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Petri Net Models for Intuitionistic Linear Logic

by

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Abstract

The connection between linear logic and Petri nets has recently been a subject of great interest. In these researches, the propositional fragment of intuitionistic linear logic with exponential ! was considered, and Petri nets were related to linear logic as follows: each place of a Petri net is regarded as an atomic proposition of linear logic, and transitions as provability relation.

Soundness and completeness of linear logic are proved for algebraic structures, called quantales. Engberg and Winskel showed soundness of linear logic for quantales induced from Petri nets, but for these quantales completeness was not valid. Soundness states that all provable properties (formulas) in linear logic hold in Petri nets, while completeness states that properties which hold in any Petri net can be proved in linear logic. When both of soundness and completeness are valid in some quantales induced from Petri nets, we can say that a property is provable in linear logic if and only if it is a common property of Petri nets, i.e., it holds in any Petri net.

Recently Engberg and Winskel have also proved completeness of a $\sqcup$-free fragment of linear logic and linear logic with distributivity. One of difficulties in proving completeness for full linear logic lies in distributivity of $\sqcap$ over $\sqcup$, i.e., $A \sqcap (B \sqcup C) \Rightarrow (A \sqcap B) \sqcup (A \sqcap C)$, which does not hold in linear logic. The quantales constructed by Engberg and Winskel are distributive lattices, i.e., distributivity is always valid. Therefore, to prove completeness using their quantales, we have to deal with the $\sqcup$-free fragment or to add the distributivity to linear logic as an axiom. However these are not what we intend to do. Although there should be argument about which of full linear logic or a logic with distributivity is appropriate for representing properties of Petri nets, we here concentrate on proving completeness for full linear logic. To find adequate logics for which the models of Engberg and Winskel are complete is another interesting problem.

In this thesis, we first construct non-distributive quantales, i.e., quantales in which distributivity is not always valid, from Petri nets, and prove completeness of linear logic without exponential for the quantales. In linear logic, exponential ! is added to compensate the absence of the rules of weakening and contraction. For example, $!A$ indicates that we may extract as many data of type A as we like, i.e., a datum of type $!A$ is a finite collection of data of type A. For Petri nets, we can regard a place with exponential ! as a place which can supply arbitrary many but finite resources (tokens, in petri net terminology) by firing transitions. We extend the construction of the quantales to those with exponential, and prove completeness of linear logic for the quantales. It means that properties which hold in any Petri nets with such exponential places can be proved in linear logic.

We also give an impression on the meaning of the logic on the proposed Petri net model, comparing with that by Engberg and Winskel.

Contents

Chapter 1

Introduction

Linear Logic, introduced by J. Y. Girard in 1987 [19], is interesting from a purely logical point of view, and potentially of considerable interest for computer science [20, 24, 26, 27, 30, 45]. Linear logic (intuitionistic, classical and predicate) are obtained by deleting the contraction and the weakening rules from standard sequent calculus formulations of corresponding logics. Linear logic may be viewed as an example of a *resource conscious* logic, where the formulas represent types of resource, and resources cannot be used ad libitum. That is to say, asserting a sequent $A, A \Rightarrow B$ means something like: we use two data (resources) of type A to obtain one datum of type B. In Gentzen-style sequent calculus for intuitionistic logic, a sequent $A_1, \ldots, A_n \Rightarrow A$ is written to mean that the formula A is deducible from the assumption formulas $A_1, \ldots, A_n$ (we shall use capital Greek letters as an abbreviation for a sequence of formulas). The calculus has the two structural rules for adding a vacant assumption and removing of a duplicate of assumption,

$$\frac{\Gamma \Rightarrow B}{\Gamma, A \Rightarrow B} \text{ (weakening)},$$

$$\frac{\Gamma, A, A \Rightarrow B}{\Gamma, A \Rightarrow B} \text{ (contraction)}.$$

In the presence of these rules the following two rules for conjunction

$$\frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A \sqcap B} \text{ (1)},$$

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \sqcap B} \text{ (2)}$$

become interderivable in the sense that the first rule can be derived from the second by weakening, and the second from the first by contraction. In intuitionistic linear logic these rules (weakening and contraction) are deleted and the rule of (1) and (2) are no longer interderivable. Without them, propositions cannot be introduced arbitrarily into a list of assumption and a duplication in the list cannot be removed. It is in this sense that linear logic is a resource conscious logic.

In the years 1960-1962, Carl Adam Petri defined Petri nets which is a general purpose mathematical model for describing relations existing between conditions and events [40]. Petri nets consist of two types of elements, places and transitions. Each place models a process in terms of types of resources, and can hold arbitrary nonnegative multiplicity.

Each transition represents a state transition rule, i.e., how those resources are consumed or produced by actions. They are described using the notion of multisets. A multiset over a set P is a function, $m : P \longrightarrow \mathcal{N}$ [9, 18, 32, 41].

The connection between linear logic and Petri nets has recently been a subject of great interest [7, 8, 15, 16, 17, 29, 34, 35]. Girard's linear logic has a great deal of interest in how might be useful in the theory of parallelism. In these researches, the propositional fragment of intuitionistic linear logic with exponential ! [47] was considered, and Petri nets were related to linear logic as follows: each place of a Petri net is regarded as an atomic propositions of linear logic, and transitions as provability relation. In the sequel, we shall simply use the word "linear logic" to denote the propositional fragment of intuitionistic linear logic.

Soundness and completeness of linear logic are proved for algebraic structures, called quantale [1, 5, 19, 21, 22, 39, 47]. Engberg and Winskel [15] showed soundness of linear logic for quantales induced from Petri nets, but for these quantales completeness was not valid. Soundness theorem states that all provable properties (formulas) in linear logic hold in Petri nets, while completeness theorem states that properties which hold in any Petri net can be proved in linear logic. When both of soundness and completeness are valid in some quantales induced from Petri nets, we can say that a property is provable in linear logic if and only if it is a common property of Petri nets, i.e., it holds in any Petri net.

Recently Engberg and Winskel have also proved completeness of a $\sqcup$-free fragment of linear logic and linear logic with distributivity [16, 17]. One of difficulties in proving completeness for full linear logic lies in distributivity of $\sqcap$ over $\sqcup$. Although the following proof shows that

$$(A \sqcap B) \sqcup (A \sqcap C) \Rightarrow A \sqcap (B \sqcup C)$$

is derivable in linear logic,

$$\dfrac{\dfrac{A \Rightarrow A}{A \sqcap B \Rightarrow A}(\sqcap 1 \Rightarrow) \quad \dfrac{\dfrac{B \Rightarrow B}{B \Rightarrow B \sqcup C}(\Rightarrow \sqcup 1)}{A \sqcap B \Rightarrow B \sqcup C}(\sqcap 2 \Rightarrow)}{A \sqcap B \Rightarrow A \sqcap (B \sqcup C)}(\Rightarrow \sqcap) \qquad (a),$$

$$\dfrac{\dfrac{A \Rightarrow A}{A \sqcap C \Rightarrow A}(\sqcap 1 \Rightarrow) \quad \dfrac{\dfrac{C \Rightarrow C}{C \Rightarrow B \sqcup C}(\Rightarrow \sqcup 2)}{A \sqcap C \Rightarrow B \sqcup C}(\sqcap 2 \Rightarrow)}{A \sqcap C \Rightarrow A \sqcap (B \sqcup C)}(\Rightarrow \sqcap) \qquad (b),$$

and from (a) and (b), we can get

$$\dfrac{A \sqcap B \Rightarrow A \sqcap (B \sqcup C) \quad A \sqcap C \Rightarrow A \sqcap (B \sqcup C)}{(A \sqcap B) \sqcup (A \sqcap C) \Rightarrow A \sqcap (B \sqcup C)}(\sqcup \Rightarrow),$$

we cannot prove the sequent

$$A \sqcap (B \sqcup C) \Rightarrow (A \sqcap B) \sqcup (A \sqcap C).$$

That is, the distributivity of $\sqcap$ over $\sqcup$ does not hold in linear logic. With the contraction and weakening rules, we can prove this as follows:

$$
\cfrac{\cfrac{\cfrac{A \Rightarrow A}{A, B \Rightarrow A}\ (\text{weakening}) \quad \cfrac{B \Rightarrow B}{A, B \Rightarrow B}\ (\text{weakening})}{A, B \Rightarrow A \sqcap B}\ (\Rightarrow \sqcap)}{A, B \Rightarrow (A \sqcap B) \sqcup (A \sqcap C)}\ (\Rightarrow \sqcup 1) \qquad (a),
$$

$$
\cfrac{\cfrac{\cfrac{A \Rightarrow A}{A, C \Rightarrow A}\ (\text{weakening}) \quad \cfrac{C \Rightarrow C}{A, C \Rightarrow C}\ (\text{weakening})}{A, C \Rightarrow A \sqcap C}\ (\Rightarrow \sqcap)}{A, C \Rightarrow (A \sqcap B) \sqcup (A \sqcap C)}\ (\Rightarrow \sqcup 2) \qquad (b),
$$

and from (a) and (b), we can get

$$
\cfrac{\cfrac{\cfrac{A, B \Rightarrow (A \sqcap B) \sqcup (A \sqcap C) \quad A, C \Rightarrow (A \sqcap B) \sqcup (A \sqcap C)}{A, B \sqcup C \Rightarrow (A \sqcap B) \sqcup (A \sqcap C)}\ (\sqcup \Rightarrow)}{A \sqcap (B \sqcup C), A \sqcap (B \sqcup C) \Rightarrow (A \sqcap B) \sqcup (A \sqcap C)}\ (\sqcap 1 \text{ and } \sqcap 2 \Rightarrow)}{A \sqcap (B \sqcup C) \Rightarrow (A \sqcap B) \sqcup (A \sqcap C)}\ (\text{contraction}) .
$$

The quantales constructed in [15, 16, 17] are distributive lattices. Therefore, to prove completeness using their quantales, we have to deal with the $\sqcup$-free fragment or to add the distributivity to linear logic as an axiom. However these are not what we intend to do.

There is a family of substructural logics, such as relevant logics, in which distributivity is valid. It has been pointed out that semantic considerations of substructural logics tend toward validating distribution, and proof theoretic considerations tend toward invalidating it (see [13]). Although there should be argument about which of full linear logic or a logic with distributivity is appropriate for representing properties of Petri nets, we here concentrate on proving completeness for full linear logic. To find adequate logics for which the models of [15, 16, 17] are complete is another interesting problem.

The key element is the way of the construction of quantales. When we construct the quantales, we use a closure operation. We introduce two closure operations C_1 and C_2. Let $\mathbf{X} = \langle M, \leq, \cdot, e \rangle$ be a preordered commutative monoid. We define two closure operations C_1 and C_2 on $\mathbf{P}(\mathbf{X})$. C_1 is an operation on $\mathbf{P}(\mathbf{X})$ such that

$$C_1 X := \{y \in M \mid \exists x \in X\ (y \leq x)\},$$

and C_2 is an operation on $\mathbf{P}(\mathbf{X})$ such that

$$C_2 X := (X^{\rightarrow})^{\leftarrow},$$

where

$$X^{\rightarrow} := \{y \in M \mid \forall x \in X (x \leq y)\}$$

and

$$X^{\leftarrow} := \{y \in M \mid \forall x \in X (y \leq x)\}.$$

C_1 is used in[15, 16, 17], and C_2, which is called the MacNeille completion of X [23, 31], is used in this thesis. In the quantales constructed from Petri nets using C_1, distributivity is always valid. But in the quantales constructed from Petri nets using C_2, distributivity is not always valid.

In this thesis, we first construct non-distributive quantales, i.e., quantales in which distributivity is not always valid, from Petri nets, and prove completeness of linear logic without exponential for the quantales.

Moreover, we extend the quantales to the quantales with exponential. In linear logic, exponential ! is added to compensate the absence of the rules of weakening and contraction. For example, $!A$ indicates that we may extract as many data of type A as we like, i.e., a datum of type $!A$ is a finite collection of data of type A. For Petri nets, we can regard a place with exponential ! as a place which can supply arbitrary many but finite resources (tokens, in Petri net terminology) by firing transitions. We extend the construction of the quantales to those with exponential, and prove completeness of linear logic for the quantales. It means that properties which hold in any Petri nets with such exponential places can be proved in linear logic.

There are two approaches to construct a Petri net model in which completeness of linear logic holds; one is introducing distributivity in the logic, and the other is the approach taken in this thesis, i.e., making a non-distributive model by using appropriate closure operation.

From the practical point of view, the former approach may be useful in dealing with properties of Petri net. The quantale constructed in this thesis seems strange from a Petri net point of view, because the closure operation takes both of forwards and backwards reachability into account. However, the motivation of this thesis is to find an answer to the following problem: Is there a Petri net model which is sound and complete for full linear logic? And the problem has been solved affirmatively.

