g(a) \leq k(c)$. Similarly, if $k(c') \leq h(b')$ holds for $b' \in B$ and $c' \in C$ then there exists some $a' \in A$ such that $k(c') \leq k \circ g(a')$ and $h \circ f(a') \leq h(b')$ hold. Therefore, $\mathsf{V}(\mathbf{L})$ has the superCHD(γ).

Conversely, suppose that $\mathsf{V}(\mathbf{L})$ has the superCHD(γ). Clearly, by Theorem 30, $\mathbf{L}$ has the SDIP. We assume that $\Gamma, \Sigma \vdash_{\mathbf{L}} \varphi\backslash\psi$ holds for a set of formulas $\Gamma \cup \Sigma \cup \{\varphi, \psi\}$ and let $Y = \mathrm{var}(\Gamma \cup \{\varphi\})$, $Z = \mathrm{var}(\Sigma \cup \{\psi\})$ and $X = Y \cap Z$. In the same way as the proof of Theorem 30, we introduce quotient algebras $\mathbf{Fm}(X)/\Delta, \mathbf{Fm}(Y)/\Gamma$ and $\mathbf{Fm}(Z)/\Sigma$, and homomorphisms $f : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Y)/\Gamma$ and $g : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Z)/\Sigma$. By the superCHD($\gamma$), there exist an algebra $\mathbf{D}$ in $\mathsf{V}(\mathbf{L})$ and homomorphisms $h : \mathbf{Fm}(Y)/\Gamma \to \mathbf{D}$ and $k : \mathbf{Fm}(Z)/\Sigma \to \mathbf{D}$ such that $h \circ f = k \circ g$, (γ) and the following hold;

> for all $b \in \mathbf{Fm}(Y)/\Gamma$ and $c \in \mathbf{Fm}(Z)/\Sigma$, if $h(b) \leq k(c)$ $(k(c) \leq h(b))$ then there exists $a \in \mathbf{Fm}(X)/\Delta$ such that $h(b) \leq h \circ f(a)$ and $k \circ g(a) \leq k(c)$ ($k(c) \leq k \circ g(a)$ and $h \circ f(a) \leq h(b)$, respectively).

We construct a valuation w over $\mathbf{D}$ for formulas from $\mathbf{Fm}(Y \cup Z)$ in the same way as the proof of Theorem 16. Then, $\Gamma, \Sigma \vdash_{\mathbf{L}} \varphi\backslash\psi$ implies $\mathbf{D}, w \models \varphi\backslash\psi$, and hence $h(\varphi/\equiv_\Gamma) \leq k(\psi/\equiv_\Sigma)$ holds. Since $\varphi/\equiv_\Gamma \in \mathbf{Fm}(Y)/\Gamma$ and $\psi/\equiv_\Sigma \in \mathbf{Fm}(Z)/\Sigma$, by the third condition of the superCHD(γ), there exists some $\delta/\equiv_\Delta$ in $\mathbf{Fm}(X)/\Delta$ such that $h(\varphi/\equiv_\Gamma) \leq h \circ f(\delta/\equiv_\Delta) = h(\delta/\equiv_\Gamma)$ and $k(\delta/\equiv_\Sigma) =$

$k \circ g(\delta/\equiv_\Delta) \leq k(\psi/\equiv_\Sigma)$. We show that $h(\varphi/\equiv_\Gamma) \leq h(\delta/\equiv_\Gamma)$ implies $\Gamma, \Sigma \vdash_{\mathbf{L}} \varphi\backslash\delta$. Suppose that $h(\varphi/\equiv_\Gamma) \leq h(\delta/\equiv_\Gamma)$. Then we have $1_{\mathbf{D}} \leq h((\varphi\backslash\delta)/\equiv_\Gamma)$. Since $(\varphi\backslash\delta)/\equiv_\Gamma \in \mathbf{Fm}(Y)/\Gamma$, by the condition (γ) there exists some $\delta'/\equiv_\Delta \in \mathbf{Fm}(X)/\Delta$ such that

$$\begin{cases} (\varphi\backslash\delta)/\equiv_\Gamma \in \mathrm{Fg}_{\mathbf{Fm}(Y)/\Gamma}(f(\delta'/\equiv_\Delta)) = \mathrm{Fg}_{\mathbf{Fm}(Y)/\Gamma}(\delta'/\equiv_\Gamma) \\ \text{and} \\ 1_{\mathbf{Fm}(Z)/\Sigma} \leq g(\delta'/\equiv_\Delta) = (\delta'/\equiv_\Sigma). \end{cases}$$

Hence $\delta', \Gamma \vdash_{\mathbf{L}} \varphi\backslash\delta$ and $\Sigma \vdash_{\mathbf{L}} \delta'$ follow from them, respectively. Therefore, $\Gamma, \Sigma \vdash_{\mathbf{L}} \varphi\backslash\delta$ holds. Similarly, we can show that $k(\delta/\equiv_\Sigma) \leq k(\psi/\equiv_\Sigma)$ implies $\Gamma, \Sigma \vdash_{\mathbf{L}} \delta\backslash\psi$. Thus, $\mathbf{L}$ has both the SDIP and the nonsepSCIP. □

THEOREM 41 *For each substructural logic* $\mathbf{L}$*,* $\mathbf{L}$ *has both the SDIP and the nonsepEqSCIP iff* $\mathsf{V}(\mathbf{L})$ *has the strongCHD(*γ*).*

Proof. Suppose that $\mathbf{L}$ has both the SDIP and the nonsepEqSCIP, and let $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$ be homomorphisms for $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathsf{V}(\mathbf{L})$. By the SDIP, we can construct an algebra $\mathbf{D}$ and homomorphisms $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ satisfying the CHD(γ) as we have done in the proof of Theorem 30. We show that they satisfy also the third condition of the strongCHD(γ), namely, $h(B) \cap k(C) = h \circ f(A)$. Clearly, $h \circ f(A) \subseteq h(B) \cap k(C)$ holds since $f(A) \subseteq B, g(A) \subseteq C$ and $h \circ f = k \circ g$. Conversely, to show that $h(B) \cap k(C) \subseteq h \circ f(A)$, let $d \in h(B) \cap k(C)$, namely for some $b \in B$ and $c \in C$, $d = h(b) = k(c)$. Then, there exist some formulas $\varphi \in \mathbf{Fm}(Y)$ and $\psi \in \mathbf{Fm}(Z)$ such that $b = \eta_{\mathbf{B}}(\varphi)$ and $c = \eta_{\mathbf{C}}(\psi)$. From the definitions of h, k and $\equiv$ in the proof of Theorem 30 (in fact, the proof of Theorem 16), we can see that the condition $h(b) = k(c)$ is equivalent to $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$. By the nonsepEqSCIP, there is a formula δ with $\mathrm{var}(\delta) \subseteq X$ such that $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} (\varphi\backslash\delta) \wedge (\delta\backslash\varphi)$ and $\Gamma_{\mathbf{B}}, \Gamma_{\mathbf{C}} \vdash_{\mathbf{L}} (\psi\backslash\delta) \wedge (\delta\backslash\psi)$. Let $a = \eta_{\mathbf{A}}(\delta) \in A$. Then, we have $d = h(b) = h(\eta_{\mathbf{B}}(\varphi)) = (\varphi/\equiv) = (\delta/\equiv) = h(\eta_{\mathbf{B}}(\delta)) = h \circ f(\eta_{\mathbf{A}}(\delta)) = h \circ f(a)$, which implies $d \in h \circ f(\mathbf{A})$. Thus, $\mathsf{V}(\mathbf{L})$ has the strongCHD(γ).

Conversely, suppose that $\mathsf{V}(\mathbf{L})$ has the strongCHD(γ). Clearly, by Theorem 30, $\mathbf{L}$ has the SDIP. We assume that $\Gamma, \Sigma \vdash_{\mathbf{L}} (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$ holds for a set of formulas $\Gamma \cup \Sigma \cup \{\varphi, \psi\}$ and let $Y = \mathrm{var}(\Gamma \cup \{\varphi\})$, $Z = \mathrm{var}(\Sigma \cup \{\psi\})$ and $X = Y \cap Z$. In the same way as the proof of Theorem 30, we introduce quotient algebras $\mathbf{Fm}(X)/\Delta, \mathbf{Fm}(Y)/\Gamma$ and $\mathbf{Fm}(Z)/\Sigma$, and homomorphisms $f : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Y)/\Gamma$ and $g : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Z)/\Sigma$. By the strongCHD($\gamma$), there exist an algebra $\mathbf{D}$ in $\mathsf{V}(\mathbf{L})$ and homomorphisms $h : \mathbf{Fm}(Y)/\Gamma \to \mathbf{D}$ and $k : \mathbf{Fm}(Z)/\Sigma \to \mathbf{D}$ satisfying $h \circ f = k \circ g$, (γ) and $h(\mathbf{Fm}(Y)/\Gamma) \cap k(\mathbf{Fm}(Z)/\Sigma) = h \circ f(\mathbf{Fm}(X)/\Delta)$. We construct a valuation w over $\mathbf{D}$ for formulas from $\mathbf{Fm}(Y \cup Z)$ in the same way as the proof of Theorem 16. Then, our assumption $\Gamma, \Sigma \vdash_{\mathbf{L}} (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$ implies $\mathbf{D}, w \models (\varphi\backslash\psi) \wedge (\psi\backslash\varphi)$, and hence $h(\varphi/\equiv_\Gamma) = k(\psi/\equiv_\Sigma)$ holds. Since $\varphi/\equiv_\Gamma \in \mathbf{Fm}(Y)/\Gamma$ and $\psi/\equiv_\Sigma \in \mathbf{Fm}(Z)/\Sigma$, the conditions $h \circ f = k \circ g$ and $h(\mathbf{Fm}(Y)/\Gamma) \cap k(\mathbf{Fm}(Z)/\Sigma) = h \circ f(\mathbf{Fm}(X)/\Delta)$ imply that there exists some $\delta/\equiv_\Delta \in \mathbf{Fm}(X)/\Delta$ such that $h(\varphi/\equiv_\Gamma) = h \circ f(\delta/\equiv_\Delta) = h(\delta/\equiv_\Gamma)$ and $k(\psi/\equiv_\Sigma) = k \circ g(\delta/\equiv_\Delta) = k(\delta/\equiv_\Sigma)$. We show that $h(\varphi/\equiv_\Gamma) = h(\delta/\equiv_\Gamma)$ implies $\Gamma, \Sigma \vdash_{\mathbf{L}} (\varphi\backslash\delta) \wedge (\delta\backslash\varphi)$. So, supppose that $h(\varphi/\equiv_\Gamma) = h(\delta/\equiv_\Gamma)$. Then we have $1_{\mathbf{D}} \leq h(((\varphi\backslash\delta) \wedge (\delta\backslash\varphi))/\equiv_\Gamma)$. Since $((\varphi\backslash\delta) \wedge (\delta\backslash\varphi))/\equiv_\Gamma \in \mathbf{Fm}(Y)/\Gamma$, by the condition (γ) there exists some $\delta'/\equiv_\Delta \in \mathbf{Fm}(X)/\Delta$ such that

$$\begin{cases} ((\varphi\backslash\delta) \wedge (\delta\backslash\varphi))/\equiv_\Gamma \in \mathrm{Fg}_{\mathbf{Fm}(Y)/\Gamma}(f(\delta'/\equiv_\Delta)) = \mathrm{Fg}_{\mathbf{Fm}(Y)/\Gamma}(\delta'/\equiv_\Gamma) \\ \text{and} \\ 1_{\mathbf{Fm}(Z)/\Sigma} \leq g(\delta'/\equiv_\Delta) = (\delta'/\equiv_\Sigma). \end{cases}$$

Hence $\delta', \Gamma \vdash_{\mathbf{L}} (\varphi\backslash\delta) \wedge (\delta\backslash\varphi)$ and $\Sigma \vdash_{\mathbf{L}} \delta'$ follow from them, respectively. Therefore, $\Gamma, \Sigma \vdash_{\mathbf{L}} (\varphi\backslash\delta) \wedge (\delta\backslash\varphi)$ holds. Similarly, we can show that $k(\psi/\equiv_\Sigma) = k(\delta/\equiv_\Sigma)$ implies $\Gamma, \Sigma \vdash_{\mathbf{L}} (\psi\backslash\delta) \wedge (\delta\backslash\psi)$. Thus, $\mathbf{L}$ has both the SDIP and the nonsepEqSCIP. □

We summarize the algebraic characterizations of several types of interpolation properties in Figure 10.

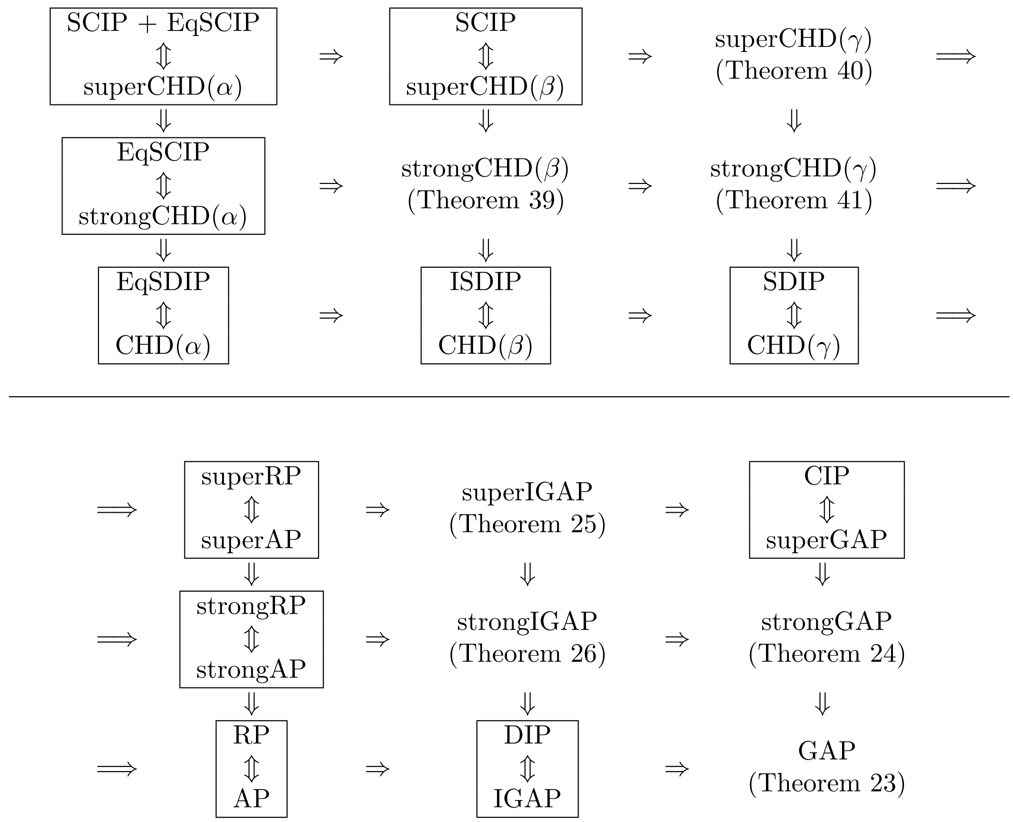

Figure 10: Algebraic characterizations of interpolation properties

6 Beth definability property

Relationships between the interpolation properties and the Beth definability properties have been discussed in many papers. See e.g. [1], [12], [30] and [33] for algebraizable logics and for abstract algebraic logic, [4] within a model-theoretic framework, and [7], [23], [24], [26] and [28] for specific classes of nonclassical logics, e.g. modal logics and superintuitionistic logics.

In this section, we discuss an algebraic characterization of the Beth definability property for substructural logics. Moreover, we clarify relationships between the Beth definability property and the strongRP.

6.1 Algebraic characterization of Beth definability property

The Beth definability property was originally defined for predicate logics. Here, we formulate its analog for propositional substructural logics. In the following, $\bar{x}$ denotes a list of distinct propositional variables, and $\alpha(\bar{x})$, $\beta(\bar{x})$ etc. mean such formulas that variables occurring in them are among $\bar{x}$. Note that, different from the notation var(α), some variables in $\bar{x}$ may not appear in $\alpha(\bar{x})$.

DEFINITION 20 *A substructural logic* **L** *has the Beth definability property (BDP) if for any formula* $\alpha(\bar{x}, x')$ *and distinct propositional variables* y *and* z*, neither of which appear in* $\bar{x}$*, if* $\alpha(\bar{x}, y), \alpha(\bar{x}, z) \vdash_{\mathbf{L}} (y\backslash z) \wedge (z\backslash y)$ *then there exists some formula* $\delta(\bar{x})$ *such that*

- $\alpha(\bar{x}, y) \vdash_{\mathbf{L}} (y\backslash\delta(\bar{x})) \wedge (\delta(\bar{x})\backslash y)$.

The BDP is called B2 in Maksimova [24]. Obviously, the logical property EqSCIP introduced in the previous section implies the BDP. Later we show a sharper connection of the BDP with interpolation properties in Figure 10. We give here another logical property which is equivalent to the BDP. In the following, $\Gamma(\bar{x})$ denotes a set of formulas such that every formulas in it is of the form $\alpha(\bar{x})$, i.e., a set of formulas whose variables are among $\bar{x}$.

LEMMA 42 *For each substructural logic* **L***, the following are equivalent.*

(1) **L** *has the BDP.*

(2) **L** *has the BDP***; for any set of formulas* $\Gamma(\bar{x}, x')$ *and distinct propositional variables* y *and* z*, neither of which appear in* $\bar{x}$*, if* $\Gamma(\bar{x}, y), \Gamma(\bar{x}, z) \vdash_{\mathbf{L}} (y\backslash z) \wedge (z\backslash y)$ *then there exists a formula* $\delta(\bar{x})$ *such that*

- $\Gamma(\bar{x}, y) \vdash_{\mathbf{L}} (y\backslash\delta(\bar{x})) \wedge (\delta(\bar{x})\backslash y)$.

Proof. It is easily shown using the fact that $\vdash_{\mathbf{L}}$ is finitary and conjunctive. □

For classical logic, and in general for modal logics, the CIP is equivalent to the BDP (Craig [5] showed that the CIP implies the BDP and the converse was shown by Maksimova in [23] and [24]). But this is not always the case. In fact, all superintuitionistic logics have the BDP, due to Kreisel [17], while there are only seven consistent superintuitionistic logics with the CIP, due to Maksimova in [20] and [21]. (See also [22] for modal logics.) [4] On the other hand, Montagna showed in [29] that not many substructural logics over Hájek's basic logic **BL** have the BDP. In fact, those logics over **BL** having the BDP are exactly superintuitionistic logics over Gödel logic. For more information on the BDP for modal and predicate logics, see [7].

DEFINITION 21 *A variety* $\mathcal{V}$ *has the ES***1*[5] *if for all* $\mathbf{A}, \mathbf{B}$ *in* $\mathcal{V}$*, for all embedding* $f : \mathbf{A} \to \mathbf{B}$ *and for all* $b_0 \in B \setminus f(A)$*, there exist an algebra* **C** *in* $\mathcal{V}$ *and embeddings* $h : \mathbf{Sg_B}(f(A) \cup \{b_0\}) \to \mathbf{C}$ *and* $k : \mathbf{Sg_B}(f(A) \cup \{b_0\}) \to \mathbf{C}$ *such that* $h \circ f = k \circ f$ *and* $h(b_0) \neq k(b_0)$*, where* $\mathbf{Sg_D}(X)$ *is the subalgebra of* **D** *generated by* X*.*

For algebraizable logics, an algebraic characterization of the BDP was given by Németi [30] and Hoogland [12]. See also Henkin, Monk and Tarski [10], Theorem 5.6.10 and Andréka, Németi and Sain [1] Theorem 58, p.213. For further study in this direction, consult also Sain [34], Hoogland [11] and [13] [6]. In [24], Maksimova gave an alternative characterization of it for modal logics. The following theorem shows that the result due to Maksimova can be extended to those for substructural logics.

THEOREM 43 *For each substructural logic* **L***,* **L** *has the BDP iff* $\mathsf{V}(\mathbf{L})$ *has the ES***1.*

[4]In [23] and [24], the BDP is defined in the implicational form, not in the deductive one as we do. But they are of course equivalent in any logic for which deduction theorem holds, e.g. in any superintuitionistic logic. Note that it is shown in [35] that the BDP fails in a wide variety of relevant logics.

[5]In [24], ES*1 is called simply ES* (*epimorphisms surjectivity*). Here, for the sake of correspondence between stronger forms of the BDP and the ES* shown later, the ES* is numbered.

[6]In [34] and [11], algebraic characterizations of a weak form of the BDP, called the *weak BDP*, are discussed. At this moment, it is not clear for us how to incorporate the weak BDP into our present framework.

Proof. First, we show that $\mathsf{V}(\mathbf{L})$ has the ES*1, by assuming the BDP of $\mathbf{L}$. By Lemma 42, it is enough to show that the BDP* implies the ES*1. Let $f : \mathbf{A} \to \mathbf{B}$ be an embedding for $\mathbf{A}, \mathbf{B}$ in $\mathsf{V}(\mathbf{L})$, $b_0 \in B \setminus f(A)$ and $\mathbf{Sg_B}(f(A) \cup \{b_0\})$ be the subalgebra of $\mathbf{B}$ generated by $f(A) \cup \{b_0\}$. We define a set of variables X by $X = \{x_a : a \in A\}$, and a mapping $\eta'_{\mathbf{A}} : X \to \mathbf{A}$ by $\eta'_{\mathbf{A}}(x_a) = a$. Then, the mapping $\eta'_{\mathbf{A}}$ is uniquely extended to a surjective homomorphism $\eta_{\mathbf{A}} : \mathbf{Fm}(X) \to \mathbf{A}$. Next, by introducing distinct new variables y_{b_0} and z_{b_0}, we define sets of variables Y and Z by $Y = X \cup \{y_{b_0}\}$ and $Z = X \cup \{z_{b_0}\}$, and define also mapping $\eta'_{\mathbf{B}1} : Y \to \mathbf{Sg_B}(f(A) \cup \{b_0\})$ and $\eta'_{\mathbf{B}2} : Z \to \mathbf{Sg_B}(f(A) \cup \{b_0\})$ by

$$\eta'_{\mathbf{B}1}(y) = \begin{cases} f(a) & \text{if } y = x_a \text{ for some } x_a \in X \\ b_0 & \text{if } y = y_{b_0} \end{cases}$$

and

$$\eta'_{\mathbf{B}2}(z) = \begin{cases} f(a) & \text{if } z = x_a \text{ for some } x_a \in X \\ b_0 & \text{if } z = z_{b_0}. \end{cases}$$

Then $\eta'_{\mathbf{B}1}$ and $\eta'_{\mathbf{B}2}$ are naturally extended to surjective homomorphisms $\eta_{\mathbf{B}1} : \mathbf{Fm}(Y) \to \mathbf{Sg_B}(f(A) \cup \{b_0\})$ and $\eta_{\mathbf{B}2} : \mathbf{Fm}(Z) \to \mathbf{Sg_B}(f(A) \cup \{b_0\})$, respectively, which satisfy $\eta_{\mathbf{B}1}(\sigma) = \eta_{\mathbf{B}2}(\sigma) = f(\eta_{\mathbf{A}}(\sigma))$ for every $\sigma \in \mathbf{Fm}(X)$. Define sets of formulas $\Gamma_1(\bar{x}, y_{b_0})$ and $\Gamma_2(\bar{x}, z_{b_0})$ by

$$\Gamma_1(\bar{x}, y_{b_0}) = \{\varphi \in \mathbf{Fm}(Y) : \eta_{\mathbf{B}1}(\varphi) \geq 1_{\mathbf{Sg_B}(f(A)\cup\{b_0\})}\},$$
$$\Gamma_2(\bar{x}, z_{b_0}) = \{\psi \in \mathbf{Fm}(Z) : \eta_{\mathbf{B}2}(\psi) \geq 1_{\mathbf{Sg_B}(f(A)\cup\{b_0\})}\},$$

where $\bar{x}$ is the list of variables in X which are used in constructions of either φ or ψ. Note that it is easily shown that both $\Gamma_1(\bar{x}, z_{b_0}) = \Gamma_2(\bar{x}, z_{b_0})$ and $\Gamma_2(\bar{x}, y_{b_0}) = \Gamma_1(\bar{x}, y_{b_0})$ hold. Thus, we can write $\Gamma_1(\bar{x}, y_{b_0})$ and $\Gamma_2(\bar{x}, z_{b_0})$ as $\Gamma(\bar{x}, y_{b_0})$ and $\Gamma(\bar{x}, z_{b_0})$, respectively. We introduce a binary relation $\equiv$ on $\mathbf{Fm}(Y \cup Z)$ by

$$\beta \equiv \gamma \text{ iff } \Gamma(\bar{x}, y_{b_0}), \Gamma(\bar{x}, z_{b_0}) \vdash_{\mathbf{L}} (\beta\backslash\gamma) \wedge (\gamma\backslash\beta).$$

Then, it is easily seen that $\equiv$ is a congruence relation on $\mathbf{Fm}(Y \cup Z)$ and that the quotient algebra $\mathbf{Fm}(Y \cup Z)/\equiv$ is a member of $\mathsf{V}(\mathbf{L})$. Let us call this algebra, $\mathbf{C}$. We will show that this $\mathbf{C}$ is a required algebra satisfying the conditions for ES*1.

Now define mappings $h : \mathbf{Sg_B}(f(A) \cup \{b_0\}) \to \mathbf{C}$ and $k : \mathbf{Sg_B}(f(A) \cup \{b_0\}) \to \mathbf{C}$ by

- $h(b) = (\varphi/\equiv)$ when $b = \eta_{\mathbf{B}1}(\varphi)$ for a formula $\varphi \in \mathbf{Fm}(Y)$,
- $k(b') = (\psi/\equiv)$ when $b' = \eta_{\mathbf{B}2}(\psi)$ for a formula $\psi \in \mathbf{Fm}(Z)$.

We prove that both h and k are well-defined embeddings. To show the well-definedness of h, suppose that $\eta_{\mathbf{B}1}(\varphi) = \eta_{\mathbf{B}1}(\varphi')$ for $\varphi, \varphi' \in \mathbf{Fm}(Y)$. Then, $\Gamma(\bar{x}, y_{b_0}) \vdash_{\mathbf{L}} (\varphi\backslash\varphi') \wedge (\varphi'\backslash\varphi)$, and hence $\varphi \equiv \varphi'$. It is easy to see that h is a homomorphism. To show that h is injective, suppose that $h(b) = h(b')$, where $b = \eta_{\mathbf{B}1}(\varphi)$ and $b' = \eta_{\mathbf{B}1}(\varphi')$ for $\varphi, \varphi' \in \mathbf{Fm}(Y)$. Then, $\varphi \equiv \varphi'$, and thus $\Gamma(\bar{x}, y_{b_0}), \Gamma(\bar{x}, z_{b_0}) \vdash_{\mathbf{L}} (\varphi\backslash\varphi') \wedge (\varphi'\backslash\varphi)$ by the definition of $\equiv$. Let s_1 be a substitution such that $s_1(x) = x$ for each $x \in X$, $s_1(y_{b_0}) = y_{b_0}$ and $s_1(z_{b_0}) = y_{b_0}$. Then, we have $s_1(\Gamma(\bar{x}, y_{b_0})), s_1(\Gamma(\bar{x}, z_{b_0})) \vdash_{\mathbf{L}} s_1((\varphi\backslash\varphi') \wedge (\varphi'\backslash\varphi))$, namely $\Gamma(\bar{x}, y_{b_0}) \vdash_{\mathbf{L}} (\varphi\backslash\varphi') \wedge (\varphi'\backslash\varphi)$. Since $\eta_{\mathbf{B}1}(\varphi^*) \geq 1_{\mathbf{Sg_B}(f(A)\cup\{b_0\})}$ holds for each $\varphi^* \in \Gamma(\bar{x}, y_{b_0})$, we have that $\eta_{\mathbf{B}1}((\varphi\backslash\varphi') \wedge (\varphi'\backslash\varphi)) \geq 1_{\mathbf{Sg_B}(f(A)\cup\{b_0\})}$, namely $b = \eta_{\mathbf{B}1}(\varphi) = \eta_{\mathbf{B}1}(\varphi') = b'$. Thus, h is an embedding. Similarly, k is shown to be a well-defined embedding. Next, we show that $h \circ f = k \circ f$. Take an arbitrary element $a \in A$. Then there exists a formula $\sigma \in \mathbf{Fm}(X)$ such that $a = \eta_{\mathbf{A}}(\sigma)$. Then, $(h \circ f)(a) = h(f(\eta_{\mathbf{A}}(\sigma))) = h(\eta_{\mathbf{B}1}(\sigma)) = (\sigma/\equiv) = k(\eta_{\mathbf{B}2}(\sigma)) = k(f(\eta_{\mathbf{A}}(\sigma))) = (k \circ f)(a)$. Thus, $h \circ f = k \circ f$. It remains to show that $h(b_0) \neq k(b_0)$. Assume that $h(b_0) = k(b_0)$ holds. Then, we have $(y_{b_0}/\equiv) = h(\eta_{\mathbf{B}1}(y_{b_0})) = h(b_0) = k(b_0) = k(\eta_{\mathbf{B}2}(z_{b_0})) = (z_{b_0}/\equiv)$, and hence $\Gamma(\bar{x}, y_{b_0}), \Gamma(\bar{x}, z_{b_0}) \vdash_{\mathbf{L}} (y_{b_0}\backslash z_{b_0}) \wedge (z_{b_0}\backslash y_{b_0})$ holds. By the BDP*, there exists some $\delta(\bar{x})$ such that $\Gamma(\bar{x}, y_{b_0}) \vdash_{\mathbf{L}} (y_{b_0}\backslash\delta(\bar{x})) \wedge (\delta(\bar{x})\backslash y_{b_0})$. Note that since $\delta(\bar{x}) \in \mathbf{Fm}(X)$, we have $\eta_{\mathbf{A}}(\delta(\bar{x})) \in A$. Let $a = \eta_{\mathbf{A}}(\delta(\bar{x}))$. Since $\eta_{\mathbf{B}1}(\varphi^*) \geq 1_{\mathbf{Sg_B}(f(A)\cup\{b_0\})}$ holds for each $\varphi^* \in \Gamma(\bar{x}, y_{b_0})$, we have that $\eta_{\mathbf{B1}}((y_{b_0}\backslash\delta(\bar{x})) \wedge (\delta(\bar{x})\backslash y_{b_0})) \geq 1_{\mathbf{Sg_B}(f(A)\cup\{b_0\})}$, which implies $b_0 = \eta_{\mathbf{B1}}(y_{b_0}) = \eta_{\mathbf{B1}}(\delta(\bar{x})) = f(\eta_{\mathbf{A}}(\delta(\bar{x}))) = f(a)$. Hence $b_0 \in f(A)$ holds. But, this contradicts the choice of b_0. Thus, $h(b_0) \neq k(b_0)$. Therefore, $\mathsf{V}(\mathbf{L})$ has the ES*1.