Of course, the significance of the result depends on the meaning of the logic in Petri nets. For the proposed Petri net model, we give an interpretation of the logic which may be acceptable from a Petri net point of view, comparing with that by Engberg and Winskel [15].

The organization of this thesis is as follows.

In Chapter 2, we review basic algebraic structures and fixed point theorem. In the discussion of this thesis, we shall often need the concepts of basic algebraic structures (for example a multiset and ordered structures) and fixed point theorem. In these notes we do not intend to go very deeply in to these; we limit ourselves to a brief description of basic algebraic structures and fixed point theorem.

In Chapter 3, we review Petri nets. First we discuss Petri net simply. Then we introduce the relation between Petri net and multiset, and reachability relation.

In Chapter 4, we discuss IL-algebras and quantales. Then we introduce two closure operations C_1 and C_2 on the algebras, which play a crucial role in the proof of completeness. Moreover, we discuss exponential and quantales with exponential.

In Chapter 5, we discuss linear logic without exponential (its syntax and semantics) and then prove soundness theorem for quantales generated by Petri nets. Next we show why we cannot prove completeness for the quantales used in [15, 16, 17]. Finally we show how to construct quantales in which the distributivity is not always valid from Petri nets, and then prove completeness of linear logic without exponential for the quantales.

In Chapter 6, we discuss linear logic with exponential (its syntax and semantics) and then prove soundness theorem for the quantales generated by Petri nets. And then we show how to construct quantales with exponential in which the distributivity is not always valid from Petri nets, and then prove completeness of linear logic with exponential for the quantales.

In Chapter 7, we give an impression on the meaning of the logic on the proposed Petri net model, comparing with that by [15]. We consider a difference of interpretations between closure operations C_1 and C_2, and we show a different interpretation of formulas under the closure operation C_2.

In Chapter 8, we consider classical quantales for classical linear logic generated by Petri nets.

Chapter 2

Preliminaries

In this chapter, we review basic algebraic structures and fixed point theorem. For background material on basic algebraic structures, see [3, 6, 14, 28, 46, 48, 49] and on fixed point theorem, see [37].

2.1 Basic Algebraic Structures

In the discussion of this thesis, we shall often need the concepts of a multiset and ordered structures. In these notes we do not intend to go very deeply into these; we limit ourselves to a brief description of the theory of multiset and the pure theory of ordered structures.

2.1.1 Multisets

Intuitively, a multiset is a set with (finite) multiplicities; there may be finitely many copies of a single element. As a formal definition we use

Definition 2.1.1 (multiset) A *multiset* over a set S is a mapping $m : S \longrightarrow \mathcal{N}$, where $m(a) = n$ means that a occurs with multiplicity n. If $m(a) = 0$, a is not an element of m (that is to say, a occurs with multiplicity 0).

The operation $+$ on multisets is defined by $(m + m')(a) = m(a) + m'(a)$ for all $a \in S$, and $[\,]$ denotes the empty multiset.

We shall denote the set of all multisets over a set S by $\mathcal{M}_{\mathrm{S}}$, and use $\{\cdots\}$ for a set and $[\cdots]$ for a multiset.

Example 2.1.2 Let a and b be elements of S. Then

- $\{a\}, \{b\}, \{a, b\}, \ldots$ are sets and

$$\{a\} = \{a, a\} \text{ and } \{a\} \cup \{a, b\} = \{a, b\},$$

- $[a], [b], [a, b], \ldots$ are multisets and

$$[a] \neq [a, a] \text{ and } [a] + [a, b] = [a, a, b].$$

2.1.2 Monoids and Ordered Structures

Definition 2.1.3 (monoid) A structure $\mathbf{M} = \langle X, \cdot, e \rangle$ is a *monoid* with the identity e if $\cdot$ is a binary operation on X and e is an element of X such that for every $a, b, c \in X$,

1. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$,
2. $a \cdot e = e \cdot a = a$.

Remark 2.1.4 When the structure satisfies only 1 of Definition 2.1.3, it is called a *semigroup*.

Definition 2.1.5 (commutative monoid) A structure $\mathbf{M} = \langle X, \cdot, e \rangle$ is a *commutative monoid* with the identity e if

1. $\langle X, \cdot, e \rangle$ is a monoid,
2. $a \cdot b = b \cdot a$ for every $a, b \in X$.

Definition 2.1.6 (partially ordered set) A structure $\mathbf{X} = \langle X, \leq \rangle$ is a *partially ordered set* if $\leq$ is a binary relation on X such that for every $a, b, c \in X$,

1. $a \leq a$ (reflexive),
2. if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitive),
3. if $a \leq b$ and $b \leq a$, then $a = b$ (antisymmetric).

Let P be a partially ordered set and $X \subseteq P$. $y \in X$ is the greatest element of X if and only if

$$\text{if } x \in X, \text{ then } x \leq y,$$

and $y' \in X$ is the least element of X if and only if

$$\text{if } x \in X, \text{ then } y' \leq x.$$

Let P be a partially ordered set and $X \subseteq P$. $y \in X$ is the maximal element of X if and only if

$$\text{if } y \leq x \text{ and } x \in X, \text{ then } x = y,$$

and $y' \in X$ is the minimal element of X if and only if

$$\text{if } x \leq y' \text{ and } x \in X, \text{ then } x = y'.$$

Let P and Q be partially ordered sets. Then a function $f : P \longrightarrow Q$ is *monotone* if and only if for all $a, b \in P$,

$$\text{if } a \leq b, \text{ then } f(a) \leq f(b).$$

So f preserves order.

We shall think of the elements of a partially ordered set as being propositions, and of $\leq$ as meaning "$\Rightarrow$", or "entails", or "is logically stronger than". Then it is precisely antisymmetry that says that if two propositions are logically equivalent (each entails the other) then they are equal: we identify them.

Example 2.1.7 Little partially ordered sets can be drawn using diagrams as follows:

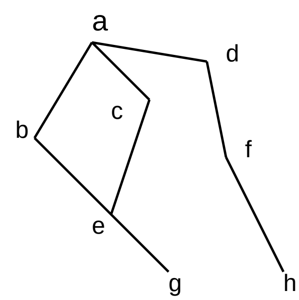

Figure 2.1: Partially ordered set.

Here each line represents an inequality. For instance, $b \leq a$, because b is at the bottom end of the line, $e \leq b$. We can deduce other inequalities, such as $f \leq f$, $e \leq a$, from the partially ordered set axioms, reflexive and transitive, i.e., $e \leq b$ and $b \leq a$, or $e \leq c$ and $c \leq a$.

Example 2.1.8 Let X be a set and $\mathcal{P}(X)$ its power set, i.e., the set of all subsets of X. Taking $\leq$ to mean $\subseteq$, $\mathcal{P}(X)$ is a partially ordered set. Antisymmetry corresponds to the extensional definition of set equality, which says that equality between sets is to be determined entirely by what elements they have.

Definition 2.1.9 (preordered set) A structure $\mathbf{X} = \langle X, \leq \rangle$ is a *preordered set* if $\leq$ is a binary relation on X such that for every $a, b, c \in X$,

1. $a \leq a$ (reflexive),

2. if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitive).

Proposition 2.1.10 *Let P be a preordered set. We define a binary relation $\equiv$ on P by*

$$a \equiv b \text{ if and only if } a \leq b \text{ and } b \leq a.$$

Then $\equiv$ is an equivalence relation, and the equivalence classes $[a]$ form a partially ordered set $P/\equiv$, with

$$[a] \leq [b] \text{ if and only if } a \leq b.$$

Definition 2.1.11 (preordered commutative monoid) A structure $\mathbf{X} = \langle X, \leq, \cdot, e \rangle$ is a *preordered commutative monoid* if,

1. $\langle X, \cdot, e \rangle$ is a commutative monoid,

2. $\langle X, \leq \rangle$ is a preordered set,

3. if $x \leq x'$ and $y \leq y'$, then $x \cdot y \leq x' \cdot y'$ for all $x, x', y, y' \in X$.

2.1.3 Meets and Joins

Thinking of $\leq$ as meaning "$\Rightarrow$", we next wish to describe what corresponds to "and" and "or".

First, we define meets, which correspond to "and".

Definition 2.1.12 (meet) Let P be a partially ordered set, $X \subseteq P$ and $y \in P$. Then y is a *meet* (or *greatest lower bound* or *infimum*) for X if and only if

1. y is a *lower bound* for X, i.e., if $x \in X$ then $y \leq x$, and
2. if z is any other lower bound for X then $z \leq y$.

In symbols, we write $y = \bigwedge X$.

Example 2.1.13 Let P be a partially ordered set, $X \subseteq P$ and $y \in P$. A meet, or a greatest lower bound y and lower bounds of X can be drawn as follows:

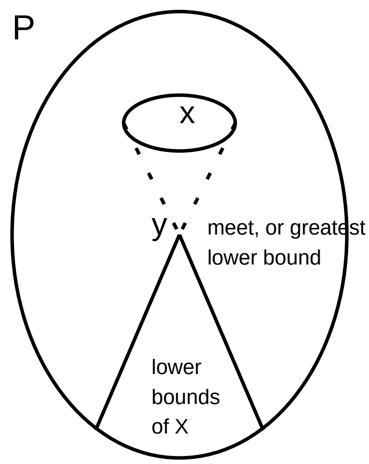

Figure 2.2: Meet and lower bounds.

- If $x \in X$, then $y \leq x$, and then y is a lower bound for X,
- if z is any other lower bound for X, then $z \leq y$.

Proposition 2.1.14 *Let P be a partially ordered set and X a subset. Then X can have at most one meet.*

Proof. Let y and y' be two meets of X. Since y is a meet and y' is a lower bound, $y' \leq y$. Similarly, $y \leq y'$. By antisymmetry, $y = y'$. ■

Next, we define joins, which correspond to "or".

Definition 2.1.15 (join) Let P be a partially ordered set, $X \subseteq P$ and $y \in P$. Then y is a *join* (or *least upper bound* or *supremum*) for X if and only if

1. y is an upper bound for X, i.e., if $x \in X$ then $y \geq x$ and
2. if z is any other upper bound for X then $z \geq y$.

In symbols, we write $y = \bigvee X$.

Remark 2.1.16 $\top$ and $\perp$ are defined as follows:

- $\top$ is the greatest element,
- $\perp$ is the least element.

$a \vee b$ and $a \wedge b$ denote $\bigvee\{a, b\}$ and $\bigwedge\{a, b\}$, respectively.

Proposition 2.1.17 *Let P be a partially ordered set. Then for all $y \in P$,*

1. *y is the empty meet if and only if it is a top (greatest) element and*
2. *y is the empty join if and only if it is a bottom (least) element.*

Proof.

1. Suppose $y = \bigwedge \emptyset$. Every $z \in P$ is a lower bound of $\emptyset$: for the condition

$$\text{if } x \in \emptyset \text{ then } z \leq x$$

is satisfied vacuously. Therefore $z \leq y$, so y is greater than every other element of P. Conversely, if y is top then it is a lower bound for $\emptyset$ (because everything is) and it is greater than all the other lower bounds (because it's greater than everything), so

$$y = \bigwedge \emptyset.$$

2. Suppose $y = \bigvee \emptyset$. Every $z \in P$ is a upper bound of $\emptyset$: for the condition

$$\text{if } x \in \emptyset \text{ then } x \leq z$$

is satisfied vacuously. Therefore $y \leq z$, so y is less than every other element of P. Conversely, if y is bottom then it is an upper bound for $\emptyset$ (because everything is) and it is less than all the other upper bounds (because it's less than everything), so

$$y = \bigvee \emptyset.$$

■

Empty meets and joins need not exist. We have already seen an example (see Figure 2.1) that had two minimal elements g and h, but no least element. A least element would have to be less than everything else. Thus this example does not have an empty join, although it does have an empty meet, namely a.

An empty meet (top) is often written as $\top$, and an empty join (bottom) as $\perp$.

Example 2.1.18 In logic, meets are conjunctions and joins are disjunctions. We are thinking of $\leq$ as meaning $\Rightarrow$.

- meets.
  - First, $P \sqcap Q = \bigwedge\{P, Q\}$, a meet for the set $\{P, Q\}$. We check
  $$P \sqcap Q \Rightarrow P \text{ and } P \sqcap Q \Rightarrow Q$$
  so that $P \sqcap Q$ is a lower bound for $\{P, Q\}$.
  - Next, if R is another lower bound for $\{P, Q\}$, in other words
  $$R \Rightarrow P \text{ and } R \Rightarrow Q,$$
  then $R \Rightarrow P \sqcap Q$.

  True is easily seen to be a top element: it holds unconditionally, so anything implies true: $P \Rightarrow true$.