We show next that the ES*1 implies the BDP by taking the contraposition. Assume that $\mathbf{L}$ does not satisfy the BDP, namely there exist some formula $\alpha(\bar{x}, x')$ and distinct propositional variables y and z, neither of which appear in $\bar{x}$, such that $\alpha(\bar{x}, y), \alpha(\bar{x}, z) \vdash_{\mathbf{L}} (y\backslash z) \wedge (z\backslash y)$ holds but there is no formula $\delta(\bar{x})$ satisfying $\alpha(\bar{x}, y) \vdash_{\mathbf{L}} (y\backslash\delta(\bar{x})) \wedge (\delta(\bar{x})\backslash y)$. Let X be a set of all variables occuring in the list $\bar{x}$ and denote sets of variables Y and Z by $Y = X \cup \{y\}$ and $Z = X \cup \{z\}$. Define $\Delta = \{\sigma \in \mathbf{Fm}(X) : \alpha(\bar{x}, y) \vdash_{\mathbf{L}} \sigma\}$, which is obviously equal to $\{\sigma \in \mathbf{Fm}(X) : \alpha(\bar{x}, z) \vdash_{\mathbf{L}} \sigma\}$. The set Δ determines a binary relation $\equiv_\Delta$ on $\mathbf{Fm}(X)$ by

$$\beta \equiv_\Delta \gamma \text{ iff } \Delta \vdash_{\mathbf{L}} (\beta\backslash\gamma) \wedge (\gamma\backslash\beta),$$

which is in fact a congruence relation. We denote the quotient algebra of $\mathbf{Fm}(X)$ determined by this $\equiv_\Delta$ as $\mathbf{Fm}(X)/\Delta$. Similarly, we can introduce quotient algebras [7] $\mathbf{Fm}(Y)/\alpha(\bar{x}, y)$ of $\mathbf{Fm}(Y)$, and $\mathbf{Fm}(Z)/\alpha(\bar{x}, z)$ of $\mathbf{Fm}(Z)$, by taking formulas $\alpha(\bar{x}, y)$ and $\alpha(\bar{x}, z)$, respectively, in the place of Δ. It is clear that all of these algebras $\mathbf{Fm}(X)/\Delta$, $\mathbf{Fm}(Y)/\alpha(\bar{x}, y)$ and $\mathbf{Fm}(Z)/\alpha(\bar{x}, z)$ are members of $\mathsf{V}(\mathbf{L})$. We define mappings $f : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Y)/\alpha(\bar{x}, y)$ and $g : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Z)/\alpha(\bar{x}, z)$ by

$$f(\sigma/\Delta) = \sigma/\alpha(\bar{x}, y) \text{ and } g(\sigma/\Delta) = \sigma/\alpha(\bar{x}, z).$$

They are shown to be well-defined embeddings, by the definition of Δ. Let s_2 be a substitution such that $s_2(x) = x$ for $x \in X$, $s_2(y) = y$ and $s_2(z) = y$. Define a mapping $w : \mathbf{Fm}(Z)/\alpha(\bar{x}, z) \to \mathbf{Fm}(Y)/\alpha(\bar{x}, y)$ by $w(\psi/\alpha(\bar{x}, z)) = s_2(\psi)/\alpha(\bar{x}, y)$. Then, it is easy to show that w is an isomorphism. We prove that $f = w \circ g$ holds. For any $\sigma \in \mathbf{Fm}(X)$, we have $f(\sigma/\Delta) = (\sigma/\alpha(\bar{x}, y)) = (s_2(\sigma)/\alpha(\bar{x}, y)) = w(\sigma/\alpha(\bar{x}, z)) = w \circ g(\sigma/\Delta)$. Note that $\mathbf{Fm}(Y)/\alpha(\bar{x}, y)$ is generated by $f(\mathbf{Fm}(X)/\Delta) \cup \{y/\alpha(\bar{x}, y)\}$. We show that $y/\alpha(\bar{x}, y) \notin f(\mathbf{Fm}(X)/\Delta)$. If not, there is some $\delta/\Delta \in \mathbf{Fm}(X)/\Delta$ such that $y/\alpha(\bar{x}, y) = f(\delta/\Delta) = \delta/\alpha(\bar{x}, y)$. Hence, $\alpha(\bar{x}, y) \vdash_{\mathbf{L}} (y\backslash\delta) \wedge (\delta\backslash y)$, namely $\alpha(\bar{x}, y) \vdash_{\mathbf{L}} (y\backslash\delta(\bar{x})) \wedge (\delta(\bar{x})\backslash y)$ holds since δ can be written as $\delta(\bar{x})$. But this contradicts our assumption. Thus $y/\alpha(\bar{x}, y) \notin f(\mathbf{Fm}(X)/\Delta)$.

Assume that $\mathsf{V}(\mathbf{L})$ has the ES*1. Then there exist some $\mathbf{C}$ in $\mathsf{V}(\mathbf{L})$ and embeddings $h : \mathbf{Fm}(Y)/\alpha(\bar{x}, y) \to \mathbf{C}$ and $k : \mathbf{Fm}(Y)/\alpha(\bar{x}, y) \to \mathbf{C}$ such that $h \circ f = k \circ f$ and $h(y/\alpha(\bar{x}, y)) \neq k(y/\alpha(\bar{x}, y))$. Now, consider a valuation u over $\mathbf{C}$ for formulas in $\mathbf{Fm}(Y \cup Z)$ defined as follows: For every $x \in Y \cup Z$

$$u(x) = \begin{cases} h(x/\alpha(\bar{x}, y)) & \text{if } x \in Y \\ k \circ w(x/\alpha(\bar{x}, z)) & \text{if } x \in Z. \end{cases}$$

The mapping u is well-defined, since if $x \in X$ then $h(x/\alpha(\bar{x}, y)) = h \circ f(x/\Delta) = k \circ f(x/\Delta) = k \circ w \circ g(x/\Delta) = k \circ w(x/\alpha(\bar{x}, z))$. As usual, u is extended to a mapping from $\mathbf{Fm}(Y \cup Z)$ to $\mathbf{C}$, which satisfies that

$$u(\theta) = \begin{cases} h(\theta/\alpha(\bar{x}, y)) & \text{if } \theta \in \mathbf{Fm}(Y) \\ k \circ w(\theta/\alpha(\bar{x}, z)) & \text{if } \theta \in \mathbf{Fm}(Z). \end{cases}$$

Since $\alpha(\bar{x}, y) \vdash_{\mathbf{L}} \alpha(\bar{x}, y)$ holds, we have $\alpha(\bar{x}, y)/\alpha(\bar{x}, y) \geq 1_{\mathbf{Fm}(Y)/\alpha(\bar{x}, y)}$. Hence both $u(\alpha(\bar{x}, y)) = h(\alpha(\bar{x}, y)/\alpha(\bar{x}, y)) \geq 1_{\mathbf{C}}$ and $u(\alpha(\bar{x}, z)) = k \circ w(\alpha(\bar{x}, z)/\alpha(\bar{x}, z)) = k(\alpha(\bar{x}, y)/\alpha(\bar{x}, y)) \geq 1_{\mathbf{C}}$ hold. By our assumption $\alpha(\bar{x}, y), \alpha(\bar{x}, z) \vdash_{\mathbf{L}} (y\backslash z) \wedge (z\backslash y)$, we have $u((y\backslash z) \wedge (z\backslash y)) \geq 1_{\mathbf{C}}$, which implies $h(y/\alpha(\bar{x}, y)) = u(y) = u(z) = k \circ w(z/\alpha(\bar{x}, z)) = k(y/\alpha(\bar{x}, y))$. But, this contradicts the condition of the ES*1, and thus $\mathsf{V}(\mathbf{L})$ does not satisfy the ES*1. Therefore, the ES*1 implies the BDP. □

[7] Note that hereafter, the symbol "/" denotes a quotient, not a right-division. In other words, "$\beta(\bar{x}, y)/\alpha(\bar{x}, y)$" means not the element of $\mathbf{Fm}(Y)$ but the congruence class of $\beta(\bar{x}, y)$ with respect to $\equiv_{\alpha(\bar{x}, y)}$.

6.2 Strong Robinson property and Beth definability property

To clarify relationships between the strongRP and the BDP, we introduce extensions of the BDP as follows. In the following, $\bar{x}_n$ denotes a list of distinct n-elements propositional variables, namely $\bar{x}_n = x_1, \ldots, x_n$.

DEFINITION 22 *For each $n \in \omega$, a substructural logic $\mathbf{L}$ has the n-Beth definability property (n-BDP) if for every formula $\alpha(\bar{x}, \bar{x}'_n)$, every pair of lists of variables $\bar{y}_n$ and $\bar{z}_n$ such that for each $1 \le i, j \le n$, y_i and z_j are distinct and none of them appear in $\bar{x}$, and every $m \le n$, if $\alpha(\bar{x}, \bar{y}_n), \alpha(\bar{x}, \bar{z}_n) \vdash_{\mathbf{L}} (y_m \backslash z_m) \wedge (z_m \backslash y_m)$ holds then there exists a formula $\delta_m(\bar{x})$ such that*

- $\alpha(\bar{x}, \bar{y}_n) \vdash_{\mathbf{L}} (y_m \backslash \delta_m(\bar{x})) \wedge (\delta_m(\bar{x}) \backslash y_m)$.

A substructural logic $\mathbf{L}$ has the projective Beth definability property [8] *(PBDP) if it has the n-BDP for all $n \in \omega$.*

For each $n \in \omega$, a substructural logic $\mathbf{L}$ has the n-BDP$^+$ if for every formula $\alpha(\bar{x}, \bar{x}'_n)$, every pair of lists of variables $\bar{y}_n$ and $\bar{z}_n$ such that for each $1 \le i, j \le n$, y_i and z_j are distinct and none of them appear in $\bar{x}$, and every $m \le n$, if $\alpha(\bar{x}, \bar{y}_n), \alpha(\bar{x}, \bar{z}_n) \vdash_{\mathbf{L}} (y_m \backslash z_l) \wedge (z_l \backslash y_m)$ holds for some $1 \le l \le n$ then there exists a formula $\delta_m(\bar{x})$ such that

- $\alpha(\bar{x}, \bar{y}_n) \vdash_{\mathbf{L}} (y_m \backslash \delta_m(\bar{x})) \wedge (\delta_m(\bar{x}) \backslash y_m)$.

A substructural logic $\mathbf{L}$ has the projective BDP$^+$ (written as PBDP$^+$) if it has the n-BDP$^+$ for all $n \in \omega$.

For each $n \in \omega$, a substructural logic $\mathbf{L}$ has the n-Beth definability property by formulas (n-BDPF) if for all formulas $\alpha(\bar{x}, \bar{x}'_n), \varphi(\bar{x}, \bar{y}_n)$ and $\psi(\bar{x}, \bar{z}_n)$ such that for each $1 \le i, j \le n$, y_i and z_j are distinct and neither of them appear in $\bar{x}$, if $\alpha(\bar{x}, \bar{y}_n), \alpha(\bar{x}, \bar{z}_n) \vdash_{\mathbf{L}} (\varphi(\bar{x}, \bar{y}_n) \backslash \psi(\bar{x}, \bar{z}_n)) \wedge (\psi(\bar{x}, \bar{z}_n) \backslash \varphi(\bar{x}, \bar{y}_n))$ then there exists a formula $\delta(\bar{x})$ such that

- $\alpha(\bar{x}, \bar{y}_n) \vdash_{\mathbf{L}} (\varphi(\bar{x}, \bar{y}_n) \backslash \delta(\bar{x})) \wedge (\delta(\bar{x}) \backslash \varphi(\bar{x}, \bar{y}_n))$.

A substructural logic $\mathbf{L}$ has the projective Beth definability property by formulas (PBDPF) if it has the n-BDPF for all $n \in \omega$.

Similar to Lemma 42, we can introduce general forms of the BDPs in Definition 22 by replacing fomulas $\alpha(\bar{x}, \bar{y})$ and $\alpha(\bar{x}, \bar{z})$ by sets of formulas $\Gamma(\bar{x}, \bar{y})$ and $\Gamma(\bar{x}, \bar{z})$, respectively, that are in fact equivalent to them.

Obviously, the 1-BDP and the 1-BDP$^+$ are nothing but the BDP. Also, for any $n \in \omega$, the n-BDPF always implies the n-BDP$^+$, which in turn always implies the n-BDP. Similarly, the PBDPF always implies the PBDP$^+$, which in turn always implies the PBDP. Note also that for any $n, m \in \omega$, if $m \le n$ then the n-BDP implies the m-BDP. For if $\alpha(\bar{x}, \bar{y}_m), \alpha(\bar{x}, \bar{z}_m) \vdash_{\mathbf{L}} (y_l \backslash z_l) \wedge (z_l \backslash y_l)$ holds for some $1 \le l \le m$ then take arbitrary lists of variables $\bar{y}_{n-m}$ and $\bar{z}_{n-m}$ such that for each $1 \le i, j \le n-m$, y_i and z_j are distinct and none of them appear in $\bar{x}, \bar{y}_m$ and $\bar{z}_m$. Then, $\alpha(\bar{x}, \bar{y}_m)$ and $\alpha(\bar{x}, \bar{z}_m)$ are expressed also as $\alpha(\bar{x}, \bar{y}_m, \bar{y}_{n-m})$ and $\alpha(\bar{x}, \bar{z}_m, \bar{z}_{n-m})$, respectively. Thus, by the n-BDP, there exists some $\delta_l(\bar{x})$ such that $\alpha(\bar{x}, \bar{y}_m, \bar{y}_{n-m}) \vdash_{\mathbf{L}} (y_l \backslash \delta_l(\bar{x})) \wedge (\delta_l(\bar{x}) \backslash y_l)$ holds. Since $\alpha(\bar{x}, \bar{y}_m, \bar{y}_{n-m}) = \alpha(\bar{x}, \bar{y}_m)$, the m-BDP follows. This holds also for both n-BDP$^+$ and n-BDPF.