- joins.
  - First, $P \sqcup Q = \bigvee\{P, Q\}$, a join for the set $\{P, Q\}$. We check
  $$P \Rightarrow P \sqcup Q \text{ and } Q \Rightarrow P \sqcup Q$$
  so that $P \sqcup Q$ is an upper bound for $\{P, Q\}$.
  - Next, if R is another upper bound for $\{P, Q\}$, in other words
  $$P \Rightarrow R \text{ and } Q \Rightarrow R,$$
  then $P \sqcup Q \Rightarrow R$.

  False is bottom because of the standard logical idea that if we assume a contradiction then we can prove anything: $false \Rightarrow P$.

Example 2.1.19 In set theory, meets are intersections and joins are unions. We work with subsets of a "universe" U, and $\leq$ is set inclusion, $\subseteq$.

- First, meets. Clearly if each X_i is a subset of U, then
  - $\bigcap_i X_i \subseteq X_i$ for all i.
  - If $Y \subseteq X_i$ for all i and $y \in Y$, then $y \in X_i$ for all i and so $y \in \bigcap_i X_i$. Therefore $Y \subseteq \bigcap_i X_i$.
  - The empty meet is the top subset, U itself.
- Next, joins. Clearly if each X_i is a subset of U, then
  - $X_i \subseteq \bigcup_i X_i$ for all i.
  - If $X_i \subseteq Y$ for all i and $x \in \bigcup_i X_i$, then $x \in X_i$ for some i and so $x \in Y$. Therefore $\bigcup_i X_i \subseteq Y$.
  - The empty join is the bottom subset, $\emptyset$.

To summarize, the relation of orders, logic and sets is the following Figure 2.3:

Orders	Logic	Sets
$\leq$	$\Rightarrow$	$\subseteq$
$=$	$\Leftrightarrow$	$=$
top, $\top$, *empty meet*, $\bigwedge \emptyset$	true	universe
bottom, $\perp$, *empty join*, $\bigvee \emptyset$	false	$\emptyset$
meet, $\bigwedge$, *greatest lower bound, glb*, *infimum, inf*	conjunction, and, $\wedge$	intersection, $\cap$
join, $\bigvee$, *least upper bound, lub*, *supremum, sup*	disjunction, or, $\vee$	union, $\cup$

Figure 2.3: The relation of orders, logic and sets.

2.1.4 Lattices

Definition 2.1.20 (lattice) A partially ordered set P is a *lattice* if and only if all two-element subsets have meets and joins.

Let $\sqsubseteq$ be a binary relation on a lattice defined by

$$x \sqsubseteq y \text{ if and only if } x \vee y = y.$$

Remark 2.1.21 Let a structure $\mathbf{L} = \langle L, \sqsubseteq \rangle$ be a *lattice*. Then for $a, b \in L$, the following are equivalent.

1. $a \sqsubseteq b$,
2. $a \vee b = b$ and
3. $a \wedge b = a$.

Proof.

- $a \sqsubseteq b$ if and only if $a \vee b = b$.

  Suppose $a \sqsubseteq b$. Then b is an upper bound of $\{a, b\}$. If c is an upper bound of $\{a, b\}$, $b \sqsubseteq c$ from definition. It means that b is the least upper bound of $\{a, b\}$, i.e., b is the supremum. Therefore $a \vee b = b$. Conversely suppose $a \vee b = b$. Then

  $$a \sqsubseteq a \vee b = b,$$

  and hence $a \sqsubseteq b$.

- $a \sqsubseteq b$ if and only if $a \wedge b = a$.

  Suppose $a \sqsubseteq b$. Then a is a lower bound of $\{a, b\}$. If c is a lower bound of $\{a, b\}$, $c \sqsubseteq a$ from definition. It means that a is the greatest lower bound of $\{a, b\}$, i.e., a is the infimum. Therefore $a \wedge b = a$. Conversely suppose $a \wedge b = a$. Then

  $$a = a \wedge b \sqsubseteq b,$$

  and hence $a \sqsubseteq b$.

■

Proposition 2.1.22 *Let a structure* $\mathbf{L} = \langle L, \vee, \wedge \rangle$ *be a lattice. Then for every* $a, b, c \in L$, *it satisfies the following conditions.*

1. $a \vee a = a$, $a \wedge a = a$ *(idempotence),*
2. $a \vee (b \vee c) = (a \vee b) \vee c$, $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ *(associativity),*
3. $a \vee b = b \vee a$, $a \wedge b = b \wedge a$ *(commutativity),*
4. $a \vee (a \wedge b) = a$, $a \wedge (a \vee b) = a$ *(absorption).*

Proof. 1 and 3 are immediate. We show 2 and 4.

1. (associativity).

   (a) $a \vee (b \vee c) = (a \vee b) \vee c$.
   Suppose that $a \vee (b \vee c) = d$. It suffices to show that d is the least upper bound of $\{a \vee b, c\}$. First, since $a \sqsubseteq d$ and $b \vee c \sqsubseteq d$, $a \sqsubseteq d$, $b \sqsubseteq d$ and $c \sqsubseteq d$. From $a \sqsubseteq d$ and $b \sqsubseteq d$, $a \vee b \sqsubseteq d$, and then d is one of the upper bound of $\{a \vee b, c\}$. Next, suppose that e is an arbitrary one of the upper bound of $\{a \vee b, c\}$. Then $a \sqsubseteq e$, $b \sqsubseteq e$ and $c \sqsubseteq e$. From $b \sqsubseteq e$ and $c \sqsubseteq e$, $b \vee c \sqsubseteq e$, and then e is also one of the upper bound of $\{a, b \vee c\}$. Since d is the least upper bound of $\{a, b \vee c\}$, $d \sqsubseteq e$. Therefore d is the least upper bound of $\{a \vee b, c\}$, and then

   $$a \vee (b \vee c) = d = (a \vee b) \vee c.$$

   (b) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.
   Suppose that $a \wedge (b \wedge c) = d$. It suffices to show that d is the greatest lower bound of $\{a \wedge b, c\}$. First, since $d \sqsubseteq a$ and $d \sqsubseteq b \wedge c$, $d \sqsubseteq a$, $d \sqsubseteq b$ and $d \sqsubseteq c$. From $d \sqsubseteq a$ and $d \sqsubseteq b$, $d \sqsubseteq a \wedge b$, and then d is one of the lower bound of $\{a \wedge b, c\}$. Next, suppose that e is an arbitrary one of the lower bound of $\{a \wedge b, c\}$. Then $e \sqsubseteq a$, $e \sqsubseteq b$ and $e \sqsubseteq c$. From $e \sqsubseteq b$ and $e \sqsubseteq c$, $e \sqsubseteq b \wedge c$, and then e is also one of the lower bound of $\{a, b \wedge c\}$. Since d is the greatest lower bound of $\{a, b \wedge c\}$, $e \sqsubseteq d$. Therefore d is the greatest lower bound of $\{a \wedge b, c\}$, and then

   $$a \wedge (b \wedge c) = d = (a \wedge b) \wedge c.$$

2. (absorption).

(a) $a \vee (a \wedge b) = a$.

$a \vee (a \wedge b) = a$ means $a \wedge b \sqsubseteq a$, and it is trivial.

(b) $a \wedge (a \vee b) = a$.

$a \wedge (a \vee b) = a$ means $a \sqsubseteq a \vee b$, and it is also trivial.

■

Definition 2.1.23 (distributive) A lattice L is *distributive* if and only if for every $a, b, c \in L$ we have

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$

i.e., $\wedge$ distributes over $\vee$, in the same way as, for numbers, multiplication distributes over addition.

Proposition 2.1.24 *In a distributive lattice L, $\vee$ also distributes over $\wedge$.*

Proof.

$$\begin{aligned}
(a \vee b) \wedge (a \vee c) &= \{(a \vee b) \wedge a\} \vee \{(a \vee b) \wedge c\} \\
&= a \vee \{c \wedge (a \vee b)\} \\
&= a \vee \{(c \wedge a) \vee (c \wedge b)\} \\
&= \{a \vee (c \wedge a)\} \vee (b \wedge c) \\
&= a \vee (b \wedge c)
\end{aligned}$$

■

Remark 2.1.25 A lattice does not satisfy the distributivity of $\vee$ and $\wedge$ in general, i.e.,

- $(a \wedge b) \vee (a \wedge c) \sqsubseteq a \wedge (b \vee c)$, but does not hold

$$a \wedge (b \vee c) \sqsubseteq (a \wedge b) \vee (a \wedge c)$$

and

- $a \vee (b \wedge c) \sqsubseteq (a \vee b) \wedge (a \vee c)$, but does not hold

$$(a \vee b) \wedge (a \vee c) \sqsubseteq a \vee (b \wedge c).$$

Example 2.1.26 If U is a set, then we have already seen that its power set $\mathcal{P}(U)$ is a lattice (it actually has all meets and joins, not just the finite ones). It is distributive.

Example 2.1.27 A partially ordered set P is *linearly* ordered if any two elements are comparable:

if $x, y \in P$ then either $x \leq y$ or $y \leq x$ (or both, if and only if $x = y$).

Such a partially ordered set has all binary meets and joins, for instance

$x \wedge y = min(x, y) = x$ if $x \leq y$,

$x \wedge y = min(x, y) = y$ if $y \leq x$.

Example 2.1.28 Two Non-examples of distributive lattices can be drawn using diagrams as follows:

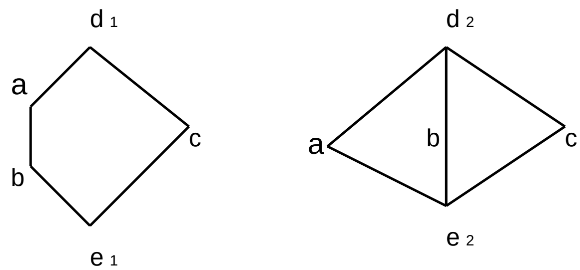

Figure 2.4: Distributive lattices.

1. Cconsider the left-hand example. First of all, it is a lattice. The nullary meets and joins are d_1 and e_1, and we know binary meets and joins of comparable elements always exist. All that is left to check are meets and joins for the incomparable pairs $\{a, c\}$ and $\{b, c\}$. These are as written in the diagram, i.e., d_1 and e_1, and they are

   - $d_1 = a \vee c = b \vee c$ and
   - $e_1 = a \wedge c = b \wedge c$

   respectively. However, the lattice is not distributive, because

   $$a \wedge (b \vee c) = a \wedge d_1 = a \neq b = b \vee e_1 = (a \wedge b) \vee (a \wedge c).$$

2. Consider the right-hand example. The nullary meets and joins are d_2 and e_2. Meets and joins for the incomparable pairs $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$ are as written in the diagram, i.e., d_2 and e_2, and they are

   - $d_2 = a \vee b = a \vee c = b \vee c$ and
   - $e_2 = a \wedge b = a \wedge c = b \wedge c$

   respectively. However, the lattice is not also distributive, because

   $$a \wedge (b \vee c) = a \wedge d_2 = a \neq e_2 = e_2 \vee e_2 = (a \wedge b) \vee (a \wedge c).$$

Proposition 2.1.29 *Let P be a partially ordered set in which every subset has a join. Then every subset has a meet.*

Proof. Let $S \subseteq P$, and let L be the set of its lower bounds. If a meet (greatest lower bound) of S exists, then it must be $\bigvee L$. Thus all we need to show is that $\bigvee L$ is a lower bound of S. But if $x \in S$ and $y \in L$, then $y \leq x$, so $\bigvee L \leq x$. ∎

Definition 2.1.30 (complete lattice) A partially ordered set P is a *complete lattice* if and only if every subset has a join and a meet.

Example 2.1.31 For a given set U, a structure $\mathbf{P(U)} = \langle \mathcal{P}(U), \cup, \cap \rangle$ forms a complete lattice, where $\cup$ and $\cap$ are the usual set theoretic operations, union and intersection.

Remark 2.1.32 In a complete lattice $\langle X, \vee \rangle$, the greatest and the least element exist: in fact

- $\top := \bigvee X$,
- $\bot := \bigwedge X$.

Remark 2.1.33 We can define a mapping $\wedge$ (the infimum) of $\mathcal{P}(X)$ into X by

$$\bigwedge Y := \bigvee \{z \mid z \sqsubseteq y \text{ for all } y \in Y\}.$$

2.2 Fixed Point Theorem

In the discussion of interpretation for the exponential !, Engberg and Winskel use the *fixed point theorem*. For an element a of a quantale (it is given in Chapter 4), as an interpretation of $!a$, they require an element x such that it is the greatest fixed point of a function. Therefore, we introduce the basic fixed point theorem here.

First we discuss monotone and fixed point, and then we show the least (greatest) fixed point theorem below.