LEMMA 44 *For each substructural logic $\mathbf{L}$, if $\mathbf{L}$ has the strongRP then it has the PBDPF.*

Proof. It is sufficient to show that the n-BDPF holds for all $n \in \omega$ by assuming the strongRP. Suppose that $\alpha(\bar{x}, \bar{y}_n), \alpha(\bar{x}, \bar{z}_n) \vdash_{\mathbf{L}} (\varphi(\bar{x}, \bar{y}_n) \backslash \psi(\bar{x}, \bar{z}_n)) \wedge (\psi(\bar{x}, \bar{z}_n) \backslash \varphi(\bar{x}, \bar{y}_n))$ holds for formulas α, φ and ψ satisfying the assumption on variables of the n-BDPF. Clearly, $\alpha(\bar{x}, \bar{y}_n) \vdash_{\mathbf{L}} \sigma(\bar{x})$ iff $\alpha(\bar{x}, \bar{z}_n) \vdash_{\mathbf{L}} \sigma(\bar{x})$ holds for each formula $\sigma(\bar{x})$. Now, by using the strongRP, we can show that there exists a formula $\delta(\bar{x})$ such that $\alpha(\bar{x}, \bar{y}_n) \vdash_{\mathbf{L}} (\varphi(\bar{x}, \bar{y}_n) \backslash \delta(\bar{x})) \wedge (\delta(\bar{x}) \backslash \varphi(\bar{x}, \bar{y}_n))$ holds. Therefore, $\mathbf{L}$

[8] The notion of the PBDP is introduced by Maksimova. In [26], the PBDP is called the PB2.

has the n-BDPF. □

Thus, the relationships among the strongRP and various forms of extensions of the BDP in Definition 22 are as shown in the following Figure 11. In particular, for superintuitionistic logics the following figure says that the CIP implies the PBDPF and hence implies the PBDP, which implies the BDP. Recall that only seven consistent superintuitionistic logics have the CIP while all superintuitionistic logics have the BDP. Maksimova proved in [27] that exactly 16 superintuitionistic logics have the PBDP.

$$
\begin{array}{ccccccccccc}
\text{strongRP} & & & & & & & & & & \\
\Downarrow & & & & & & & & & & \\
\text{PBDPF} & \Rightarrow & \cdots & \Rightarrow & n\text{-BDPF} & \Rightarrow & \cdots & \Rightarrow & 2\text{-BDPF} & \Rightarrow & 1\text{-BDPF} \\
\Downarrow & & & & \Downarrow & & & & \Downarrow & & \Downarrow \\
\text{PBDP}^+ & \Rightarrow & \cdots & \Rightarrow & n\text{-BDP}^+ & \Rightarrow & \cdots & \Rightarrow & 2\text{-BDP}^+ & \Rightarrow & 1\text{-BDP}^+ \\
\Downarrow & & & & \Downarrow & & & & \Downarrow & & \Updownarrow \\
\text{PBDP} & \Rightarrow & \cdots & \Rightarrow & n\text{-BDP} & \Rightarrow & \cdots & \Rightarrow & 2\text{-BDP} & \Rightarrow & \text{BDP}
\end{array}
$$

Figure 11: Relations among the strongRP and the BDPs

To give algebraic characterizations of variants of the BDPs in Definition 22, we strengthen the ES*1 as follows.

DEFINITION 23 *For each $n \in \omega$, a variety $\mathcal{V}$ has the ES$^*n$ if for each $\mathbf{A}, \mathbf{B} \in \mathcal{V}$, each embedding $f : \mathbf{A} \to \mathbf{B}$ and each $b_0, \ldots, b_{n-1} \in B \setminus f(A)$, there exist an algebra $\mathbf{C}$ in $\mathcal{V}$ and embeddings $h : \mathbf{B}' \to \mathbf{C}$ and $k : \mathbf{B}' \to \mathbf{C}$ such that $h \circ f = k \circ f$ and $h(b_i) \neq k(b_i)$ for all $i < n$, where $\mathbf{B}'$ is the subalgebra of $\mathbf{B}$ generated by $f(A) \cup \{b_0, \ldots, b_{n-1}\}$.*

A variety $\mathcal{V}$ has the SES [9] if for each $\mathbf{A}, \mathbf{B} \in \mathcal{V}$ and each embedding $f : \mathbf{A} \to \mathbf{B}$, there exist an algebra $\mathbf{C}$ in $\mathcal{V}$ and embeddings $h : \mathbf{B} \to \mathbf{C}$ and $k : \mathbf{B} \to \mathbf{C}$ such that $h \circ f = k \circ f$ and $h(b) \neq k(b)$ for all $b \in B \setminus f(A)$.

For each $n \in \omega$, a variety $\mathcal{V}$ has the ES$^{**}n$ if for each $\mathbf{A}, \mathbf{B} \in \mathcal{V}$, each embedding $f : \mathbf{A} \to \mathbf{B}$ and each $b_0, \ldots, b_{n-1} \in B \setminus f(A)$, there exist an algebra $\mathbf{C}$ in $\mathcal{V}$ and embeddings $h : \mathbf{B}' \to \mathbf{C}$ and $k : \mathbf{B}' \to \mathbf{C}$ such that $h \circ f = k \circ f$ and $h(b_i) \neq k(b_j)$ for all $i, j < n$, where $\mathbf{B}'$ is the subalgebra of $\mathbf{B}$ generated by $f(A) \cup \{b_0, \ldots, b_{n-1}\}$.[10]

For each $n \in \omega$, a variety $\mathcal{V}$ has the ES$^{***}n$ if for each $\mathbf{A}, \mathbf{B} \in \mathcal{V}$, each embedding $f : \mathbf{A} \to \mathbf{B}$ and each $b_0, \ldots, b_{n-1} \in B \setminus f(A)$, there exist an algebra $\mathbf{C}$ in $\mathcal{V}$ and embeddings $h : \mathbf{B}' \to \mathbf{C}$ and $k : \mathbf{B}' \to \mathbf{C}$ such that $h \circ f = k \circ f$ and $h(B') \cap k(B') = h \circ f(A)$, where $\mathbf{B}'$ is the subalgebra of $\mathbf{B}$ generated by $f(A) \cup \{b_0, \ldots, b_{n-1}\}$.

A variety $\mathcal{V}$ has the ES† if for each $\mathbf{A}, \mathbf{B} \in \mathcal{V}$ and each embedding $f : \mathbf{A} \to \mathbf{B}$, there exist an algebra $\mathbf{C}$ in $\mathcal{V}$ and embeddings $h : \mathbf{B} \to \mathbf{C}$ and $k : \mathbf{B} \to \mathbf{C}$ such that $h \circ f = k \circ f$ and $h(B) \cap k(B) = h \circ f(A)$.

Obiviously, for each $n \in \omega$, the ES$^{***}n$ always implies the ES$^{**}n$, which in turn always implies the ES$^*n$. Similarly, the ES† always implies the SES. Also, for all $n, m \in \omega$, if $m \leq n$ then the ES$^*n$ implies the ES$^*m$. This holds also for the ES$^{**}n$ and ES$^{***}n$. Note also that when $\mathbf{B} = \mathbf{C}$ and $f = g$ in the definition of the strongAP, it is nothing but the ES†. Thus, ES† is a special case of the strongAP.

[9] The notion of the SES is introduced in [26]

[10] Our *ES$^{**}n$* is different from ES** by Maksimova in [24]. In that paper, the ES** is defined by replacing "embedding f" with "homomorphism f" in the definition of ES* (i.e. our ES*1). It was proved there that in varieties of modal algebras, the ES** is equivalent to the ES*1. Recall that this is not the case for the AP. As shown Lemma 28, if either f or g (or both) in the definition of the AP is replaced by "homomorphism" then the only trivial variety satisfies them.

Now, we give algebraic characterizations of the n-BDP, the n-BDP$^+$ and the n-BDPF, respectively.

THEOREM 45 *For each $n \in \omega$ and substructural logic $\mathbf{L}$, $\mathbf{L}$ has the n-BDP iff $\mathsf{V}(\mathbf{L})$ has the ES$^*n$.*

Proof. We show our theorem using the proof of Theorem 43. First, we show that $\mathsf{V}(\mathbf{L})$ has the ES$^*n$, by assuming the n-BDP of $\mathbf{L}$. Let $f : \mathbf{A} \to \mathbf{B}$ be an embedding for $\mathbf{A}, \mathbf{B} \in \mathsf{V}(\mathbf{L})$, $b_0, \ldots, b_{n-1} \in B \setminus f(A)$ and $\mathbf{B}'$ be the subalgebra of $\mathbf{B}$ generated by $f(A) \cup \{b_0, \ldots, b_{n-1}\}$. Define a set of variables X and a surjective homomorphisms $\eta_{\mathbf{A}} : \mathbf{Fm}(X) \to \mathbf{A}$ in the same way as before. Different from the proof of Theorem 43, we need to introduce distinct new variables y_{b_i} and z_{b_i} with $0 \le i \le n-1$. We define sets of variables Y and Z by $Y = X \cup \{y_{b_0}, \ldots, y_{b_{n-1}}\}$ and $Z = X \cup \{z_{b_0}, \ldots, z_{b_{n-1}}\}$, respectively, so that $Y \cap Z = X$, and define also mapping $\eta'_{\mathbf{B1}} : Y \to \mathbf{B}'$ and $\eta'_{\mathbf{B2}} : Z \to \mathbf{B}'$ by

$$\eta'_{\mathbf{B1}}(y) = \begin{cases} f(a) & \text{if } y = x_a \text{ for some } x_a \in X \\ b_i & \text{if } y = y_{b_i} \end{cases}$$

and

$$\eta'_{\mathbf{B2}}(z) = \begin{cases} f(a) & \text{if } z = x_a \text{ for some } x_a \in X \\ b_i & \text{if } z = z_{b_i}, \end{cases}$$

respectively. As usual, the mappings $\eta'_{\mathbf{B1}}$ and $\eta'_{\mathbf{B2}}$ can be extended to surjective homomorphisms $\eta_{\mathbf{B1}} : \mathbf{Fm}(Y) \to \mathbf{B}'$ and $\eta_{\mathbf{B2}} : \mathbf{Fm}(Z) \to \mathbf{B}'$, respectively, such that they satisfy $\eta_{\mathbf{B1}}(\sigma) = \eta_{\mathbf{B2}}(\sigma) = f(\eta_{\mathbf{A}}(\sigma))$ for every $\sigma \in \mathbf{Fm}(X)$. Define sets of formulas $\Gamma_1(\bar{x}, \bar{y}_n)$ and $\Gamma_2(\bar{x}, \bar{z}_n)$ by

$$\Gamma_1(\bar{x}, \bar{y}_n) = \{\varphi \in \mathbf{Fm}(Y) : \eta_{\mathbf{B1}}(\varphi) \ge 1_{\mathbf{B}'}\},$$
$$\Gamma_2(\bar{x}, \bar{z}_n) = \{\psi \in \mathbf{Fm}(Z) : \eta_{\mathbf{B2}}(\psi) \ge 1_{\mathbf{B}'}\},$$

where $\bar{x}$ is a list of variables in X which are used in constructions of either φ or ψ, and $\bar{y}_n$ and $\bar{z}_n$ are lists such that $\bar{y}_n = y_{b_0}, \ldots, y_{b_{n-1}}$ and $\bar{z}_n = z_{b_0}, \ldots, z_{b_{n-1}}$. Note that it is easily shown that both $\Gamma_1(\bar{x}, \bar{z}_n) = \Gamma_2(\bar{x}, \bar{z}_n)$ and $\Gamma_2(\bar{x}, \bar{y}_n) = \Gamma_1(\bar{x}, \bar{y}_n)$ holds. Thus, we can write $\Gamma_1(\bar{x}, \bar{y}_n)$ and $\Gamma_2(\bar{x}, \bar{z}_n)$ as $\Gamma(\bar{x}, \bar{y}_n)$ and $\Gamma(\bar{x}, \bar{z}_n)$, respectively. We introduce a binary relation $\equiv$ on $\mathbf{Fm}(Y \cup Z)$ by

$$\beta \equiv \gamma \text{ iff } \Gamma(\bar{x}, \bar{y}_n), \Gamma(\bar{x}, \bar{z}_n) \vdash_{\mathbf{L}} (\beta \backslash \gamma) \wedge (\gamma \backslash \beta).$$

Then, it is easily seen that $\equiv$ is a congruence relation on $\mathbf{Fm}(Y \cup Z)$ and that the quotient algebra $\mathbf{Fm}(Y \cup Z)/\equiv$ is a member of $\mathsf{V}(\mathbf{L})$. Let us call this algebra, $\mathbf{C}$. We will show that this $\mathbf{C}$ is the required algebra satisfying the conditions for ES$^*n$.

Define mappings $h : \mathbf{B}' \to \mathbf{C}$ and $k : \mathbf{B}' \to \mathbf{C}$ by

- $h(b) = (\varphi/\equiv)$ when $b = \eta_{\mathbf{B1}}(\varphi)$ for a formula $\varphi \in \mathbf{Fm}(Y)$,
- $k(b') = (\psi/\equiv)$ when $b' = \eta_{\mathbf{B2}}(\psi)$ for a formula $\psi \in \mathbf{Fm}(Z)$.

We prove that both h and k are well-defined embeddings. To show the well-definedness of h, suppose that $\eta_{\mathbf{B1}}(\varphi) = \eta_{\mathbf{B1}}(\varphi')$ for $\varphi, \varphi' \in \mathbf{Fm}(Y)$. Then, $\Gamma(\bar{x}, \bar{y}_n) \vdash_{\mathbf{L}} (\varphi \backslash \varphi') \wedge (\varphi' \backslash \varphi)$, and hence $\varphi \equiv \varphi'$. It is easy to see that h is a homomorphism. To show that h is injective, suppose that $h(b) = h(b')$, where $b = \eta_{\mathbf{B1}}(\varphi)$ and $b' = \eta_{\mathbf{B1}}(\varphi')$ for $\varphi, \varphi' \in \mathbf{Fm}(Y)$. Then, $\varphi \equiv \varphi'$, and thus $\Gamma(\bar{x}, \bar{y}_n), \Gamma(\bar{x}, \bar{z}_n) \vdash_{\mathbf{L}} (\varphi \backslash \varphi') \wedge (\varphi' \backslash \varphi)$ by the definition of $\equiv$. Let s_1 be a substitution such that $s_1(x) = x$ for each $x \in X$, $s_1(y_{b_i}) = y_{b_i}$ and $s_1(z_{b_i}) = y_{b_i}$. Then, we have $s_1(\Gamma(\bar{x}, \bar{y}_n)), s_1(\Gamma(\bar{x}, \bar{z}_n)) \vdash_{\mathbf{L}} s_1((\varphi \backslash \varphi') \wedge (\varphi' \backslash \varphi))$, namely $\Gamma(\bar{x}, \bar{y}_n) \vdash_{\mathbf{L}} (\varphi \backslash \varphi') \wedge (\varphi' \backslash \varphi)$. Since $\eta_{\mathbf{B1}}(\varphi^*) \ge 1_{\mathbf{B}'}$ holds for each $\varphi^* \in \Gamma(\bar{x}, \bar{y}_n)$, we have that $\eta_{\mathbf{B1}}((\varphi \backslash \varphi') \wedge (\varphi' \backslash \varphi)) \ge 1_{\mathbf{B}'}$. Thus $b = \eta_{\mathbf{B1}}(\varphi) = \eta_{\mathbf{B1}}(\varphi') = b'$. Hence h is an embedding. Similarly, k is shown to be a well-defined embedding.

The condition $h \circ f = k \circ f$ is shown in the same way as the proof of Theorem 43. It remains to show that $h(b_i) \neq k(b_i)$ for all $0 \leq i \leq n-1$. Assume that $h(b_m) = k(b_m)$ holds for some $0 \leq m \leq n-1$. Then, we have $(y_{b_m}/\equiv) = h(\eta_{\mathbf{B1}}(y_{b_m})) = h(b_m) = k(b_m) = k(\eta_{\mathbf{B2}}(z_{b_m})) = (z_{b_m}/\equiv)$, and hence $\Gamma(\bar{x}, \bar{y}_n), \Gamma(\bar{x}, \bar{z}_n) \vdash_{\mathbf{L}} (y_m \backslash z_m) \wedge (z_m \backslash y_m)$ holds. By the (general form of) n-BDP, there exists some $\delta_m(\bar{x})$ such that $\Gamma(\bar{x}, \bar{y}_n) \vdash_{\mathbf{L}} (y_m \backslash \delta_m(\bar{x})) \wedge (\delta_m(\bar{x}) \backslash y_m)$. Note that since $\delta_m(\bar{x}) \in \mathbf{Fm}(X)$, we have $\eta_{\mathbf{A}}(\delta_m(\bar{x})) \in A$. Let $a = \eta_{\mathbf{A}}(\delta_m(\bar{x}))$. Since $\eta_{\mathbf{B1}}(\varphi^*) \geq 1_{\mathbf{B}'}$ holds for each $\varphi^* \in \Gamma(\bar{x}, \bar{y}_n)$, we have that $\eta_{\mathbf{B1}}((y_{b_m} \backslash \delta_m(\bar{x})) \wedge (\delta_m(\bar{x}) \backslash y_{b_m})) \geq 1_{\mathbf{B}'}$, which implies that $b_m = \eta_{\mathbf{B1}}(y_{b_m}) = \eta_{\mathbf{B1}}(\delta_m(\bar{x})) = f(\eta_{\mathbf{A}}(\delta_m(\bar{x}))) = f(a)$. Hence $b_m \in f(A)$ holds. But, this contradicts the way of choosing b_m. Therefore, $\mathsf{V}(\mathbf{L})$ has the ES$^*n$.

We show next that the ES$^*n$ implies the n-BDP by taking the contraposition. Assume that $\mathbf{L}$ does not satisfy the n-BDP, namely there exist a formula $\alpha(\bar{x}, \bar{x}'_n)$ and lists $\bar{y}_n$ and $\bar{z}_n$ where for each $1 \leq i, j \leq n$, y_i and z_j are distinct and none of them appear in $\bar{x}$, such that $\alpha(\bar{x}, \bar{y}_n), \alpha(\bar{x}, \bar{z}_n) \vdash_{\mathbf{L}} (y_m \backslash z_m) \wedge (z_m \backslash y_m)$ holds but there is no formulas $\delta_m(\bar{x})$ satisfying $\alpha(\bar{x}, \bar{y}_n) \vdash_{\mathbf{L}} (y_m \backslash \delta_m(\bar{x})) \wedge (\delta_m(\bar{x}) \backslash y_m)$ for some $1 \leq m \leq n$. Let X be a set of all variables occuring in the list $\bar{x}$, namely $X = \{\bar{x}\}$, and denote sets of variables Y and Z by $Y = X \cup \{\bar{y}_n\}$ and $Z = X \cup \{\bar{z}_n\}$, respectively, so that $Y \cap Z = X$. We define quotient algebras $\mathbf{Fm}(X)/\Delta, \mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n)$ and $\mathbf{Fm}(Z)/\alpha(\bar{x}, \bar{z}_n)$, and mappings $f : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n)$ and $g : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Z)/\alpha(\bar{x}, \bar{z}_n)$ in the similar way to the proof of Theorem 43. Then, these algebras are members of $\mathsf{V}(\mathbf{L})$, and both f and g are well-defined embeddings. Let s_2 be a substitution such that $s_2(x) = x$ for all $x \in X$, $s_2(y_i) = y_i$ for $y_i \in Y \setminus X$ and $s_2(z_j) = y_j$ for $z_j \in Z \setminus X$ (note that every member of $Y \setminus X$ and $Z \setminus X$ is already numbered). Define a mapping $w : \mathbf{Fm}(Z)/\alpha(\bar{x}, \bar{z}_n) \to \mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n)$ by $w(\psi/\alpha(\bar{x}, \bar{z}_n)) = s_2(\psi)/\alpha(\bar{x}, \bar{y}_n)$. Then, in the same way as the proof of Theorem 43, we can show that w is an isomorphism and $f = w \circ g$ holds.

Note that $\mathbf{Fm}(X)/\alpha(\bar{x}, \bar{y}_n)$ is generated by $f(\mathbf{Fm}(X)/\Delta) \cup \{y_1/\alpha(\bar{x}, \bar{y}_n), \ldots, y_n/\alpha(\bar{x}, \bar{y}_n)\}$. Now, for each $1 \leq i \leq n$, either of $y_i/\alpha(\bar{x}, \bar{y}_n) \in f(\mathbf{Fm}(X)/\Delta)$ or $y_i/\alpha(\bar{x}, \bar{y}_n) \notin f(\mathbf{Fm}(X)/\Delta)$ holds. Let us pick up all elements in $\{y_1/\alpha(\bar{x}, \bar{y}_n), \ldots, y_n/\alpha(\bar{x}, \bar{y}_n)\}$, each of which does not belong to $f(\mathbf{Fm}(X)/\Delta)$, and enumerate them as $y'_1/\alpha(\bar{x}, \bar{y}_n), \ldots, y'_k/\alpha(\bar{x}, \bar{y}_n)$. Then, obviously, $\mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n)$ is generated by $f(\mathbf{Fm}(X)/\Delta) \cup \{y'_1/\alpha(\bar{x}, \bar{y}_n), \ldots, y'_k/\alpha(\bar{x}, \bar{y}_n)\}$. Note that $y'_s/\alpha(\bar{x}, \bar{y}_n) = y_m/\alpha(\bar{x}, \bar{y}_n)$ holds for some $1 \leq s \leq k$. For, if it is not the case, $y_m/\alpha(\bar{x}, \bar{y}_n) \in f(\mathbf{Fm}(X)/\Delta)$. Then, there is some $\delta(\bar{x})/\Delta \in \mathbf{Fm}(X)/\Delta$ such that $y_m/\alpha(\bar{x}, \bar{y}_n) = f(\delta(\bar{x})/\Delta) = \delta(\bar{x})/\alpha(\bar{x}, \bar{y}_n)$. Hence $\alpha(\bar{x}, \bar{y}_n) \vdash_{\mathbf{L}} (y_m \backslash \delta(\bar{x})) \wedge (\delta(\bar{x}) \backslash y_m)$ holds. But, this contradicts our assumption.

Now, let us assume that $\mathsf{V}(\mathbf{L})$ has the ES$^*n$. Since $k \leq n$, $\mathsf{V}(\mathbf{L})$ has also the ES$^*k$. Then there exist some $\mathbf{C}$ in $\mathsf{V}(\mathbf{L})$ and embeddings $h : \mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n) \to \mathbf{C}$ and $k : \mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n) \to \mathbf{C}$ such that $h \circ f = k \circ f$ and $h(y'_t/\alpha(\bar{x}, \bar{y}_n)) \neq k(y'_t/\alpha(\bar{x}, \bar{y}_n))$ for all $1 \leq t \leq k$. Now, consider a valuation u over $\mathbf{C}$ for formulas in $\mathbf{Fm}(Y \cup Z)$ defined as follows: For every $x \in Y \cup Z$,

$$u(x) = \begin{cases} h(x/\alpha(\bar{x}, \bar{y}_n)) & \text{if } x \in Y \\ k \circ w(x/\alpha(\bar{x}, \bar{z}_n)) & \text{if } x \in Z. \end{cases}$$

The mapping u is well-defined, since if $x \in X$ then $h(x/\alpha(\bar{x}, \bar{y}_n)) = h \circ f(x/\Delta) = k \circ f(x/\Delta) = k \circ w \circ g(x/\Delta) = k \circ w(x/\alpha(\bar{x}, \bar{z}_n))$. As usual, u is extended to a mapping from $\mathbf{Fm}(Y \cup Z)$ to $\mathbf{C}$, which satisfies that

$$u(\theta) = \begin{cases} h(\theta/\alpha(\bar{x}, \bar{y}_n)) & \text{if } \theta \in \mathbf{Fm}(Y) \\ k \circ w(\theta/\alpha(\bar{x}, \bar{z}_n)) & \text{if } \theta \in \mathbf{Fm}(Z). \end{cases}$$

Since $\alpha(\bar{x}, \bar{y}_n) \vdash_{\mathbf{L}} \alpha(\bar{x}, \bar{y}_n)$ holds, we have $\alpha(\bar{x}, \bar{y}_n)/\alpha(\bar{x}, \bar{y}_n) \geq 1_{\mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n)}$. Hence, both

$$u(\alpha(\bar{x}, \bar{y}_n)) = h(\alpha(\bar{x}, \bar{y}_n)/\alpha(\bar{x}, \bar{y}_n)) \geq 1_{\mathbf{C}} \text{ and}$$
$$u(\alpha(\bar{x}, \bar{z}_n)) = k \circ w(\alpha(\bar{x}, \bar{z}_n)/\alpha(\bar{x}, \bar{z}_n)) = k(\alpha(\bar{x}, \bar{y}_n)/\alpha(\bar{x}, \bar{y}_n)) \geq 1_{\mathbf{C}}$$

hold. By our assumption $\alpha(\bar{x}, \bar{y}_n), \alpha(\bar{x}, \bar{z}_n) \vdash_{\mathbf{L}} (y_m \backslash z_m) \wedge (z_m \backslash y_m)$, we have $u((y_m \backslash z_m) \wedge (z_m \backslash y_m)) \geq 1_{\mathbf{C}}$. Hence $u(y_m) = u(z_m)$. This implies that

$$\begin{aligned} h(y'_s/\alpha(\bar{x}, \bar{y}_n)) &= h(y_m/\alpha(\bar{x}, \bar{y}_n)) \\ &= u(y_m) \\ &= u(z_m) \\ &= k \circ w(z_m/\alpha(\bar{x}, \bar{z}_n)) \\ &= k(y_m/\alpha(\bar{x}, \bar{y}_n)) \\ &= k(y'_s/\alpha(\bar{x}, \bar{y}_n)). \end{aligned}$$

But, this contradicts the condition of the ES$^*k$, and thus $\mathsf{V}(\mathbf{L})$ does not satisfy the ES$^*k$, and hence $\mathsf{V}(\mathbf{L})$ does not have the ES$^*n$. Therefore, the ES$^*n$ implies the n-BDP. □

THEOREM 46 *For each $n \in \omega$ and substructural logic $\mathbf{L}$, $\mathbf{L}$ has the n-BDP$^+$ iff $\mathsf{V}(\mathbf{L})$ has the ES$^{**}n$.*

Proof. The proof proceeds in a similar way to the proofs of Theorems 43 and 45. First, we show that $\mathsf{V}(\mathbf{L})$ has the ES$^{**}n$, by assuming the n-BDP$^+$ of $\mathbf{L}$. Let $f : \mathbf{A} \to \mathbf{B}$ be an embedding for $\mathbf{A}, \mathbf{B} \in \mathsf{V}(\mathbf{L})$, $b_0, \ldots, b_{n-1} \in B \setminus f(A)$ and $\mathbf{B}'$ be the subalgebra of $\mathbf{B}$ generated by $f(A) \cup \{b_0, \ldots, b_{n-1}\}$. Define an algebra $\mathbf{C}$ and mappings $h : \mathbf{B}' \to \mathbf{C}$ and $k : \mathbf{B}' \to \mathbf{C}$ in the same way as the proof of Theorem 45. Then, we can show that $\mathbf{C}$ is in $\mathsf{V}(\mathbf{L})$, and both h and k are well-defined embeddings satisfying $h \circ f = k \circ f$ in the same way as before. It remains to show that $h(b_i) \neq k(b_j)$ for all $0 \leq i, j < n$. Assume that $h(b_m) = k(b_l)$ holds for some $0 \leq m, l < n$. Then, we have $(y_{b_m}/\equiv) = h(\eta_{\mathbf{B1}}(y_{b_m})) = h(b_m) = k(b_l) = k(\eta_{\mathbf{B2}}(z_{b_l})) = (z_{b_l}/\equiv)$, and hence $\Gamma(\bar{x}, \bar{y}_n), \Gamma(\bar{x}, \bar{z}_n) \vdash_{\mathbf{L}} (y_m \backslash z_l) \wedge (z_l \backslash y_m)$ holds. By the (general form of) n-BDP$^+$, there exists some $\delta_m(\bar{x})$ such that $\Gamma(\bar{x}, \bar{y}_n) \vdash_{\mathbf{L}} (y_m \backslash \delta_m(\bar{x})) \wedge (\delta_m(\bar{x}) \backslash y_m)$. Note that since $\delta_m(\bar{x}) \in \mathbf{Fm}(X)$, we have $\eta_{\mathbf{A}}(\delta_m(\bar{x})) \in A$. Let $a = \eta_{\mathbf{A}}(\delta_m(\bar{x}))$. Since $\eta_{\mathbf{B1}}(\varphi^*) \geq 1_{\mathbf{B}'}$ holds for each $\varphi^* \in \Gamma(\bar{x}, \bar{y}_n)$, we have that $\eta_{\mathbf{B1}}((y_{b_m} \backslash \delta_m(\bar{x})) \wedge (\delta_m(\bar{x}) \backslash y_{b_m})) \geq 1_{\mathbf{B}'}$, which implies that $b_m = \eta_{\mathbf{B1}}(y_{b_m}) = \eta_{\mathbf{B1}}(\delta_m(\bar{x})) = f(\eta_{\mathbf{A}}(\delta_m(\bar{x}))) = f(a)$. Hence $b_m \in f(A)$ holds. But, this contradicts the way of choosing b_m. Therefore, $\mathsf{V}(\mathbf{L})$ has the ES$^{**}n$.

We show next that the ES$^{**}n$ implies the n-BDP$^+$ by taking the contraposition. Assume that $\mathbf{L}$ does not satisfy the n-BDP$^+$, namely there exist a formula $\alpha(\bar{x}, \bar{x}'_n)$ and lists $\bar{y}_n$ and $\bar{z}_n$, where for each $1 \leq i, j \leq n$, y_i and z_j are distinct and none of them appear in $\bar{x}$, such that $\alpha(\bar{x}, \bar{y}_n), \alpha(\bar{x}, \bar{z}_n) \vdash_{\mathbf{L}} (y_m \backslash z_l) \wedge (z_l \backslash y_m)$ holds but there is no formulas $\delta_m(\bar{x})$ satisfying $\alpha(\bar{x}, \bar{y}_n) \vdash_{\mathbf{L}} (y_m \backslash \delta_m(\bar{x})) \wedge (\delta_m(\bar{x}) \backslash y_m)$ for some $1 \leq m, l \leq n$. In the same way as the proof of Theorem 45, define quotient algebras $\mathbf{Fm}(X)/\Delta, \mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n)$ and $\mathbf{Fm}(Z)/\alpha(\bar{x}, \bar{z}_n)$, and mappings $f : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n)$ and $g : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Z)/\alpha(\bar{x}, \bar{z}_n)$. Then, these algebras are members of $\mathsf{V}(\mathbf{L})$, and both f and g are well-defined embeddings. Also, define a mapping $w : \mathbf{Fm}(Z)/\alpha(\bar{x}, \bar{z}_n) \to \mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n)$ in the same way. Then, w is an isomorphism and $f = w \circ g$ holds.