- monotone:

  Given a complete lattice L, a function $h : L \longrightarrow L$ is *monotone* if and only if

  $$\text{if } x \sqsubseteq y, \text{ then } h(x) \sqsubseteq h(y) \text{ for all } x, y \in L.$$

- fixed point:

  An element $x \in L$ is a *fixed point* of h if and only if

  $$x = h(x).$$

Theorem 2.2.1 (fixed point) *Let L be a complete lattice and h be a monotone mapping on L. Then h has a fixed point.*

Proof. Suppose

$$D = \{x \in L \mid h(x) \sqsubseteq x\}$$

and maximum of L is 1. Then D is not empty since $1 \in D$ is immediate, and so $\wedge D$ exists. Therefore, we may assume $a = \wedge D$, and it suffices to show that a is a fixed point of h. Suppose that $x \in D$, then $a \sqsubseteq x$. Since h is monotone,

$$h(a) \sqsubseteq h(x) \sqsubseteq x,$$

and hence $h(a)$ is a lower bound of D. Since a is the infimum of D,

$$h(a) \sqsubseteq a,$$

and using monotone of h, $h(h(a)) \sqsubseteq h(a)$. This means $h(a) \in D$. Since the infimum of D is a,

$$a \sqsubseteq h(a).$$

It follows that $h(a) = a$, and hence a $(= \wedge D)$ is a fixed point of h. ■

Theorem 2.2.2 (least fixed point) *Let L be a complete lattice and h be a monotone mapping on L. Then h has the least fixed point:*

$$\bigwedge\{x \in L \mid h(x) \sqsubseteq x\}.$$

Proof. Suppose

$$D = \{x \in L \mid h(x) \sqsubseteq x\}.$$

Since we have shown that h has at least one fixed point in Theorem 2.2.1, we show a fixed point a in 2.2.1 is the least fixed point. Suppose that b is a fixed point of h, then $h(b) = b$, and hence $b \in D$. Since a is the infimum of D, $a \sqsubseteq b$, and hence a is the least fixed point. ■

Theorem 2.2.3 (greatest fixed point) *Let L be a complete lattice and h be a monotone mapping on L. Then h has a fixed point, and especially,*

$$\bigvee\{x \in L \mid x \sqsubseteq h(x)\}$$

is the greatest fixed point of L.

Proof. We can prove similarly. Suppose

$$D = \{x \in L \mid x \sqsubseteq h(x)\}$$

and minimum of L is $\perp$. Then D is not empty since $\perp \in D$ is immediate, and so $\vee D$ exists. Therefore, we may assume $a = \vee D$, and it suffices to show that a is a fixed point of h. Suppose that $x \in D$, then $x \sqsubseteq a$. Since h is monotone,

$$x \sqsubseteq h(x) \sqsubseteq h(a),$$

and hence $h(a)$ is an upper bound of D. Since a is the supremum of D,

$$a \sqsubseteq h(a),$$

and using monotone of h, $h(a) \sqsubseteq h(h(a))$. This means $h(a) \in D$. Since the supremum of D is a,

$$h(a) \sqsubseteq a.$$

It follows that $a = h(a)$, and hence $a(= \vee D)$ is a fixed point of h.

Next we show a fixed point a is the greatest fixed point. Suppose that b is a fixed point of h, then $b = h(b)$, and hence $b \in D$. Since a is the supremum of D, $b \sqsubseteq a$, and hence a is the greatest fixed point. ■

Remark 2.2.4 In chapter 4, when we discuss the definition of the exponential on the complete lattice L, we use the greatest fixed point theorem.

The monotone mapping h is

$$h : x \longrightarrow a \wedge 1 \wedge (x \bullet x).$$

Then the greatest fixed point p is defined as

$$\bigvee \{x \in L \mid x \sqsubseteq a \wedge 1 \wedge (x \bullet x)\}.$$

Chapter 3

Petri Nets

In this chapter, we discuss Petri nets. For background material on Petri nets, see [40, 41] and on the relation between Petri nets and multisets, see [9, 18, 32].

3.1 Petri Nets and Multisets

Petri nets is a general purpose mathematical model for describing relations existing between conditions and events.

Petri nets consist of two types of elements, places and transitions. Each place models a process in terms of types of resources, and can hold arbitrary nonnegative multiplicity. Each transition represents a state transition rule, i.e., how those resources are consumed or produced by actions. They are described using the notion of multisets.

First we define Petri nets, and then we discuss the relation between Petri nets and multisets with some examples.

Definition 3.1.1 (Petri net) A *Petri net* N is a quadruple $\langle P, T, {}^\bullet(-), (-)^\bullet \rangle$ such that

1. P is a set (of places),
2. T is a set (of transitions),
3. ${}^\bullet(-), (-)^\bullet$ are mappings of T into $\mathcal{M}_{\mathrm{P}}$ (i.e., the set of all multisets over a set P), where for $t \in T$,
   (a) ${}^\bullet(t)$ is called the pre-multiset of t and
   (b) $(t)^\bullet$ is called the post-multiset of t

   respectively. Each element of $\mathcal{M}_{\mathrm{P}}$ is called a marking, and in the sequel, we shall use simply $\mathcal{M}$ for $\mathcal{M}_{\mathrm{P}}$.

A *firing* of transitions transforms the given marking into another one; firing a single transition t subtracts (adds) n from (to) the mark m in place A if there is an arrow with label n from A to t (t to A). Firing of transitions gives the result of firing transitions in some other.

Graphically we can represent a Petri net by drawing the places as circles, the transitions as squares, and an arrow from place A to transition t (from transition t to place A) labeled with $n \in \mathcal{N} \setminus 0$ if $(A, t) \in^\bullet (-)$ $((t, A) \in (-)^\bullet)$ with multiplicity n ($n = 1$ usually omitted) (respectively). A marking can be indicated in a graphical representation of a Petri net by inscribing the multiplicities in the circles.

For understanding of the relation between Petri nets and multisets, We give the following examples.

Example 3.1.2 Consider the following two nets, net-1 and net-2.

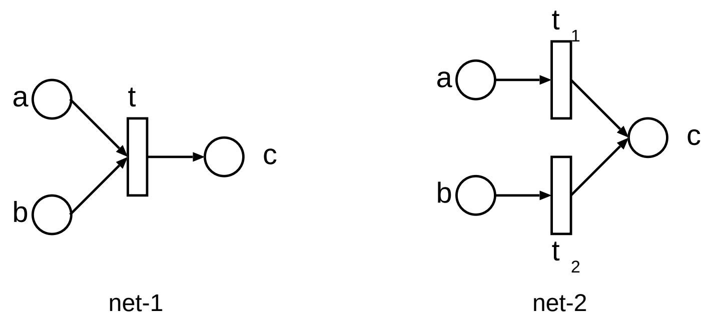

Figure 3.1: Petri net - I.

- net-1:

  We show a Petri net N $= \langle P, T, ^\bullet(-), (-)^\bullet \rangle$ with $P = \{a, b, c\}$ and $T = \{t\}$. Pre-multiset $^\bullet(t)$ is $[a, b]$ and post-multiset $(t)^\bullet$ is $[c]$ respectively. Graphically this becomes like net-1.

  - Suppose that there are one token in a and in b respectively. A firing of transition t changes the marking from $[a, b]$ to $[c]$.
  - Suppose that there are two tokens in a and in b respectively, then a firing of transition t changes the marking from $[a, a, b, b]$ to $[a, b, c]$.

- net-2:

  We show a Petri net N $= \langle P, T, ^\bullet(-), (-)^\bullet \rangle$ with $P = \{a, b, c\}$ and $T = \{t_1, t_2\}$. Pre-multiset $^\bullet(t_1)$ and $^\bullet(t_2)$ are $[a]$ and $[b]$, and post-multiset $(t_1)^\bullet$ and $(t_2)^\bullet$ are $[c]$ respectively. Graphically this becomes like net-2.

  - Suppose that there are one token in a and in b respectively. A firing of transition t_1 changes the marking from $[a, b]$ to $[b, c]$, and a firing of transition t_2 changes the marking from $[b, c]$ to $[c, c]$.
  - Suppose that there are two tokens in a and one token in b. A firing of transition t_1 changes the marking from $[a, a, b]$ to $[a, b, c]$, and a firing of transition t_2 changes the marking from $[a, b, c]$ to $[a, c, c]$.

Example 3.1.3 Consider the following net, net-3.

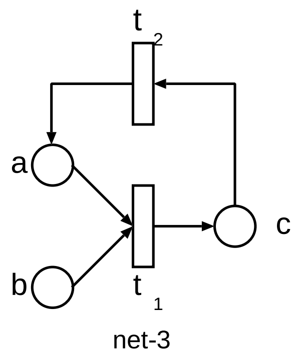

Figure 3.2: Petri net - II.

We show a Petri net $\mathrm{N} = \langle P, T, {}^\bullet(-), (-)^\bullet \rangle$ with $P = \{a, b, c\}$ and $T = \{t_1, t_2\}$. Pre-multiset ${}^\bullet(t_1)$ and ${}^\bullet(t_2)$ are $[a, b]$ and $[c]$, and post-multiset $(t_1)^\bullet$ and $(t_2)^\bullet$ are $[c]$ and $[a]$ respectively. Graphically this becomes like net-3.

Suppose that there are two tokens in a and one token in b. A firing of transition t_1 changes the marking from $[a, a, b]$ to $[a, c]$, and a firing of transition t_2 changes the marking from $[a, c]$ to $[a, a]$.

3.2 Reachability Relation

Next, we discuss the reachability relation $\rhd$. It is the reflexive and transitive relation defined as follows.

Definition 3.2.1 (reachability relation) Let $\mathrm{N} = \langle P, T, {}^\bullet(-), (-)^\bullet \rangle$ be a Petri net. Then we define a relation $\rhd$ on $\mathcal{M}$ called the *reachability relation* of N as follows:

1. For $t \in T$, let $[t\rangle$ be a relation on $\mathcal{M}$ such that

$$m\,[t\rangle\,m' \text{ if and only if } m = m'' +^\bullet t \text{ and } t^\bullet + m'' = m'$$

for some $m'' \in \mathcal{M}$.

2. Let $\rhd$ be a relation on $\mathcal{M}$ such that

$$m \rhd m' \text{ if and only if } m\,[t_1\rangle\,m_1\,[t_2\rangle\,m_2\,[t_3\rangle \cdots [t_n\rangle\,m_n = m'$$

for some $t_1, t_2, \ldots, t_n \in T$, $m_1, m_2, \ldots, m_n \in \mathcal{M}$ and $n \geq 0$.

We show the reachability relation $\rhd$ using the net-3 of Figure 3.2.

Example 3.2.2 Consider the net-3 of Figure 3.2.

- Suppose that there are one token in a and in b. A firing of transition t_1 changes the marking from $[a, b]$ to $[c]$, and a firing of transition t_2 changes the marking from $[c]$ to $[a]$. This means that

$$[a, b]\,[t_1\rangle\,[c]\,[t_2\rangle\,[a]$$

and we have

$$[a, b] \rhd [a].$$

- Next suppose that there are one token in a and two tokens in b. A firing of transition t_1 changes the marking from $[a, b, b]$ to $[b, c]$, and a firing of transition t_2 changes the marking from $[b, c]$ to $[a, b]$. Then a firing of transition t_1 changes the marking from $[a, b]$ to $[c]$, and a firing of transition t_2 changes the marking from $[c]$ to $[a]$. This means that
$$[a, b, b]\,[t_1\rangle\,[b, c]\,[t_2\rangle\,[a, b]\,[t_1\rangle\,[c]\,[t_2\rangle\,[a]$$
and we have
$$[a, b, b] \rhd [a].$$

For a Petri net N, we define structures $\mathbf{M}_{\mathrm{N}} = \langle \mathcal{M}, +, [\,] \rangle$ and $\mathbf{X}_{\mathrm{N}} = \langle \mathcal{M}, \rhd, +, [\,] \rangle$.

Proposition 3.2.3 *A structure* $\mathbf{M}_{\mathrm{N}} = \langle \mathcal{M}, +, [\,] \rangle$ *is a commutative monoid.*

Proof. In fact, we can show that a structure defined as above satisfies the conditions of Definition 2.1.5, for every $m, m', m'' \in \mathcal{M}$,

1. $m + (m' + m'') = (m + m') + m''$,
2. $m + [\,] = [\,] + m = m$,
3. $m + m' = m' + m$.

■

Proposition 3.2.4 *A structure* $\mathbf{X}_{\mathrm{N}} = \langle \mathcal{M}, \rhd, +, [\,] \rangle$ *is a preordered commutative monoid.*

Proof. In fact, we can show that a structure defined as above satisfies the conditions of Definition 2.1.11, for every $m, m', m'' \in \mathcal{M}$,

1. $\langle \mathcal{M}, +, [\,] \rangle$ is a commutative monoid,
2. a structure $\langle \mathcal{M}, \rhd \rangle$ holds Definition 2.1.9 since
   (a) $m \rhd m$,
   (b) if $m \rhd m'$ and $m' \rhd m''$, then $m \rhd m''$,
3. if $x \rhd x'$ and $y \rhd y'$, then $x + y \rhd x' + y'$.

■

We show the conditions 2(b) and 3 of Proposition 3.2.4 with the example net-3 of Figure 3.2.

Example 3.2.5 Consider the net-3 of Figure 3.2.

- Suppose that $m = [a, a, a, b, b, b]$, and then $m' = [c, c, c]$ and $m'' = [a, a, a]$, i.e.,
$$[a, a, a, b, b, b] \rhd [c, c, c] \text{ and } [c, c, c] \rhd [a, a, a].$$
Since $m \rhd m' = [a, a, a, b, b, b] \rhd [c, c, c]$ and $m' \rhd m'' = [c, c, c] \rhd [a, a, a]$, then $[a, a, a, b, b, b] \rhd [a, a, a]$, i.e.,
$$\text{if } m \rhd m' \text{ and } m' \rhd m'', \text{ then } m \rhd m''.$$

- Suppose that $x = [a, b]$ and $y = [a, a, b, b]$, and then $x' = [c]$ and $y' = [c, c]$, i.e.,

  $$[a, b] \rhd [c] \text{ and } [a, a, b, b] \rhd [c, c].$$

  Since $x + y = [a, b] + [a, a, b, b] = [a, a, a, b, b, b]$ and $x' + y' = [c] + [c, c] = [c, c, c]$, then $[a, a, a, b, b, b] \rhd [c, c, c]$, i.e.,

  $$x + y \rhd x' + y'.$$

Chapter 4

Quantales and Closure Operations

In this chapter, we discuss IL-algebras (intuitionistic linear algebras) and quantales, and introduce closure operations on the algebras. Moreover, we discuss exponential ! and quantales with exponential. For background material on the algebras, see [2, 4, 38, 39, 42, 43, 50].

4.1 IL-algebras and Quantales

Algebraic semantics for linear logic, as presented here, is for linear logic what Boolean-valued models are for classical logic, and Heyting-valued models for intuitionistic logic.

4.1.1 IL-algebras

Definition 4.1.1 (IL-algebra) A structure $\mathbf{A} = \langle A, \wedge, \vee, \bot, \multimap, \bullet, 1\rangle$ is an IL-*algebra* if

1. $\langle A, \wedge, \vee, \bot\rangle$ is a *lattice* with the least element $\bot$,
2. $\langle A, \bullet, 1\rangle$ is a commutative monoid with unit 1,
3. if $x \sqsubseteq x'$, $y \sqsubseteq y'$, then $x \bullet y \sqsubseteq x' \bullet y'$ and $x' \multimap y \sqsubseteq x \multimap y'$,
4. $x \bullet y \sqsubseteq z$ if and only if $x \sqsubseteq y \multimap z$ for all $x, y, z \in A$.

Lemma 4.1.2 *In any IL-algebra* $\mathbf{A} = \langle A, \wedge, \vee, \bot, \multimap, \bullet, 1\rangle$*, for all* x, y, z *in* A*,*

1. $z \bullet (x \vee y) = (z \bullet x) \vee (z \bullet y)$*, and moreover, if the join* $\bigvee_{i\in I} y_i$ *exists, then* $x \bullet \bigvee_{i\in I} y_i = \bigvee_{i\in I}(x \bullet y_i)$*,*
2. $x \multimap (y \multimap z) = x \bullet y \multimap z$*,*
3. $\bot \multimap \bot$ *is top of* A $(\top := \bot \multimap \bot)$*.*

Proof.

1. Suppose that $z \bullet (x \vee y) \sqsubseteq v$. Since $x \vee y \sqsubseteq z \multimap v$, $x \sqsubseteq z \multimap v$ and $y \sqsubseteq z \multimap v$. Therefore $x \bullet z \sqsubseteq v$ and $y \bullet z \sqsubseteq v$, and hence $(z \bullet x) \vee (z \bullet y) \sqsubseteq v$. We can prove the converse similarly.

2. Suppose that $u \sqsubseteq x \multimap (y \multimap z)$. Then $u \bullet x \sqsubseteq y \multimap z$, and hence $u \bullet x \bullet y \sqsubseteq z$. Therefore $u \sqsubseteq x \bullet y \multimap z$. Similarly we can prove the converse.

3. Since $\bot \sqsubseteq x \multimap \bot$, $\bot \bullet x \sqsubseteq \bot$. Therefore $x \sqsubseteq \bot \multimap \bot$, and hence $\top = \bot \multimap \bot$.

■

Proposition 4.1.3 A definition of IL-algebra is obtained replacing Definition 3 of 4.1.1 by $z \bullet (x \vee y) = (z \bullet x) \vee (z \bullet y)$.

Proof. Assume Definition 1, 2 and 4 of 4.1.1 and $z \bullet (x \vee y) = (z \bullet x) \vee (z \bullet y)$. If $x \sqsubseteq x'$, then $x \vee x' = x'$. Therefore

$$\begin{aligned} z \bullet (x \vee x') &= (z \bullet x) \vee (z \bullet x') \\ &= z \bullet x', \end{aligned}$$

and hence $z \bullet x \sqsubseteq z \bullet x'$. Also assume $x \sqsubseteq x'$. Since $z \sqsubseteq x' \multimap y$ if and only if $z \bullet x' \sqsubseteq y$, $z \bullet x \sqsubseteq y$. Therefore $z \sqsubseteq x \multimap y$, and so $x' \multimap y \sqsubseteq x \multimap y$. ■

Definition 4.1.4 (complete IL-algebra) A structure $\mathbf{A} = \langle A, \multimap, \vee, \wedge, \bullet, 1\rangle$ is a *complete* IL-*algebra* if

1. $\langle A, \wedge, \vee\rangle$ is a complete lattice,

2. $\langle A, \bullet, 1\rangle$ is a commutative monoid with unit 1,

3. $(\bigvee x_i) \bullet y = \bigvee(x_i \bullet y)$ for all $x_i, y \in A$,

4. $x \bullet y \sqsubseteq z$ if and only if $x \sqsubseteq y \multimap z$ for all $x, y, z \in A$.

Proposition 4.1.5 *Let* $\mathbf{M} = \langle M, \cdot, e\rangle$ *be a commutative monoid with the identity* e*, and for each* $X, Y \subseteq M$*, define sets* $X \bullet Y$ *and* $Y \multimap Z$ *of* M *by*

1. $X \bullet Y := \{x \cdot y \mid x \in X, y \in Y\}$,

2. $Y \multimap Z := \{x \in M \mid x \cdot y \in Z \text{ for all } y \in Y\}$.

Then the structure

$$\mathbf{P}(\mathbf{M}) = \langle \mathcal{P}(M), \multimap, \cup, \cap, \bullet, \{e\}\rangle,$$

where $\cup$ *and* $\cap$ *are the usual set-theoretic operations, is a complete IL-algebra.*

Proof.

1. 1 and 2 of Definition 4.1.4 are trivial.

2. For 3 of Definition 4.1.4, we show that

$$(\bigcup X_i) \bullet Y = \bigcup(X_i \bullet Y).$$

Suppose that $x \in (\bigcup X_i) \bullet Y$. Then $x = y \cdot z$ for some $y \in \bigcup X_i$ and some $z \in Y$. Therefore $y \in X_i$ for some i, and so $x \in X_i \bullet Y$. Since $x_i \bullet Y \subseteq \bigcup(X_i \bullet Y)$, $x \in \bigcup(x_i \bullet Y)$. Conversely suppose that $X_j \subseteq \bigcup X_i$. Then $X_j \bullet Y \subseteq (\bigcup X_i) \bullet Y$. Therefore $\bigcup(X_j \bullet Y) \subseteq (\bigcup X_i) \bullet Y$.

3. For Definition 4 of 4.1.4, we show that

$$X \bullet Y \subseteq Z \text{ if and only if } X \subseteq Y \multimap Z.$$

If $x \in X$, then $x \cdot y \in X \bullet Y$ for all $y \in Y$. Therefore $x \cdot y \in Z$, and so $x \in Y \multimap Z$. Thus $X \subseteq Y \multimap Z$. Conversely for all $x \in X$ and $y \in Y$, since $X \subseteq Y \multimap Z$, $x \cdot y \in Z$. Therefore $X \bullet Y \subseteq Z$.

■

Remark 4.1.6 The structure $\mathbf{P}(\mathbf{M}) = \langle \mathcal{P}(M), \multimap, \cup, \cap, \bullet, \{e\} \rangle$ satisfies a law not generally valid in IL-algebras: distributivity of the lattice operations.

The simplest way to see that distributivity of the lattice operations does not hold in general, is to verify that the sequent

$$A \sqcap (B \sqcup C) \Rightarrow (A \sqcap B) \sqcup (A \sqcap C)$$

is not derivable, since this means that the IL-algebra constructed from Intuitionistic linear logic or classical linear logic by means of the Lindenbaum construction does not obey distributivity.

4.1.2 Quantales

Definition 4.1.7 (commutative quantale) A structure $\mathbf{Q} = \langle Q, \vee, \bullet, 1 \rangle$ is *a commutative quantale* if

1. $\langle Q, \vee \rangle$ is a complete lattice,
2. $\langle Q, \bullet, 1 \rangle$ is a commutative monoid,
3. $(\bigvee x_i) \bullet y = \bigvee (x_i \bullet y)$ for all $x_i, y \in Q$.

We shall use simply quantale for commutative quantale with unit.

Remark 4.1.8 Define a binary operation $\multimap$ on $\mathbf{Q}$ by

$$y \multimap z := \bigvee \{x \mid x \bullet y \sqsubseteq z\}.$$

Then

$$x \sqsubseteq y \multimap z \text{ if and only if } x \bullet y \sqsubseteq z.$$

Proposition 4.1.9 *Let* $\mathbf{M} = \langle M, \cdot, e \rangle$ *be a commutative monoid with the identity* e*, and for each* $X, Y \subseteq M$*, define a subset* $X \bullet Y$ *of* M *by*

$$X \bullet Y := \{x \cdot y \mid x \in X, y \in Y\}.$$

Then the structure

$$\mathbf{P}(\mathbf{M}) = \langle \mathcal{P}(M), \cup, \bullet, \{e\} \rangle$$

is a quantale.

Proof.

1. 1 and 2 of Definition 4.1.7 are trivial.

2. For 3 of Definition 4.1.7, we show that
$$(\bigcup X_i) \bullet Y = \bigcup (X_i \bullet Y).$$
Suppose that $x \in (\bigcup X_i) \bullet Y$. Then $x = y \cdot z$ for some $y \in \bigcup X_i$ and some $z \in Y$. Therefore $y \in X_i$ for some i, and so $x \in X_i \bullet Y$. Since $x_i \bullet Y \subseteq \bigcup (X_i \bullet Y)$, $x \in \bigcup (x_i \bullet Y)$. Conversely suppose that $X_j \subseteq \cup X_i$. Then $X_j \bullet Y \subseteq (\bigcup X_i) \bullet Y$. Therefore $\bigcup (X_j \bullet Y) \subseteq (\bigcup X_i) \bullet Y$.

■

Remark 4.1.10 *In the quantale* $\mathbf{P}(\mathbf{M})$*,*
$$\begin{aligned} Y \multimap\!\!\bullet Z &:= \bigcup \{X \mid X \bullet Y \subseteq Z\} \\ &= \{x \in M \mid x \cdot y \in Z \textit{ for all } y \in Y\}. \end{aligned}$$

Remark 4.1.11 It is easy to show that a complete IL-algebra is just a quantale, in which $y \multimap\!\!\bullet z$ is defined by
$$y \multimap\!\!\bullet z := \bigvee \{x \mid x \bullet y \sqsubseteq z\}.$$

Proposition 4.1.12 *A structure is a complete IL-algebra if and only if it is a quantale.*

Proof. It is trivial that if a structure is a complete IL-algebra, then it is a quantale. We show that if a structure is a quantale, then it is a complete IL-algebra. Define
$$y \multimap\!\!\bullet z := \bigvee \{x \mid x \bullet y \sqsubseteq z\}.$$
Then we show that a commutative quantale is always a complete IL-algebra.

1. 1, 2 and 3 of Definition 4.1.4 are trivial.