Let us pick up all elements in $\{y_1/\alpha(\bar{x}, \bar{y}_n), \ldots, y_n/\alpha(\bar{x}, \bar{y}_n)\}$, each of which does not belong to $f(\mathbf{Fm}(X)/\Delta)$, and enumerate them as $y'_1/\alpha(\bar{x}, \bar{y}_n), \ldots, y'_k/\alpha(\bar{x}, \bar{y}_n)$. Then, in the same way as the proof of Theorem 45, we can show that $\mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n)$ is generated by $f(\mathbf{Fm}(X)/\Delta) \cup \{y'_1/\alpha(\bar{x}, \bar{y}_n), \ldots, y'_k/\alpha(\bar{x}, \bar{y}_n)\}$ and $y'_s/\alpha(\bar{x}, \bar{y}_n) = y_m/\alpha(\bar{x}, \bar{y}_n)$ holds for some $1 \leq s \leq k$.

Assume that $\mathsf{V}(\mathbf{L})$ has the ES$^{**}n$. Since $k \leq n$, $\mathsf{V}(\mathbf{L})$ has also the ES$^{**}k$. Then there exist some $\mathbf{C}$ in $\mathsf{V}(\mathbf{L})$ and embeddings $h : \mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n) \to \mathbf{C}$ and $k : \mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n) \to \mathbf{C}$ such that $h \circ f = k \circ f$ and $h(y'_{t_1}/\alpha(\bar{x}, \bar{y}_n)) \neq k(y'_{t_2}/\alpha(\bar{x}, \bar{y}_n))$ for all $1 \leq t_1, t_2 \leq k$. Construct a valuation u over $\mathbf{C}$ for formulas in $\mathbf{Fm}(Y \cup Z)$ in the same way as the proof of Theorem 45. By our assumption

$\alpha(\bar{x}, \bar{y}_n), \alpha(\bar{x}, \bar{z}_n) \vdash_{\mathbf{L}} (y_m \backslash z_l) \wedge (z_l \backslash y_m)$, we have $u((y_m \backslash z_l) \wedge (z_l \backslash y_m)) \geq 1_{\mathbf{C}}$, which implies that

$$\begin{aligned} h(y'_s/\alpha(\bar{x}, \bar{y}_n)) &= h(y_m/\alpha(\bar{x}, \bar{y}_n)) \\ &= u(y_m) \\ &= u(z_l) \\ &= k \circ w(z_l/\alpha(\bar{x}, \bar{z}_n)) \\ &= k(y_l/\alpha(\bar{x}, \bar{y}_n)). \end{aligned}$$

Case 1: $y_l/\alpha(\bar{x}, \bar{y}_n) \in f(\mathbf{Fm}(X)/\Delta)$

In this case, $k(y_l/\alpha(\bar{x}, \bar{y}_n)) = h(y_l/\alpha(\bar{x}, \bar{y}_n))$ holds since $h \circ f = k \circ f$. Hence, we have $h(y'_s/\alpha(\bar{x}, \bar{y}_n)) = h(y_l/\alpha(\bar{x}, \bar{y}_n))$. Since h is injective, we have $y'_s/\alpha(\bar{x}, \bar{y}_n) = y_l/\alpha(\bar{x}, \bar{y}_n) \in f(\mathbf{Fm}(X)/\Delta)$. But, this is a contradiction.

Case 2: $y_l/\alpha(\bar{x}, \bar{y}_n) \notin f(\mathbf{Fm}(X)/\Delta)$

In this case, there is some $1 \leq t \leq k$ such that $y_l/\alpha(\bar{x}, \bar{y}_n) = y'_t/\alpha(\bar{x}, \bar{y}_n)$. Then, we have $h(y'_s/\alpha(\bar{x}, \bar{y}_n)) = k(y_l/\alpha(\bar{x}, \bar{y}_n)) = k(y'_t/\alpha(\bar{x}, \bar{y}_n))$. But, this contradicts to the condition for the ES$^{**}k$.

Thus, in both cases, $\mathsf{V}(\mathbf{L})$ does not satisfy the ES$^{**}k$, and hence $\mathsf{V}(\mathbf{L})$ does not have the ES$^{**}n$. Therefore, the ES$^{**}n$ implies the n-BDP$^+$. □

THEOREM 47 *For each $n \in \omega$ and substructural logic $\mathbf{L}$, $\mathbf{L}$ has the n-BDPF iff $\mathsf{V}(\mathbf{L})$ has the ES$^{***}n$.*

Proof. Once again, the proof proceeds in a similar way to the proofs of Theorems 43 and 45. First, we show that $\mathsf{V}(\mathbf{L})$ has the ES$^{***}n$, by assuming the n-BDPF of $\mathbf{L}$. Let $f : \mathbf{A} \to \mathbf{B}$ be an embedding for $\mathbf{A}, \mathbf{B} \in \mathsf{V}(\mathbf{L})$, $b_0, \ldots, b_{n-1} \in B \setminus f(A)$ and $\mathbf{B}'$ be the subalgebra of $\mathbf{B}$ generated by $f(A) \cup \{b_0, \ldots, b_{n-1}\}$. Define an algebra $\mathbf{C}$ and mappings $h : \mathbf{B}' \to \mathbf{C}$ and $k : \mathbf{B}' \to \mathbf{C}$ in the same way as the proof of Theorem 45. Then, we can show that $\mathbf{C}$ is in $\mathsf{V}(\mathbf{L})$, and both h and k are well-defined embeddings satisfying $h \circ f = k \circ f$ in the same way as before. It remains to show that $h(B') \cap k(B') = h \circ f(A)$ holds. Note that $h \circ f(A) \subseteq h(B') \cap k(B')$ holds since $f(A) \subseteq B'$ and $h \circ f = k \circ f$. To show the converse inclusion, let $c \in h(B') \cap k(B')$, namely $c = h(b) = k(b')$ for some $b, b' \in B'$. Then, there exist some formulas $\varphi(\bar{x}, \bar{y}_n)$ and $\psi(\bar{x}, \bar{z}_n)$ such that $b = \eta_{\mathbf{B1}}(\varphi(\bar{x}, \bar{y}_n))$ and $b' = \eta_{\mathbf{B2}}(\psi(\bar{x}, \bar{z}_n))$. By the definitions of h, k and $\equiv$, we can see that the condition $h(b) = k(b')$ is equivalent to $\Gamma(\bar{x}, \bar{y}_n), \Gamma(\bar{x}, \bar{z}_n) \vdash_{\mathbf{L}} (\varphi(\bar{x}, \bar{y}_n) \backslash \psi(\bar{x}, \bar{z}_n)) \wedge (\psi(\bar{x}, \bar{z}_n) \backslash \varphi(\bar{x}, \bar{y}_n))$. By the (general form of) n-BDPF, there exists some $\delta(\bar{x})$ such that $\Gamma(\bar{x}, \bar{y}_n) \vdash_{\mathbf{L}} (\varphi(\bar{x}, \bar{y}_n) \backslash \delta(\bar{x})) \wedge (\delta(\bar{x}) \backslash \varphi(\bar{x}, \bar{y}_n))$. Let $a = \eta_{\mathbf{A}}(\delta(\bar{x})) \in A$. Then, we have $c = h(b) = h(\eta_{\mathbf{B1}}(\varphi(\bar{x}, \bar{y}_n))) = (\varphi(\bar{x}, \bar{y}_n)/\equiv) = (\delta(\bar{x})/\equiv) = h(\eta_{\mathbf{B1}}(\delta(\bar{x}))) = (h \circ f)(\eta_{\mathbf{A}}(\delta(\bar{x}))) = h \circ f(a) \in h \circ f(A)$. Therefore, $\mathsf{V}(\mathbf{L})$ has the ES$^{***}n$.

Conversely, suppose that $\mathsf{V}(\mathbf{L})$ has the ES$^{***}n$, and $\alpha(\bar{x}, \bar{y}_n), \alpha(\bar{x}, \bar{z}_n) \vdash_{\mathbf{L}} (\varphi(\bar{x}, \bar{y}_n) \backslash \psi(\bar{x}, \bar{z}_n)) \wedge (\psi(\bar{x}, \bar{z}_n) \backslash \varphi(\bar{x}, \bar{y}_n))$ holds for formulas α, φ and ψ, where for each $1 \leq i, j \leq n$, y_i and z_j are distinct and none of them appear in $\bar{x}$. In the same way as the proof of Theorem 45, define quotient algebras $\mathbf{Fm}(X)/\Delta, \mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n)$ and $\mathbf{Fm}(Z)/\alpha(\bar{x}, \bar{z}_n)$, and mappings $f : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n)$ and $g : \mathbf{Fm}(X)/\Delta \to \mathbf{Fm}(Z)/\alpha(\bar{x}, \bar{z}_n)$. Then, these algebras are members of $\mathsf{V}(\mathbf{L})$, and both f and g are well-defined embeddings. Also, define a mapping $w : \mathbf{Fm}(Z)/\alpha(\bar{x}, \bar{z}_n) \to \mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n)$ in the same way. Then, w is an isomorphism and $f = w \circ g$ holds. Note that $\mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n)$ is generated by $f(\mathbf{Fm}(X)/\Delta) \cup \{y_1/\alpha(\bar{x}, \bar{y}_n), \ldots, y_n/\alpha(\bar{x}, \bar{y}_n)\}$.

Case 1: $y_i/\alpha(\bar{x}, \bar{y}_n) \in f(\mathbf{Fm}(X)/\Delta)$ for all $1 \leq i \leq n$

In this case, there exist some $\sigma_i/\Delta \in \mathbf{Fm}(X)/\Delta$ such that $y_i/\alpha(\bar{x}, \bar{y}_n) = f(\sigma_i/\Delta) = \sigma_i/\alpha(\bar{x}, \bar{y}_n)$ holds for all $1 \leq i \leq n$. Thus, $\alpha(\bar{x}, \bar{y}_n) \vdash_{\mathbf{L}} (y_i \backslash \sigma_i) \wedge (\sigma_i \backslash y_i)$ holds for all $1 \leq i \leq n$. By the replacement theorem, we have $\alpha(\bar{x}, \bar{y}_n) \vdash_{\mathbf{L}} (\varphi(\bar{x}, \bar{y}_n) \backslash \varphi(\bar{x}, \sigma_1, \ldots, \sigma_n)) \wedge (\varphi(\bar{x}, \sigma_1, \ldots, \sigma_n) \backslash \varphi(\bar{x}, \bar{y}_n))$.

Note that all variables occurring in $\varphi(\bar{x}, \sigma_1, \ldots, \sigma_n)$ are in $\bar{x}$ since $\sigma_i \in \mathbf{Fm}(X)$ for all $1 \leq i \leq n$. Let us denote $\varphi(\bar{x}, \sigma_1, \ldots, \sigma_n)$ by $\delta(\bar{x})$. Then, $\alpha(\bar{x}, \bar{y}_n) \vdash_{\mathbf{L}} (\varphi(\bar{x}, \bar{y}_n)\backslash\delta(\bar{x})) \wedge (\delta(\bar{x})\backslash\varphi(\bar{x}, \bar{y}_n))$ holds, and hence $\mathbf{L}$ has the n-BDPF.

Case 2: $y_i/\alpha(\bar{x}, \bar{y}_n) \notin f(\mathbf{Fm}(X)/\Delta)$ for some $1 \leq i \leq n$

Let us pick up all elements in $\{y_1/\alpha(\bar{x}, \bar{y}_n), \ldots, y_n/\alpha(\bar{x}, \bar{y}_n)\}$, each of which does not belong to $f(\mathbf{Fm}(X)/\Delta)$, and enumerate them as $y'_1/\alpha(\bar{x}, \bar{y}_n), \ldots, y'_k/\alpha(\bar{x}, \bar{y}_n)$. Then, clearly, $\mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n)$ is generated by $f(\mathbf{Fm}(X)/\Delta) \cup \{y'_1/\alpha(\bar{x}, \bar{y}_n), \ldots, y'_k/\alpha(\bar{x}, \bar{y}_n)\}$. By the ES$^{***}n$, $\mathsf{V}(\mathbf{L})$ has also the ES$^{***}k$ since $k \leq n$. Thus, there exist some $\mathbf{C}$ in $\mathsf{V}(\mathbf{L})$ and embeddings $h : \mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n) \to \mathbf{C}$ and $k : \mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n) \to \mathbf{C}$ such that both $h \circ f = k \circ f$ and $h(\mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n)) \cap k(\mathbf{Fm}(Y)/\alpha(\bar{x}, \bar{y}_n)) = h \circ f(\mathbf{Fm}(X)/\Delta)$ hold. Construct a valuation u over $\mathbf{C}$ for formulas in $\mathbf{Fm}(Y \cup Z)$ in the same way as the proof of Theorem 45. By our assumption $\alpha(\bar{x}, \bar{y}_n), \alpha(\bar{x}, \bar{z}_n) \vdash_{\mathbf{L}} (\varphi(\bar{x}, \bar{y}_n)\backslash\psi(\bar{x}, \bar{z}_n)) \wedge (\psi(\bar{x}, \bar{y}_n)\backslash\varphi(\bar{x}, \bar{y}_n))$, we have that $u((\varphi(\bar{x}, \bar{y}_n)\backslash\psi(\bar{x}, \bar{z}_n)) \wedge (\psi(\bar{x}, \bar{z}_n)\backslash\varphi(\bar{x}, \bar{y}_n))) \geq 1_{\mathbf{C}}$ holds. Thus,

$$h(\varphi(\bar{x}, \bar{y}_n)/\alpha(\bar{x}, \bar{y}_n)) = k \circ w(\psi(\bar{x}, \bar{z}_n)/\alpha(\bar{x}, \bar{z}_n)) = k(\psi(\bar{x}, \bar{y}_n)/\alpha(\bar{x}, \bar{y}_n))$$

holds. By the condition of the ES$^{***}k$, there exists some $\delta(\bar{x})/\Delta \in \mathbf{Fm}(X)/\Delta$ such that

$$h(\varphi(\bar{x}, \bar{y}_n)/\alpha(\bar{x}, \bar{y}_n)) = (h \circ f)(\delta(\bar{x})/\Delta) = h(\delta(\bar{x})/\alpha(\bar{x}, \bar{y}_n)).$$

Since h is injective, we have that $\varphi(\bar{x}, \bar{y}_n)/\alpha(\bar{x}, \bar{y}_n) = \delta(\bar{x})/\alpha(\bar{x}, \bar{y}_n)$, which implies $\alpha(\bar{x}, \bar{y}_n) \vdash_{\mathbf{L}} (\varphi(\bar{x}, \bar{y}_n)\backslash\delta(\bar{x})) \wedge (\delta(\bar{x})\backslash\varphi(\bar{x}, \bar{y}_n))$. Therefore, $\mathbf{L}$ has the n-BDPF. □

Next, we give an algebraic characterization of the PBDP. The following theorem is an extension of the result due to Maksimova [26]. In the setting of algebraic logic, Hoogland proved this for algebraizable logics in [12].