2. For 4 of Definition 4.1.4, we show that
$$x \bullet y \sqsubseteq z \text{ if and only if } x \sqsubseteq y \multimap\!\!\bullet z.$$
Suppose that $y \multimap\!\!\bullet z := \bigvee \{u | u \bullet y \sqsubseteq z\}$. If $x \bullet y \sqsubseteq z$, then $x \in \{u \mid u \bullet y \sqsubseteq z\}$, and hence $x \sqsubseteq y \multimap\!\!\bullet z$. Conversely if $x \sqsubseteq y \multimap\!\!\bullet z$, then
$$\begin{aligned} x \bullet y &\sqsubseteq (y \multimap\!\!\bullet z) \bullet y \\ &= (\bigvee \{u \mid u \bullet y \sqsubseteq z\}) \bullet y \\ &= \bigvee \{u \bullet y \mid u \bullet y \sqsubseteq z\} \\ &\sqsubseteq z, \end{aligned}$$
and hence $x \bullet y \sqsubseteq z$.

■

The proposition shows that complete IL-algebras and quantales amount to the same thing.

Corollary 4.1.13 *A structure* $\mathbf{P}(\mathbf{M}_{\mathrm{N}}) := \langle \mathcal{P}(\mathcal{M}), \cup, \bullet, \{[\,]\} \rangle$ *is a quantale, where*
$$X \bullet Y := \{m + m' \mid m \in X, m' \in Y\}.$$
(Note that a structure $\mathbf{M}_{\mathrm{N}} = \langle \mathcal{M}, +, [\,] \rangle$ *is a commutative monoid.)*

Remark 4.1.14 *In the quantale* $\mathbf{P}(\mathbf{M}_{\mathrm{N}})$*,*
$$\begin{aligned} Y \multimap\!\!\bullet Z &:= \bigcup \{X \mid X \bullet Y \subseteq Z\} \\ &= \{m \in \mathcal{M} \mid m + m' \in Z \textit{ for all } m' \in Y\}. \end{aligned}$$

4.2 Closure Operations on IL-algebras and Quantales

Next, we discuss closure operations on the IL-algebras and the quantales which play a crucial role in the proof of completeness.

4.2.1 Closure Operation

Definition 4.2.1 (closure operation) An operation C on a quantale $\mathbf{Q} = \langle Q, \vee, \bullet, 1 \rangle$ is a *closure operation* on Q if

1. $x \sqsubseteq Cx$,
2. if $x \sqsubseteq y$ then $Cx \sqsubseteq Cy$,
3. $CCx \sqsubseteq Cx$,
4. $Cx \bullet Cy \sqsubseteq C(x \bullet y)$.

An element x of Q is C*-closed* if $x = Cx$ holds. $C(Q)$ denotes the set of all C-closed elements of Q.

Lemma 4.2.2 *$C(Cx \vee Cy) = C(x \vee y)$ holds for every $x, y \in Q$.*

Proof. Since $C(x \vee y) \sqsubseteq C(Cx \vee Cy)$ is trivial, we show that

$$C(Cx \vee Cy) \sqsubseteq C(x \vee y).$$

$x \sqsubseteq x \vee y$ and $y \sqsubseteq x \vee y$, then $Cx \sqsubseteq C(x \vee y)$ and $Cy \sqsubseteq C(x \vee y)$. Therefore $(Cx \vee Cy) \sqsubseteq C(x \vee y)$, and hence

$$\begin{aligned} C(Cx \vee Cy) &\sqsubseteq CC(x \vee y) \\ &= C(x \vee y). \end{aligned}$$

■

Lemma 4.2.3 *$C(Cx \bullet Cy) = C(x \bullet y)$ holds for every $x, y \in Q$.*

Proof. Since $C(x \bullet y) \sqsubseteq C(Cx \bullet Cy)$ is trivial, we show that

$$C(Cx \bullet Cy) \sqsubseteq C(x \bullet y).$$

Since $Cx \bullet Cy \sqsubseteq C(x \bullet y)$ (4 of Definition 4.2.1),

$$\begin{aligned} C(Cx \bullet Cy) &\sqsubseteq CC(x \bullet y) \\ &= C(x \bullet y). \end{aligned}$$

■

4.2.2 Closure Operation on IL-algebras

Proposition 4.2.4 *If C is a closure operation on an IL-algebra $\mathbf{A} = \langle A, \wedge, \vee, \perp, -\!\!\bullet, \bullet, 1\rangle$, then*

$$\mathbf{C(A)} = \langle C(A), \wedge, \vee_C, C\perp, -\!\!\bullet, \bullet_C, C1\rangle$$

is also an IL-algebra, where $\vee_C$ and $\bullet_C$ are defined by

1. $\vee_C x_i := C(\vee x_i)$,
2. $x \bullet_C y := C(x \bullet y)$.

Proof. We show that a structure $\langle C(A), \wedge, \vee_C, C\perp, -\!\!\bullet, \bullet_C, C1\rangle$ defined as above holds Definition 4.1.1. The proof of that if $a, b \in C(Q)$, then $a \wedge b \in C(Q)$, $a \vee_C b \in C(Q)$, $a \bullet_C b \in C(Q)$ and $a -\!\!\bullet b \in C(Q)$, and 1 of definition will be shown later (see the proof of Proposition 4.2.5. Now we show 2, 3 and 4 of definition.

1. We show that a structure $\langle C(A), \wedge, \vee_C, C\perp, -\!\!\bullet, \bullet_C, C1\rangle$ is a commutative monoid with unit 1.
   (a) For $a \bullet_C (b \bullet_C c) = (a \bullet_C b) \bullet_C c$, $a \bullet_C (b \bullet_C c) = C(a \bullet C(b \bullet c)) = C(Ca \bullet C(b \bullet c)) = C(a \bullet (b \bullet c)) = C((a \bullet b) \bullet c) = C(C(a \bullet b) \bullet Cc) = C(C(a \bullet b) \bullet c) = C((a \bullet_c b) \bullet C) = (a \bullet_C b) \bullet_C c$.
   (b) For $a \bullet_C C1 = C1 \bullet_C a = a$, $C1 \bullet_C a = C(C1 \bullet a) = C(C1 \bullet Ca) = C(1 \bullet a) = C(a \bullet 1) = C(Ca \bullet C1) = C(a \bullet C1) = a \bullet_C C1$, and $C(1 \bullet a) = Ca = a$. Therefore $C1$ is the unit.
   (c) For $a \bullet_C b = b \bullet_C a$, $a \bullet_C b = C(a \bullet b) = C(b \bullet a) = b \bullet_C a$.
2. We show $z \bullet_C (x \vee_C y) = (z \bullet_C x) \vee_C (z \bullet_C y)$.

$$\begin{aligned} z \bullet_C (x \vee_C y) &= C(z \bullet (x \vee_C y)) \\ &= C(z \bullet C(x \vee y)) \\ &= C(Cz \bullet C(x \vee y)) \\ &= C(z \bullet (x \vee y)) \\ &= C((z \bullet x) \vee (z \bullet y)) \\ &= C(C(z \bullet x) \vee C(z \bullet y)) \\ &= (z \bullet_C x) \vee_C (z \bullet_C y). \end{aligned}$$

3. We show $x \bullet_C y \sqsubseteq z$ if and only if $x \sqsubseteq y -\!\!\bullet z$.

$$\begin{aligned} x \bullet_C y &= C(x \bullet y) \\ &= C(Cx \bullet Cy) \\ &\sqsubseteq Cz, \end{aligned}$$

then $Cx \bullet Cy \sqsubseteq Cz$, and hence $x \sqsubseteq y -\!\!\bullet z$.

∎

4.2.3 Closure Operation on Quantales

Proposition 4.2.5 *If C is a closure operation on a quantale $\mathbf{Q} = \langle Q, \vee, \bullet, 1\rangle$, then*

$$\mathbf{C}(\mathbf{Q}) = \langle C(Q), \vee_C, \bullet_C, C1\rangle$$

is also a quantale, where $\vee_C$ and $\bullet_C$ are defined by

1. $\vee_C x_i := C(\vee x_i)$,
2. $x \bullet_C y := C(x \bullet y)$.

Proof. First we show that if $a, b \in C(Q)$, then $a \wedge b \in C(Q)$, $a \vee_C b \in C(Q)$, $a \bullet_C b \in C(Q)$ and $a \multimap b \in C(Q)$.

- For $a \wedge b \in C(Q)$, since $a \wedge b \sqsubseteq a$, $C(a \wedge b) \sqsubseteq Ca = a$. Similarly since $a \wedge b \sqsubseteq b$, $C(a \wedge b) \sqsubseteq Cb = b$. Therefore $C(a \wedge b) \sqsubseteq a \wedge b \sqsubseteq C(a \wedge b)$, and so $a \wedge b \in C(Q)$.
- For $a \vee_C b \in C(Q)$, $C(a \vee_C b) = C(C(a \vee b)) = C(a \vee b) = a \vee_C b$. Therefore $a \vee_C b \in C(Q)$.
- For $a \bullet_C b \in C(Q)$, $C(a \bullet_C b) = C(C(a \bullet b)) = C(a \bullet b) = a \bullet_C b$. Therefore $a \bullet_C b \in C(Q)$.
- For $a \multimap b \in C(Q)$,

$$\begin{aligned} C(a \multimap b) \bullet_C a &= C(a \multimap b) \bullet_C Ca \\ &= C(C(a \multimap b) \bullet Ca) \\ &= C((a \multimap b) \bullet a) \\ &\sqsubseteq b \end{aligned}$$

(since we can get $(a \multimap b) \bullet a \sqsubseteq b$ from $a \multimap b \sqsubseteq a \multimap b$). Therefore

$$\begin{aligned} C(a \multimap b) &\sqsubseteq a \multimap b \\ &\sqsubseteq C(a \multimap b), \end{aligned}$$

and so $a \multimap b \in C(Q)$.

Next we show that a structure $\mathbf{C(Q)} = \langle C(Q), \vee_C, \bullet_C, C1\rangle$ defined as above holds Definition 4.1.7.

1. $\langle C(Q), \vee_C\rangle$ is a complete lattice.

   We show that a structure $\langle C(Q), \vee_C\rangle$ is a lattice and holds Definition 2.1.30. It is enough to prove for $\vee_C$.

   First we show that $\langle C(Q), \vee_C\rangle$ holds Definition 2.1.20.

   (a) For $a \vee_C a = a$, $a \vee_C a = C(a \vee a) = Ca = a$.

(b) For associative $a \vee_C (b \vee_C c) = (a \vee_C b) \vee_C c$,

$$\begin{aligned} a \vee_C (b \vee_C c) &= C(a \vee C(b \vee c)) \\ &= C(Ca \vee C(b \vee c)) \\ &\sqsubseteq C(C(a \vee (b \vee c))) \\ &= C(a \vee (b \vee c)) \\ &\sqsubseteq C(a \vee C(b \vee c)) \\ &= a \vee_C (b \vee_C c). \end{aligned}$$

Therefore $a \vee_C (b \vee_C c) = C(a \vee (b \vee c)) = C((a \vee b) \vee c) = (a \vee_C b) \vee_C c$.

(c) For $a \vee_C b = b \vee_C a$, $a \vee_C b = C(a \vee b) = C(b \vee a) = b \vee_C a$.

(d) For $a \vee_C (a \wedge b) = a$, $a \vee_C (a \wedge b) = C(a \vee (a \wedge b)) = Ca = a$. For $a \wedge (a \vee_C b) = a$,

$$\begin{aligned} a &\sqsupseteq a \wedge (a \vee_C b) \\ &= a \wedge C(a \vee b) \\ &\sqsupseteq a \wedge (a \vee b) \\ &= a. \end{aligned}$$

Next we show that the maximum element : $\top$ and the minimum element : $C\perp$ exist.

(a) $\top \in C(Q)$ since $C\top \sqsubseteq \top \sqsubseteq C\top$.

(b) Since $\perp \sqsubseteq a$ for all $a \in Q$, $C\perp \sqsubseteq Ca = a$.

2. $\langle C(Q), \bullet_C, C1 \rangle$ is a commutative monoid.

We show that a structure $\langle C(Q), \bullet_C, C1 \rangle$ is a monoid and holds Definition 2.1.5, for every $a, b, c \in C(Q)$.

First we show that $\langle C(Q), \bullet_C, C1 \rangle$ holds Definition 2.1.3.

(a) For $a \bullet_C (b \bullet_C c) = (a \bullet_C b) \bullet_C c$, $a \bullet_C (b \bullet_C c) = C(a \bullet C(b \bullet c)) = C(Ca \bullet C(b \bullet c)) = C(a \bullet (b \bullet c)) = C((a \bullet b) \bullet c) = C(C(a \bullet b) \bullet Cc) = C(C(a \bullet b) \bullet c) = C((a \bullet_c b) \bullet C) = (a \bullet_C b) \bullet_C c$.

(b) For $a \bullet_C C1 = C1 \bullet_C a = a$, $C1 \bullet_C a = C(C1 \bullet a) = C(C1 \bullet Ca) = C(1 \bullet a) = C(a \bullet 1) = C(Ca \bullet C1) = C(a \bullet C1) = a \bullet_C C1$, and $C(1 \bullet a) = Ca = a$. Therefore $C1$ is the unit.

Next we show that $\langle C(Q), \bullet_C, C1 \rangle$ holds 2 of Definition 2.1.5.

For $a \bullet_C b = b \bullet_C a$, $a \bullet_C b = C(a \bullet b) = C(b \bullet a) = b \bullet_C a$.

3. We show that $\bullet_C$ distributes over $\vee_C$.

$$\begin{aligned} (\bigvee_C S) \bullet_C b &= C(C(\bigvee S) \bullet b) \\ &= C(C(\bigvee S) \bullet Cb) \end{aligned}$$

$$\begin{aligned}
&= C(\bigvee S \bullet b) \\
&= C(\bigvee_{a\in S} (a \bullet b)) \\
&= C(\bigvee_{a\in S} C(a \bullet b)) \\
&= \bigvee_{a\in S_C} (a \bullet_C b).
\end{aligned}$$

Therefore for every $S \subseteq C(Q)$ and $a \in S$, $(\bigvee_C S) \bullet_C b = \bigvee_{a\in S_C} (a \bullet_C b)$.

4. We show that for every $a, b, c \in C(Q)$, $a \bullet_C b \sqsubseteq c$ if and only if $a \sqsubseteq b \multimap\!\bullet c$. Suppose that $a \bullet_C b \sqsubseteq c$.

$$\begin{aligned}
a \bullet b &= Ca \bullet Cb \\
&\sqsubseteq C(a \bullet b) \\
&= a \bullet_C b \\
&\sqsubseteq c,
\end{aligned}$$

and hence $a \sqsubseteq b \multimap\!\bullet c$. Conversely, suppose that $a \sqsubseteq b \multimap\!\bullet c$. Then $a \bullet b \sqsubseteq c$. Therefore $a \bullet_C b = C(a \bullet b) \sqsubseteq Cc = c$, and hence $a \bullet_C b \sqsubseteq c$. Therefore for every $a, b, c \in C(A)$, $a \bullet_C b \sqsubseteq c$ if and only if $a \sqsubseteq b \multimap\!\bullet c$.

$\blacksquare$

Lemma 4.2.6 *In the quantale* $\mathbf{C(Q)}$, $x \wedge y$ *and* $x \multimap\!\bullet y$ *are* C*-closed whenever* x *and* y *are* C*-closed. Hence operations* $\wedge$ *and* $\multimap\!\bullet$ *coincide with the original operations on* $\mathbf{Q}$.

Proof.

1. $Cx \multimap\!\bullet Cy \sqsubseteq C(Cx \multimap\!\bullet Cy)$ is trivial. We show $C(Cx \multimap\!\bullet Cy) \sqsubseteq Cx \multimap\!\bullet Cy)$.

$$C(Cx \multimap\!\bullet Cy) \sqsubseteq Cx \multimap\!\bullet Cy \text{ if and only if } C(Cx \multimap\!\bullet Cy) \bullet Cx \sqsubseteq Cy.$$

But

$$\begin{aligned}
C(Cx \multimap\!\bullet Cy) \bullet Cx &\sqsubseteq C(C(Cx \multimap\!\bullet Cy) \bullet Cx) \\
&= C((Cx \multimap\!\bullet Cy) \bullet Cx) \\
&\sqsubseteq CCy \\
&= Cy,
\end{aligned}$$

using $(u \multimap\!\bullet v) \bullet u \sqsubseteq v$ and 1, 2, 3 and 4 of Definition 4.2.1.

2. $Cx \wedge Cy \sqsubseteq C(Cx \wedge Cy)$ is trivial. We show $C(Cx \wedge Cy) \sqsubseteq Cx \wedge Cy$. $Cx \wedge Cy \sqsubseteq Cx$ and $Cx \wedge Cy \sqsubseteq Cy$, hence $C(Cx \wedge Cy) \sqsubseteq Cx$ and $C(Cx \wedge Cy) \sqsubseteq Cy$, therefore $C(Cx \wedge Cy) \sqsubseteq Cx \wedge Cy$.

$\blacksquare$

4.2.4 Closure Operation C_1 and C_2

Let $\mathbf{X} = \langle M, \leq, \cdot, e\rangle$ be a preordered commutative monoid. We define two closure operations C_1 and C_2 on $\mathbf{P}(\mathbf{X})$.

Proposition 4.2.7 *Let $\mathbf{X} = \langle M, \leq, \cdot, e\rangle$ be a preordered commutative monoid and define an operation $\downarrow$ on $\mathbf{P}(\mathbf{X})$ such that*

$$\downarrow X := \{y \in M \mid \exists x \in X\,(y \leq x)\}.$$

Then it is easy to see that $\downarrow$ is a closure operation on the quantale $\mathbf{P}(\mathbf{X}) = \langle \mathcal{P}(M), \cup, \bullet, \{e\}\rangle$ (see the proof of Proposition 4.2.10).

Definition 4.2.8 Let $\mathbf{X} = \langle M, \leq, \cdot, e\rangle$ be a preordered commutative monoid and define an operation C_1 on $\mathbf{P}(\mathbf{X})$ by

$$C_1 X :=\downarrow X.$$

Definition 4.2.9 Let $\mathbf{X} = \langle M, \leq, \cdot, e\rangle$ be a preordered commutative monoid and define two operations $\rightarrow$ and $\leftarrow$ on $\mathbf{P}(\mathbf{X})$ by

$$X^{\rightarrow} := \{y \in M \mid \forall x \in X(x \leq y)\},$$

$$X^{\leftarrow} := \{y \in M \mid \forall x \in X(y \leq x)\},$$

and let C_2 be the operation on $\mathbf{P}(\mathbf{X})$ defined by

$$C_2 X := (X^{\rightarrow})^{\leftarrow}.$$

(C_2 is called the MacNeille completion of X, see [14, 23, 31, 36]).

We can easily show the following propositions (see e.g. [47]).

Proposition 4.2.10 *C_1 is a closure operation on the quantale*

$$\mathbf{P}(\mathbf{X}) = \langle \mathcal{P}(M), \cup, \bullet, \{e\}\rangle.$$

Proof. We show that a function C_1 defined as above holds Definition 4.2.1.

1. If $x \in X$, then $x \leq x$, and hence $\exists y \in X\,(x \leq y)$. Therefore $x \in C_1X$, and so $X \subseteq C_1X$.

2. If $z \in C_1X$, then $\exists x \in X\,(z \leq x)$, and since $X \subseteq Y$, so $\exists x \in Y\,(z \leq x)$. Therefore $z \in C_1Y$, and hence $C_1X \subseteq C_1Y$.

3. If $x \in C_1C_1X$, then $\exists y \in C_1X\,(x \leq y)$ and $\exists z \in X\,(y \leq z)$. Therefore $x \leq z$, so $x \leq C_1X$, and hence $C_1C_1X \subseteq C_1X$. Since $C_1X \subseteq C_1C_1X$ (1 of definition), $C_1C_1X = C_1X$.

4. We show that if $x \in C_1X$ and $y \in C_1Y$, then $x \cdot y \in C_1(X \bullet Y)$. Since $x \in C_1X$, $\exists x' \in X\,(x \leq x')$. Since $y \in C_1Y$, $\exists y' \in Y\,(y \leq y')$. By definition, since $x \cdot y \leq x' \cdot y'$, $x \cdot y \in X \bullet Y$. Therefore $\exists x' \cdot \exists y' \in X \bullet Y\,(x \cdot y \leq x' \cdot y')$. Thus $x \cdot y \in C_1(X \bullet Y)$, and hence $C_1X \bullet C_1Y \subseteq C_1(X \bullet Y)$.

■

Proposition 4.2.11 *C_2 is a closure operation on the quantale*

$$\mathbf{P(X)} = \langle \mathcal{P}(M), \cup, \bullet, \{e\} \rangle.$$

Proof. We show that a function C_2 defined as above holds Definition 4.2.1.

1. If $x \in X$, then $\forall y \in X^{\rightarrow} (x \leq y)$. Therefore $x \in (X^{\rightarrow})^{\leftarrow} = C_2X$, and hence $X \subseteq C_2X$.

2. First we show that
$$\text{if } X \subseteq Y, \text{ then } Y^{\rightarrow} \subseteq X^{\rightarrow} \cdots (1).$$
If $z \in Y^{\rightarrow}$, then $\forall y \in Y (y \leq z)$. Therefore $\forall x \in X (x \leq z)$, and hence $z \in X^{\rightarrow}$. Next we show that
$$\text{if } X \subseteq Y, \text{ then } Y^{\leftarrow} \subseteq X^{\leftarrow} \cdots (2).$$
If $z \in Y^{\leftarrow}$, then $\forall y \in Y (z \leq y)$. Therefore $\forall x \in X (z \leq x)$, and hence $z \in X^{\leftarrow}$. By (1) and (2), if $X \subseteq Y$, then $Y^{\rightarrow} \subseteq X^{\rightarrow}$. and if $Y^{\rightarrow} \subseteq X^{\rightarrow}$, then $(X^{\rightarrow})^{\leftarrow} \subseteq (Y^{\rightarrow})^{\leftarrow}$. Therefore if $X \subseteq Y$ then $(X^{\rightarrow})^{\leftarrow} \subseteq (Y^{\rightarrow})^{\leftarrow}$, and hence $C_2X \subseteq C_2Y$.

3. $X \subseteq (X^{\rightarrow})^{\leftarrow}$ by 1, then by 2,
$$((X^{\rightarrow})^{\leftarrow})^{\rightarrow} = (C_2X)^{\rightarrow} \subseteq X^{\rightarrow} \cdots (1).$$
If $x \in X^{\rightarrow}$, then by definition, $\forall y \in (X^{\rightarrow})^{\leftarrow} (y \leq x)$. Therefore $x \in ((X^{\rightarrow})^{\leftarrow})^{\rightarrow}$, and hence
$$X^{\rightarrow} \subseteq ((X^{\rightarrow})^{\leftarrow})^{\rightarrow} = (C_2X)^{\rightarrow} \cdots (2).$$
By (1) and (2), $((X^{\rightarrow})^{\leftarrow})^{\rightarrow} = (C_2X)^{\rightarrow} = X^{\rightarrow}$. Therefore $C_2C_2X = ((C_2X)^{\rightarrow})^{\leftarrow} = (X^{\rightarrow})^{\leftarrow} = C_2X$, then $C_2C_2X = C_2X$.

4. We show that
$$\text{if } x \in C_2X \text{ and } y \in C_2Y, \text{ then } x \cdot y \in C_2(X \bullet Y).$$
Suppose that $z \in (X \bullet Y)^{\rightarrow}$. Then $\forall u \in X$ and $\forall v \in Y (u \cdot v \leq z)$, and hence $\forall u \in X (u \leq v \multimap\!\!\bullet z)$. Since $u \in X$ is arbitrary, $\forall v \in Y (v \multimap\!\!\bullet z \in X^{\rightarrow})$. Therefore $x \leq v \multimap\!\!\bullet z$, and hence we have $v \cdot x = x \cdot v \leq z$. Thus $v \leq x \multimap\!\!\bullet z$. Since $v \in Y$ is arbitrary, we have $x \multimap\!\!\bullet z \in Y^{\rightarrow}$, and hence $y \leq x \multimap\!\!\bullet z$. Therefore $x \cdot y = y \cdot x \leq z$, and hence $x \cdot y \in ((X \bullet Y)^{\rightarrow})^{\leftarrow} = C_2(X \bullet Y)$. Thus $C_2X \bullet C_2Y \subseteq C_2(X \bullet Y)$.

■

Remark 4.2.12 C_1 is the closure operation used in [15]. In the quantales constructed from Petri nets using C_1, since C_1-closed sets are downwards closed, for m, $m' \in \mathcal{M}$

$$C_1(m) \subseteq C_1(m') \text{ if and only if } m \rhd m'.$$

Therefore C_1 adequates for the reachability relation of Petri nets. Similarly, since every C_2-closed set is downwards closed, C_2 also adequates for the reachability relation.

Lemma 4.2.13 *Suppose that a structure* $\mathbf{P}(\mathbf{M}_\mathrm{N}) := \langle \mathcal{P}(\mathcal{M}), \cup, \bullet, \{[\,]\} \rangle$ *is a quantale constructed from a Petri net using* C_2 *closure operation. Then for all* $x \in \mathcal{M}$, *the following holds.*

$$C_2(\{x\}) = \{y \mid y \sqsubseteq x\}.$$

Proof.