THEOREM 48 *For each substructural logic* $\mathbf{L}$*,* $\mathbf{L}$ *has the PBDP iff* $\mathsf{V}(\mathbf{L})$ *has the SES.*

Proof. We show our theorem using the proofs of Theorems 43 and 45. First, we show that $\mathsf{V}(\mathbf{L})$ has the SES, by assuming the PBDP of $\mathbf{L}$. Let $f : \mathbf{A} \to \mathbf{B}$ be an embedding for $\mathbf{A}, \mathbf{B}$ in $\mathsf{V}(\mathbf{L})$. Define a set of variables X and a surjective homomorphism $\eta_{\mathbf{A}} : \mathbf{Fm}(X) \to \mathbf{A}$ in the same way as before. Differently to the proof of Theorem 45, we need to define sets of variables Y and Z by $Y = X \cup \{y_b : b \in B \setminus f(A)\}$ and $Z = X \cup \{z_b : b \in B \setminus f(A)\}$, respectively, so that $Y \cap Z = X$, and mappings $\eta'_{\mathbf{B1}} : Y \to \mathbf{B}$ and $\eta'_{\mathbf{B2}} : Z \to \mathbf{B}$ by

$$\eta'_{\mathbf{B1}}(y) = \begin{cases} f(a) & \text{if } y = x_a \text{ for some } x_a \in X \\ b' & \text{if } y = y_{b'} \text{ for some } b' \in B \setminus f(A) \end{cases}$$

and

$$\eta'_{\mathbf{B2}}(z) = \begin{cases} f(a) & \text{if } z = x_a \text{ for some } x_a \in X \\ b' & \text{if } z = z_{b'} \text{ for some } b' \in B \setminus f(A), \end{cases}$$

respectively. As usual, the mappings $\eta'_{\mathbf{B1}}$ and $\eta'_{\mathbf{B2}}$ can be extended to surjective homomorphisms $\eta_{\mathbf{B1}} : \mathbf{Fm}(Y) \to \mathbf{B}$ and $\eta_{\mathbf{B2}} : \mathbf{Fm}(Z) \to \mathbf{B}$, respectively, such that they satisfy $\eta_{\mathbf{B1}}(\sigma) = \eta_{\mathbf{B2}}(\sigma) = f(\eta_{\mathbf{A}}(\sigma))$ for every $\sigma \in \mathbf{Fm}(X)$. Define sets of formulas Γ and Σ by

$$\Gamma = \{\varphi \in \mathbf{Fm}(Y) : \eta_{\mathbf{B1}}(\varphi) \geq 1_{\mathbf{B}}\},$$
$$\Sigma = \{\psi \in \mathbf{Fm}(Z) : \eta_{\mathbf{B2}}(\psi) \geq 1_{\mathbf{B}}\}.$$

We introduce a binary relation $\equiv$ on $\mathbf{Fm}(Y \cup Z)$ by

$$\beta \equiv \gamma \text{ iff } \Gamma, \Sigma \vdash_{\mathbf{L}} (\beta\backslash\gamma) \wedge (\gamma\backslash\beta).$$

Then, it is easily seen that $\equiv$ is a congruence relation on $\mathbf{Fm}(Y \cup Z)$ and that the quotient algebra $\mathbf{Fm}(Y \cup Z)/\equiv$ is a member of $\mathsf{V}(\mathbf{L})$. Let us call this algebra, $\mathbf{C}$. We will show that this $\mathbf{C}$ is a required algebra satisfying the conditions for the SES.

Define mappings $h : \mathbf{B} \to \mathbf{C}$ and $k : \mathbf{B} \to \mathbf{C}$ by

- $h(b) = (\varphi/\equiv)$ when $b = \eta_{\mathbf{B}1}(\varphi)$ for a formula $\varphi \in \mathbf{Fm}(Y)$,
- $k(b') = (\psi/\equiv)$ when $b' = \eta_{\mathbf{B}2}(\psi)$ for a formula $\psi \in \mathbf{Fm}(Z)$.

We prove that both h and k are well-defined embeddings. To show the well-definedness of h, suppose that $\eta_{\mathbf{B}1}(\varphi) = \eta_{\mathbf{B}1}(\varphi')$ for $\varphi, \varphi' \in \mathbf{Fm}(Y)$. Then, $\Gamma \vdash_{\mathbf{L}} (\varphi\backslash\varphi') \wedge (\varphi'\backslash\varphi)$, and hence $\varphi \equiv \varphi'$. It is easy to see that h is a homomorphism. To show that h is injective, suppose that $h(b) = h(b')$, where $b = \eta_{\mathbf{B}1}(\varphi)$ and $b' = \eta_{\mathbf{B}1}(\varphi')$ for $\varphi, \varphi' \in \mathbf{Fm}(Y)$. Then, $\varphi \equiv \varphi'$, and thus $\Gamma, \Sigma \vdash_{\mathbf{L}} (\varphi\backslash\varphi') \wedge (\varphi'\backslash\varphi)$ by the definition of $\equiv$. Let s_1 be a substitution such that $s_1(x) = x$ for $x \in X$, $s_1(y_b) = y_b$ for $y_b \in Y \setminus X$ and $s_1(z_{b'}) = y_{b'}$ for $z_{b'} \in Z \setminus X$. Note that for any $\psi \in \Sigma$, $s_1(\psi) \in \Gamma$ holds. Then, $\Gamma, \Sigma \vdash_{\mathbf{L}} (\varphi\backslash\varphi') \wedge (\varphi'\backslash\varphi)$ implies $s_1(\Gamma), s_1(\Sigma) \vdash_{\mathbf{L}} s_1((\varphi\backslash\varphi') \wedge (\varphi'\backslash\varphi))$, and hence $\Gamma \vdash_{\mathbf{L}} (\varphi\backslash\varphi') \wedge (\varphi'\backslash\varphi)$ holds. Thus, we have $b = \eta_{\mathbf{B}1}(\varphi) = \eta_{\mathbf{B}1}(\varphi') = b'$, and hence h is an embedding. Similarly, k is shown to be a well-defined embedding.

The condition $h \circ f = k \circ f$ is shown in the same way as the proof of Theorem 45. It remains to show that $h(b) \neq k(b)$ for every $b \in B \backslash f(A)$. Assume that there exists some $b_0 \in B \backslash f(A)$ such that $h(b_0) = k(b_0)$. Then, we have $(y_{b_0}/\equiv) = h(\eta_{\mathbf{B}1}(y_{b_0})) = h(b_0) = k(b_0) = k(\eta_{\mathbf{B}2}(z_{b_0})) = (z_{b_0}/\equiv)$, and hence $\Gamma, \Sigma \vdash_{\mathbf{L}} (y_{b_0}\backslash z_{b_0}) \wedge (z_{b_0}\backslash y_{b_0})$. Since $\vdash_{\mathbf{L}}$ is finitary and conjunctive, there exist some $\varphi_i \in \Gamma$ with $1 \leq i \leq n$ and $\psi_j \in \Sigma$ with $1 \leq j \leq m$ such that $\bigwedge_{i=1}^n \varphi_i, \bigwedge_{j=1}^m \psi_j \vdash_{\mathbf{L}} (y_{b_0}\backslash z_{b_0}) \wedge (z_{b_0}\backslash y_{b_0})$ holds. Let $\bar{x}$ be a list of variables from X, each of which appears in either of $\bigwedge_{i=1}^n \varphi_i$ or $\bigwedge_{j=1}^m \psi_j$. Also, let $\bar{y}_l$ be a list of variables from $Y \setminus X$, each of which either is equal to y_{b_0} or appears in $\bigwedge_{i=1}^n \varphi_i$. Similarly, let $\bar{z}_g$ be a list of variables from $Z \setminus X$, each of which is equal to z_{b_0} or appears to $\bigwedge_{j=1}^m \psi_j$. Then $\bigwedge_{i=1}^n \varphi_i, \bigwedge_{j=1}^m \psi_j \vdash_{\mathbf{L}} (y_{b_0}\backslash z_{b_0}) \wedge (z_{b_0}\backslash y_{b_0})$ is expressed also as

(*) $\bigwedge_{i=1}^n \varphi_i(\bar{x}, \bar{y}_l), \bigwedge_{j=1}^m \psi_j(\bar{x}, \bar{z}_g) \vdash_{\mathbf{L}} (y_{b_0}\backslash z_{b_0}) \wedge (z_{b_0}\backslash y_{b_0})$.

Define a formula $\alpha(\bar{x}, \bar{y}_t)$ by $\alpha(\bar{x}, \bar{y}_t) = \bigwedge_{i=1}^n \varphi_i(\bar{x}, \bar{y}_l) \wedge \bigwedge_{j=1}^m \psi_j(\bar{x}, s_1(\bar{z}_g))$, where $\bar{y}_t$ is a list of variables in either of $\bar{y}_l$ or $s_1(\bar{z}_g)$. Let s_2 be a substitution such that $s_2(x) = x$ for $x \in X$, $s_2(y_b) = z_b$ for $y_b \in Y \setminus X$ and $s_2(z_{b'}) = z_{b'}$ for $z_{b'} \in Z \setminus X$. We define $\alpha(\bar{x}, \bar{z}_t)$ by $\alpha(\bar{x}, s_2(\bar{y}_t))$, i.e. $\bigwedge_{i=1}^n \varphi_i(\bar{x}, s_2(\bar{y}_l)) \wedge \bigwedge_{j=1}^m \psi_j(\bar{x}, s_2 \circ s_1(\bar{z}_g))$, which is equal to $\bigwedge_{i=1}^n \varphi_i(\bar{x}, s_2(\bar{y}_l)) \wedge \bigwedge_{j=1}^m \psi_j(\bar{x}, \bar{z}_g)$. Then, the condition (*) implies $\alpha(\bar{x}, \bar{y}_t), \alpha(\bar{x}, \bar{z}_t) \vdash_{\mathbf{L}} (y_{b_0}\backslash z_{b_0}) \wedge (z_{b_0}\backslash y_{b_0})$.

By the PBDP, there exists a formula $\delta(\bar{x})$ such that $\alpha(\bar{x}, \bar{y}_t) \vdash_{\mathbf{L}} (y_{b_0}\backslash \delta(\bar{x})) \wedge (\delta(\bar{x})\backslash y_{b_0})$ holds. Note that $\Gamma \vdash_{\mathbf{L}} \alpha(\bar{x}, \bar{y}_t)$ holds. For, $\varphi_i(\bar{x}, \bar{y}_l) \in \Gamma$ for each $1 \leq i \leq n$, and also $\psi_j(\bar{x}, s_1(\bar{z}_g)) \in \Gamma$ for each $1 \leq j \leq m$ as $\eta_{\mathbf{B}1}(\psi_j(\bar{x}, s_1(\bar{z}_g))) = \eta_{\mathbf{B}2}(\psi_j(\bar{x}, \bar{z}_g)) \geq 1_{\mathbf{B}}$. Thus, $\Gamma \vdash_{\mathbf{L}} (y_{b_0}\backslash \delta(\bar{x})) \wedge (\delta(\bar{x})\backslash y_{b_0})$, which implies that $b_0 = \eta_{\mathbf{B}1}(y_{b_0}) = \eta_{\mathbf{B}1}(\delta(\bar{x})) = f(\eta_{\mathbf{A}}(\delta(\bar{x})))$. Since $\eta_{\mathbf{A}}(\delta(\bar{x})) \in A$, we have $b_0 \in f(A)$. But, this contradicts to the choice of b_0. Therefore, $\mathsf{V}(\mathbf{L})$ has the SES.

Conversely, suppose that $\mathsf{V}(\mathbf{L})$ has the SES. It is easily shown that $\mathsf{V}(\mathbf{L})$ has also the ES$^*n$ for all $n \in \omega$. By Theorem 45, $\mathbf{L}$ has the n-BDP for all $n \in \omega$. Therefore, $\mathbf{L}$ has the PBDP. □

The following result gives an algebraic characterization of the PBDPF. Moreover, it is shown that the PBDPF is equivalent to the PBDP$^+$.

THEOREM 49 *For each substructural logic* $\mathbf{L}$*, the following are equivalent.*

(1) $\mathbf{L}$ *has the PBDPF.*

(2) $\mathbf{L}$ *has the PBDP*$^+$*.*

(3) $\mathsf{V}(\mathbf{L})$ *has the ES*†*.*

Proof. (3) ⇒ (1). Suppose that $\mathsf{V}(\mathbf{L})$ has the ES†. Then, it is easily seen that $\mathsf{V}(\mathbf{L})$ has also the ES$^{***}n$ for all $n \in \omega$. By Theorem 47, $\mathbf{L}$ has the n-BDPF for all $n \in \omega$. Therefore, $\mathbf{L}$ has the PBDPF.

(1) ⇒ (2). Obvious.

(2) ⇒ (3). We show this direction using the proof of Theorem 48. Let $f : \mathbf{A} \to \mathbf{B}$ be an embedding for $\mathbf{A}, \mathbf{B}$ in $\mathsf{V}(\mathbf{L})$. Define an algebra $\mathbf{C}$ and mapping $h : \mathbf{B} \to \mathbf{C}$ and $k : \mathbf{B} \to \mathbf{C}$ in the same way as the proof of Theorem 48. Then, we can show that $\mathbf{C}$ is in $\mathsf{V}(\mathbf{L})$, and both h and k are well-defined embeddings satifying $h \circ f = k \circ f$ in the same way as before. It remains to show that $h(B) \cap k(B) = h \circ f(A)$ holds. Note that $h \circ f(A) \subseteq h(B) \cap k(B)$ holds since $f(A) \subseteq B$ and $h \circ f = k \circ f$. To show the converse inclusion, let $c \in h(B) \cap k(B)$, namely $c = h(b_1) = k(b_2)$ for some $b_1, b_2 \in B$. Note that both $b_1 = \eta_{\mathbf{B}1}(y_{b_1})$ and $b_2 = \eta_{\mathbf{B}2}(z_{b_2})$ hold. From the definitions of h, k and $\equiv$, we can see that the condition $h(b_1) = k(b_2)$ is equivalent to $\Gamma, \Sigma \vdash_{\mathbf{L}} (y_{b_1} \backslash z_{b_2}) \wedge (z_{b_2} \backslash y_{b_1})$. Since $\vdash_{\mathbf{L}}$ is finitary and conjunctive, there exist some $\varphi_i \in \Gamma$ with $1 \leq i \leq n$ and $\psi_j \in \Sigma$ with $1 \leq j \leq m$ such that $\bigwedge_{i=1}^n \varphi_i, \bigwedge_{j=1}^m \psi_j \vdash_{\mathbf{L}} (y_{b_1} \backslash z_{b_2}) \wedge (z_{b_2} \backslash y_{b_1})$ holds. Let $\bar{x}$ be a list of variables from X, each of which appears in either of $\bigwedge_{i=1}^n \varphi_i$ or $\bigwedge_{j=1}^m \psi_j$. Also, let $\bar{y}_l$ be a list of variables from $Y \setminus X$, each of which is equal to y_{b_1} or appears in $\bigwedge_{i=1}^n \varphi_i$. Similarly, let $\bar{z}_g$ be a list of variables from $Z \setminus X$, each of which is equal to z_{b_2} or appears to $\bigwedge_{j=1}^m \psi_j$. Then $\bigwedge_{i=1}^n \varphi_i, \bigwedge_{j=1}^m \psi_j \vdash_{\mathbf{L}} (y_{b_1} \backslash z_{b_2}) \wedge (z_{b_2} \backslash y_{b_1})$ is expressed also as

$$(^{**})\ \bigwedge_{i=1}^n \varphi_i(\bar{x}, \bar{y}_l), \bigwedge_{j=1}^m \psi_j(\bar{x}, \bar{z}_g) \vdash_{\mathbf{L}} (y_{b_1} \backslash z_{b_2}) \wedge (z_{b_2} \backslash y_{b_1}).$$

Define formulas $\alpha(\bar{x}, \bar{y}_t)$ and $\alpha(\bar{x}, \bar{z}_t)$ in the same way as before. Then, the condition (**) implies $\alpha(\bar{x}, \bar{y}_t), \alpha(\bar{x}, \bar{z}_t) \vdash_{\mathbf{L}} (y_{b_1} \backslash z_{b_2}) \wedge (z_{b_2} \backslash y_{b_1})$.

By the PBDP$^+$, there exists some formula $\delta(\bar{x})$ such that $\alpha(\bar{x}, \bar{y}_t) \vdash_{\mathbf{L}} (y_{b_1} \backslash \delta(\bar{x})) \wedge (\delta(\bar{x}) \backslash y_{b_1})$. Then, in the same way as before, we can show that $c \in h \circ f(A)$. Therefore, $\mathsf{V}(\mathbf{L})$ has the ES†. □

We summarize the algebraic characterizations of BDPs and their relationships in Figure 12. It is shown in [28] that only sixteen superintuitionistic logics have the PBDP. Since every superintuitionistic logic has the BDP (in [17]), the BDP does not imply the PBDPF, in general.

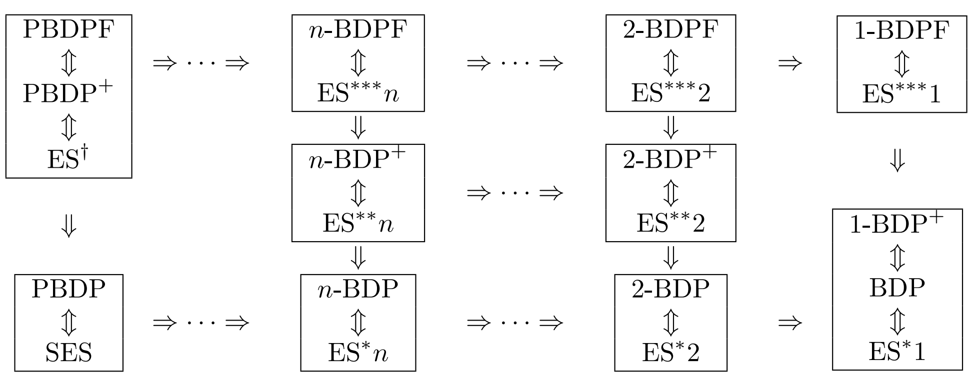

Figure 12: Algebraic characterizations of BDPs

In the following, we show a relationship between the strongRP, the RP and the PBDPF. We employ here the method due to Maksimova. [11]

[11] Theorem 4 in [24] shows the corresponding algebraic result, i.e, the strongAP is equivalent to AP with ES* for varieties of modal logics. But the proof of the implication of the strongAP from AP with ES* given there contains a gap. But the theorem stands as it is, as a corrected proof is given recently by Maksimova (by a private communication). We owe the proof of Theorem 50 to her revised proof. Our study of various extensions of the BDP and ES*1 in the present Section 6.2 was started initially in order to fill a gap in Maksimova's proof.

THEOREM 50 *For each substructural logic* $\mathbf{L}$*,* $\mathbf{L}$ *has the strongRP iff it has both the RP and the BDP.*

Proof. It is easily seen that the strongRP implies both the RP and the BDP. Thus, we show only the converse direction. By Theorems 16, 20, and 43, it is enough to show that if $\mathsf{V}(\mathbf{L})$ has both the AP and the ES*1 then it has also the the strongAP.

First, we show that if $\mathsf{V}(\mathbf{L})$ has both the AP and the ES*1 then it has also the following;

(*) for any $\mathbf{A}, \mathbf{B} \in \mathsf{V}(\mathbf{L})$, each embedding $f : \mathbf{A} \to \mathbf{B}$ and $b \in B \setminus f(A)$, there exist an $\mathbf{E} \in \mathsf{V}(\mathbf{L})$ and embeddings $i : \mathbf{B} \to \mathbf{E}$ and $j : \mathbf{B} \to \mathbf{E}$ such that $i \circ f = j \circ f$ and $i(b) \neq j(b)$.

Let $f : \mathbf{A} \to \mathbf{B}$ be an embedding for $\mathbf{A}, \mathbf{B} \in \mathsf{V}(\mathbf{L})$ and $b \in B \setminus f(A)$. Denote by $\mathbf{B}'$ a subalgebra of $\mathbf{B}$ generated by $f(A) \cup \{b\}$. Then, by ES*1, there exist some algebra $\mathbf{B}_1$ in $\mathsf{V}(\mathbf{L})$ and embeddings $\alpha_1 : \mathbf{B}' \to \mathbf{B}_1$ and $\alpha_2 : \mathbf{B}' \to \mathbf{B}_1$ such that $\alpha_1 \circ f = \alpha_2 \circ f$ and $\alpha_1(b) \neq \alpha_2(b)$. Let $e : \mathbf{B}' \to \mathbf{B}$ be the identity embedding. Since $\alpha_1 : \mathbf{B}' \to \mathbf{B}_1$ and $e : \mathbf{B}' \to \mathbf{B}$ are embeddings, by the AP, there exist some $\mathbf{C}_1$ in $\mathsf{V}(\mathbf{L})$ and embeddings $\beta_1 : \mathbf{B}_1 \to \mathbf{C}_1$ and $\gamma_1 : \mathbf{B} \to \mathbf{C}_1$ such that $\beta_1 \circ \alpha_1 = \gamma_1 \circ e$. Similarly, from $\alpha_2 : \mathbf{B}' \to \mathbf{B}_1, e : \mathbf{B}' \to \mathbf{B}$ and the AP, there exist some $\mathbf{C}_2$ in $\mathsf{V}(\mathbf{L})$ and embeddings $\beta_2 : \mathbf{B}_1 \to \mathbf{C}_2$ and $\gamma_2 : \mathbf{B} \to \mathbf{C}_2$ such that $\beta_2 \circ \alpha_2 = \gamma_2 \circ e$. Now $\beta_1 : \mathbf{B}_1 \to \mathbf{C}_1$ and $\beta_2 : \mathbf{B}_1 \to \mathbf{C}_2$ are embeddings, by the AP again, there exist some $\mathbf{E}$ in $\mathsf{V}(\mathbf{L})$ and embeddings $\delta_1 : \mathbf{C}_1 \to \mathbf{E}$ and $\delta_2 : \mathbf{C}_2 \to \mathbf{E}$ such that $\delta_1 \circ \beta_1 = \delta_2 \circ \beta_2$. Then, $\delta_1 \circ \gamma_1$ and $\delta_2 \circ \gamma_2$ are embeddings from $\mathbf{B}$ into $\mathbf{E}$. Let $i = \delta_1 \circ \gamma_1$ and $j = \delta_2 \circ \gamma_2$. We show that $i \circ f = j \circ f$ but $i(b) \neq j(b)$. For any $a \in A$,

$$\begin{aligned} i \circ f(a) &= \delta_1 \circ \gamma_1 \circ f(a) \\ &= \delta_1 \circ \gamma_1 \circ e \circ f(a) \\ &= \delta_1 \circ \beta_1 \circ \alpha_1 \circ f(a) \\ &= \delta_1 \circ \beta_1 \circ \alpha_2 \circ f(a) \\ &= \delta_2 \circ \beta_2 \circ \alpha_2 \circ f(a) \\ &= \delta_2 \circ \gamma_2 \circ e \circ f(a) \\ &= \delta_2 \circ \gamma_2 \circ f(a) \\ &= j \circ f(a). \end{aligned}$$

Thus, $i \circ f = j \circ f$ holds. On the other hand,

$$\begin{aligned} &i(b) = \delta_1 \circ \gamma_1(b) = \delta_1 \circ \gamma_1 \circ e(b) = \delta_1 \circ \beta_1 \circ \alpha_1(b) = \delta_2 \circ \beta_2 \circ \alpha_1(b) \quad \text{and} \\ &j(b) = \delta_2 \circ \gamma_2(b) = \delta_2 \circ \gamma_2 \circ e(b) = \delta_2 \circ \beta_2 \circ \alpha_2(b) \end{aligned}$$

hold. Since $\delta_2 \circ \beta_2$ is an embedding and $\alpha_1(b) \neq \alpha_2(b)$, we have $i(b) \neq j(b)$. Hence, $\mathsf{V}(\mathbf{L})$ satisfies the condition (*).

Now, we show that $\mathsf{V}(\mathbf{L})$ has the strongAP. Let $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$ be embeddings, $b \in B \setminus f(A)$ and $c \in C \setminus g(A)$ for $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ in $\mathsf{V}(\mathbf{L})$. Then, by the above argument, there exist some $\mathbf{E}$ in $\mathsf{V}(\mathbf{L})$ and embeddings $i : \mathbf{B} \to \mathbf{E}$ and $j : \mathbf{B} \to \mathbf{E}$ such that $i \circ f = j \circ f$ and $i(b) \neq j(b)$. Since $i \circ f$ is an embedding from $\mathbf{A}$ to $\mathbf{E}$, by the AP, there exist some $\mathbf{D}_{(b,c)}$ in $\mathsf{V}(\mathbf{L})$ and embeddings $l_{(b,c)} : \mathbf{E} \to \mathbf{D}_{(b,c)}$ and $k_{(b,c)} : \mathbf{C} \to \mathbf{D}_{(b,c)}$ such that $l_{(b,c)} \circ i \circ f = k_{(b,c)} \circ g$. Since $i(b) \neq j(b)$, we have $l_{(b,c)} \circ i(b) \neq l_{(b,c)} \circ j(b)$ by the injectivity of l. Therefore, either $l_{(b,c)} \circ i(b) \neq k_{(b,c)}(c)$ or $l_{(b,c)} \circ j(b) \neq k_{(b,c)}(c)$ holds. If the former holds then define an embedding $h_{(b,c)} : \mathbf{B} \to \mathbf{D}_{(b,c)}$ by $h_{(b,c)} = l_{(b,c)} \circ i$. Otherwise, let $h_{(b,c)}$ by $h_{(b,c)} = l_{(b,c)} \circ j$. Then, it is easily seen that $h_{(b,c)} \circ f = k_{(b,c)} \circ g$ holds, but $h_{(b,c)}(b) \neq k_{(b,c)}(c)$. Thus, in general for every $b' \in B \setminus f(A)$ and $c' \in C \setminus g(A)$, we can get some $\mathbf{D}_{(b',c')}$ in $\mathsf{V}(\mathbf{L})$ and embeddings $h_{(b',c')} : \mathbf{B} \to \mathbf{D}_{(b',c')}$ and $k_{(b',c')} : \mathbf{C} \to \mathbf{D}_{(b',c')}$ such that $h_{(b',c')} \circ f = k_{(b',c')} \circ g$ and $h_{(b',c')}(b') \neq k_{(b',c')}(c')$. Let $\mathbf{D}$ be the direct product $\Pi_{(b,c) \in B \setminus f(A) \times C \setminus f(A)} \mathbf{D}_{(b,c)}$ for all possible pairs $(b, c) \in B \setminus f(A) \times C \setminus f(A)$. Clearly, $\mathbf{D}$ is in $\mathsf{V}(\mathbf{L})$ since $\mathsf{V}(\mathbf{L})$ is a variety. We define mappings $h : \mathbf{B} \to \mathbf{D}$ by $h(x)\langle(b, c)\rangle = h_{(b,c)}(x)$, namely the (b, c)-th coordinate of $h(x)$ is $h_{(b,c)}(x)$ which is in $\mathbf{D}_{(b,c)}$. Similarly, define $k : \mathbf{C} \to \mathbf{D}$ by $k(y)\langle(b, c)\rangle = k_{(b,c)}(y)$. Then, both h and k are embeddings. For any $a \in A$, $h(f(a))\langle(b, c)\rangle = h_{(b,c)} \circ f(a)$, which is equal either to $l_{(b,c)} \circ i \circ f(a)$ or to $l_{(b,c)} \circ j \circ f(a)$. Since $i \circ f = j \circ f$ and

$l_{(b,c)} \circ i \circ f = k_{(b,c)} \circ g$, we have that $h_{(b,c)} \circ f(a) = k_{(b,c)} \circ g(a) = k(g(a))\langle(b,c)\rangle$. Hence, $h \circ f = k \circ g$ holds. It remains to show that $h(B) \cap k(C) = h \circ f(A)$. Note that $h \circ f(A) \subseteq h(B) \cap k(C)$ holds since $f(A) \subseteq B$, $g(A) \subseteq C$ and $h \circ f = k \circ g$. To show converse inclusion, let $b^* \in B \setminus f(A)$ and $c^* \in C \setminus g(A)$. Then, we have $h(b^*)\langle(b^*,c^*)\rangle = h_{(b^*,c^*)}(b^*) \neq k_{(b^*,c^*)}(c^*) = k(c^*)\langle(b^*,c^*)\rangle$, and hence $h(b^*) \neq k(c^*)$. Thus, $h(B) \cap k(C) = h \circ f(A)$. Therefore, $\mathsf{V}(\mathbf{L})$ has the strongAP. □

Intuitively, Theorem 50 means that the condition **(strong)** of the definition of the strongAP is equivalent to the ES^*1. Note that since the strongAP implies the $\mathrm{ES}^\dagger$, all ESs (therefore, all BDPs) in Figure 12 are mutually equivalent whenever a variety has the AP. Note also that the strongAP is equivalent to the AP for every variety of Heyting algebras. Hence, in this case, the AP implies the $\mathrm{ES}^\dagger$. In other words, the RP implies the PBDPF for every superintuitionistic logics.

strongRP		RP + BDP
$\Updownarrow$	$\Leftrightarrow$	$\Updownarrow$
strongAP		AP + ES^*1

Figure 13: Relation among the strongRP, the RP and the BDP

7 Future work

A lot of work must be done in future to solve many of the important problems put forward in this paper. The most important is to show whether implications among these algebraic/logical properties introduced in this paper are proper or not. Of course, their answers depend on what kind of logics we are concerned with. For example, some non-implicational and non-equivalence results are obtained for modal and superintuitionistic logics in e.g. [24], [28], and for the abstract model-theoretic setting in [4] and [31]. Even if we restrict our attention only to substructural logics, the following would be challenging questions.

- Does the CIP imply the DIP? If not, find a substructural logic which has the CIP but not the DIP. Note that the CIP always implies the DIP for commutative cases, and also that the CIP does not always imply the SDIP (see Proposition 10).
- Does the CIP imply the strongRP or the superRP? See Figure 10. (The same question is put for multi-modal logics in [19].)
- Is SDIP (ISDIP, EqSDIP) + BDP equivalent to SDIP (ISDIP, EqSDIP) + nonsepEqSCIP? (cf. Theorem 50) More strongly, is the BDP equivalent to the nonsepEqSCIP?

Another direction for future work is to show how far our results can be generalized. As a matter of fact, as only a few properties specific to **FL**-algebras or residuated lattices are used in our discussion, it is quite likely that results in the present paper can be extended to those in a more general setting of many other varieties and also of abstract algebraic logic.

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