- First we show that if $m \in C_2(\{x\})$, then $m \sqsubseteq x$. Suppose that $m \in C_2(\{x\})$. Since $\sqsubseteq$ is reflexive, then $x \in \{x\}^{\rightarrow}$, and hence $m \sqsubseteq x$.
- Next we show that if $m \sqsubseteq x$, then $m \in C_2(\{x\})$. Suppose that $m \sqsubseteq x$. If $m' \in x^{\rightarrow}$, then $x \sqsubseteq m'$. Since $\sqsubseteq$ is transitive, then $m \sqsubseteq m'$, and hence

$$m \in (\{x\}^{\rightarrow})^{\leftarrow} = C_2(\{x\}).$$

$\blacksquare$

4.3 Quantales with Exponential

We extend the quantales given above to the quantales with exponential, which correspond to the logical connective !.

The following definition of quantales with exponential will naturally arise from the syntactic properties of ! in linear logic.

4.3.1 Exponential on Quantales

Definition 4.3.1 (exponential) Let $\mathbf{Q} = \langle Q, \vee, \bullet, 1 \rangle$ be a quantale. An *exponential* ! on $\mathbf{Q}$ is an unary operator such that

1. $!x \sqsubseteq x$ for all $x \in Q$,
2. if $!x \sqsubseteq y$, then $!x \sqsubseteq !y$ for all $x, y \in Q$,
3. $1 = !\top$,
4. $!x \bullet !y = !(x \wedge y)$ for all $x, y \in Q$.

Lemma 4.3.2 Clause 2 of the definition is equivalent to the following:

$$\text{if } x \sqsubseteq y, \text{ then } !x \sqsubseteq !y \text{ and } !x \sqsubseteq !!y.$$

Proof. We show that if $!x \sqsubseteq y$, then $!x \sqsubseteq !y$. Since $!x \sqsubseteq y$, $!!x \sqsubseteq !y$, and since $x \sqsubseteq x$, $!x \sqsubseteq !!x$. Therefore $!x \sqsubseteq !y$. Next we show if $x \sqsubseteq y$, then $!x \sqsubseteq !y$ and $!x \sqsubseteq !!y$. Since $x \sqsubseteq y$, $!x \sqsubseteq y$ by 1 of the definition, and hence $!x \sqsubseteq !y$ and $!x \sqsubseteq !!y$ by 2 of the definition. $\blacksquare$

Lemma 4.3.3 In every quantale with exponential, the following holds:

1. $!!x = !x$,
2. $1 = !1$.

Proof.

1. $!!x \sqsubseteq !x$ by 1 of the definition. Since $!x \sqsubseteq !x$, $!x \sqsubseteq !!x$ by 2 of the definition.

2. $!1 \sqsubseteq 1$ by 1 of definition. Since $1 = !\top$ and $1 \sqsubseteq 1$, $!\top = 1 \sqsubseteq 1$, and hence $1 \sqsubseteq !1$ by 2 of the definition.

■

Lemma 4.3.4 In every quantale with exponential, the following holds:

1. $!x \sqsubseteq 1$,

2. $!x \sqsubseteq !x \bullet !x$,

3. $!x \bullet !y \sqsubseteq !(!x \bullet !y)$,

4. if $x \sqsubseteq y$, then $!x \sqsubseteq !y$.

Proof.

1. Since $x \sqsubseteq \top$, $!x \sqsubseteq \top$ by 1 of the definition, and $!x \sqsubseteq !\top$ by 2 of the definition, so $!x \sqsubseteq 1$ by 3 of the definition.

2. $!x = !(x \wedge x) = !x \bullet !x$ by 4 of the definition.

3. $!x \bullet !y = !(x \wedge y)$ by 4 of the definition. Since $!(x \wedge y) \sqsubseteq !(x \wedge y)$, $!(x \wedge y) \sqsubseteq !!(x \wedge y)$ by 2 of the definition. Therefore $!x \bullet !y \sqsubseteq !!(x \wedge y) = !(!x \bullet !y)$.

4. Since we have $!x \sqsubseteq y$ from $x \sqsubseteq y$ by 1 of the definition, $!x \sqsubseteq !y$ by 2 of the definition.

■

4.3.2 FS-quantales and Quantales with Exponential

Here we discuss two definitions of the quantale with exponential !. The former definition is used in [15, 16, 17]. Using this definition, we could not prove the completeness of linear logic for Petri net models with exponential. Therefore we use the latter definition and prove it.

In this thesis, we call the quantale defined by the former definition "FS-quantale with exponential", and call the quantale defined by the latter definition simply "quantale with exponential".

1. **FS-quantales with exponential**

   In linear logic, given a proposition a, the assertion of $!a$ has the possibility of being instantiated by the proposition a, 1 or $!a * !a$.

   The definition of FS-quantales with exponential will naturally arise from the following syntactic properties of ! in linear logic, the rules of (1) and (2). The formal definition of exponential ! of linear logic will be given in Chapter 6.

- From the rules of $!a \Rightarrow a$, $!a \Rightarrow 1$ and $!a \Rightarrow !a * !a$,

$$!a \Rightarrow a \sqcap 1 \sqcap !a * !a \cdots (1),$$

- from the rule of free storage,

$$\frac{b \Rightarrow a \quad b \Rightarrow 1 \quad b \Rightarrow b * b}{b \Rightarrow !a} \cdots (2).$$

As an interpretation of $!a$, for an element a of a lattice, we require an element x from (1) such that

$$x \sqsubseteq a \wedge 1 \wedge (x \bullet x).$$

This will not in general characterize a unique value of the lattice. For instance taking x to be the bottom element $\perp$ of the lattice will always do. How ever from (2) it follows that any x satisfying $x \sqsubseteq a \wedge 1 \wedge (x \bullet x)$ should be below $!a$. Therefore $!a$ should be the greatest fixed point of the monotone mapping

$$x \longrightarrow a \wedge 1 \wedge (x \bullet x)$$

in the complete lattice given together with the quantale.

(In Chapter 2, we have shown that for a complete lattice L and a monotone mapping h on L, there exists fixed point, and especially,

$$\bigvee \{x \in L \mid x \sqsubseteq h(x)\}$$

is the greatest fixed point of L.)

Such a solution ensures the soundness of the proof rules extended by those for !a.

We define FS-quantale with exponential $\mathbf{Q}^! = \langle Q, \vee, \bullet, 1, ! \rangle$ as follows:

Definition 4.3.5 (commutative FS-quantale with exponential) A quantale with exponential $\mathbf{Q}^! = \langle Q, \vee, \bullet, 1, ! \rangle$ is a *commutative FS-quantale with exponential* if

$$b \sqsubseteq a, b \sqsubseteq 1 \text{ and } b \sqsubseteq b \bullet b \text{ implies } b \sqsubseteq !a.$$

We shall use simply *FS-quantale with exponential* for commutative FS-quantale with exponential.

Proposition 4.3.6 *Let* $\mathbf{Q} = \langle Q, \vee, \bullet, 1 \rangle$ *be a quantale, and for each* $a \in Q$*, define an operator ! on* Q *by*

$$!a := \bigvee \{x \in Q | x \sqsubseteq a \wedge 1 \wedge (x \bullet x)\}.$$

Then $\mathbf{Q}^! = \langle Q, \vee, \bullet, 1, ! \rangle$ *is an FS-quantale with exponential.*

Proof. It is immediate from the definition of exponential. ∎

2. **quantales with exponential**

   This definition of $!a$ is different from one in [15, 16, 17]. They define $!a$ for the linear logic with the rule of free storage [47], and to be the greatest fixed point of a mapping corresponding to the free storage rule.

   We define quantale with exponential $\mathbf{Q}^!_{\mathrm{F}} = \langle Q, \vee, \bullet, 1, !, \mathrm{F}\rangle$ as follows:

Definition 4.3.7 (commutative quantale with exponential) Let $\mathbf{Q} = \langle Q, \vee, \bullet, 1\rangle$ be a quantale, and F be a subset of Q such that

(a) if $x, y \in \mathrm{F}$, then $x \bullet y \in \mathrm{F}$,

(b) $x \bullet x = x$ for all $x \in \mathrm{F}$,

(c) $1 \in \mathrm{F}$ and $x \sqsubseteq 1$ for all $x \in \mathrm{F}$.

Then a structure $\mathbf{Q}^!_{\mathrm{F}} = \langle Q, \vee, \bullet, 1, !, \mathrm{F}\rangle$ is a *commutative quantale with exponential*

$$!a = \bigvee\{x \in \mathrm{F} \mid x \sqsubseteq a\}.$$

We shall use simply *quantale with exponential* for commutative quantale with exponential.

Proposition 4.3.8 *$!a = \bigvee\{x \in \mathrm{F} \mid x \sqsubseteq a\}$ defines an exponential over* $\mathbf{Q}$.

Proof. We show that $!a = \bigvee\{x \in \mathrm{F} \mid x \sqsubseteq a\}$ holds Definition 4.3.1.

(a) $!x \sqsubseteq x$ is immediate from condition of definition. Since $!x = \bigvee\{y \in \mathrm{F} \mid y \sqsubseteq x\}$, then $!x \sqsubseteq x$.

(b) We show that if $!x \sqsubseteq y$, then $!x \sqsubseteq !y$. Suppose that $!x \sqsubseteq y$. From definition

$$!x = \bigvee\{z \in F \mid z \sqsubseteq x\} \text{ and}$$

$$!y = \bigvee\{z \in F \mid z \sqsubseteq y\}.$$

Since $!x = \bigvee\{z \in F \mid z \sqsubseteq x\} \sqsubseteq y$, if $z \sqsubseteq x$, then $z \sqsubseteq y$, for all $z \in F$. Therefore we have $!x \sqsubseteq !y$.

(c) We show $1 = !\top$. Since

$$!\top = \bigvee\{x \in F \mid x \sqsubseteq \top\},$$

and from condition of the definition, it is immediate. Because since $1 \in F$ and $x \sqsubseteq 1$ for all $x \in F$, $!\top \sqsubseteq 1$, and since $1 \sqsubseteq \top$ and $1 \in F$, $1 \sqsubseteq !\top$.

(d) We show that $!x \bullet !x = !(x \wedge y)$ for all $x, y \in Q$.

- First we show that $!x \bullet !y \sqsubseteq !(x \wedge y)$. Since $!y \sqsubseteq !\top = 1$,

$$\begin{aligned} !x \bullet !y &\sqsubseteq !x \bullet !\top \\ &= !x \bullet 1 \\ &= !x. \end{aligned}$$

Therefore $!x \bullet !y \sqsubseteq !x$, and similarly $!x \bullet !y \sqsubseteq !y$. Since $!x \sqsubseteq x$ and $!y \sqsubseteq y$ from 1 of definition, $!x \bullet !y \sqsubseteq x$ and $!x \bullet !y \sqsubseteq y$, and then $!x \bullet !y \sqsubseteq x \wedge y$. Also

$$\begin{aligned}
!x \bullet !y &= \bigvee_{z,z' \in F} \{z \bullet z' \mid z \sqsubseteq x, z' \sqsubseteq y\} \\
&\quad \text{(by distributivity of } \vee \text{ over } \bullet) \\
&\sqsubseteq \bigvee_{z,z' \in F} \{z \bullet z' \mid z \sqsubseteq !x, z' \sqsubseteq !y\} \\
&\quad \text{(since } z \in F \text{ and } z \sqsubseteq u \text{ implies } z \sqsubseteq !u) \\
&\sqsubseteq \bigvee \{z \in F \mid z \sqsubseteq !x \bullet !y\} \\
&= !(!x \bullet !y),
\end{aligned}$$

hence $!x \bullet !y \sqsubseteq !(!x \bullet !y) \sqsubseteq !(x \wedge y)$.

- Next we show the converse, $!(x \wedge y) \sqsubseteq !x \bullet !y$. On the other hand,

$$!(x \wedge y) = \bigvee \{z \in F \mid z \sqsubseteq x \wedge y\}.$$

For all $z \in Q$, if $z \sqsubseteq x \wedge y$, then $z \sqsubseteq x$ and $z \sqsubseteq y$, which in turn implies $z \sqsubseteq !x$ and $z \sqsubseteq !y$, and hence $z = z \bullet z \sqsubseteq !x \bullet !y$. Therefore $!(x \wedge y) \sqsubseteq !x \bullet !y$.

■

Chapter 5

Intuitionistic Linear Logic without Exponential

In this chapter, we will first describe syntax and semantics of linear logic without exponential, and prove soundness theorem for quantales. Then we will prove completeness of linear logic without exponential for quantales induced from Petri nets by using the closure operation C_2.

In this thesis, we follow notation of linear logic in [47]. Therefore we have kept Girard's symbols $1, \top$, and replaced $\otimes, \oplus, \&$ by $*, \sqcup, \sqcap$ respectively, and interchanged $\perp$ and 0. For technical background on linear logic, see [19, 21, 22].

5.1 Syntax

5.1.1 Formulas

The *language* of linear logic without exponential has an alphabet consisting of

propositional variables: $a, b, c, \ldots$,

propositional constants: $1, \top, \perp$,

connectives: $*, \sqcup, \sqcap, \multimap$ and

auxiliary symbols: (,).

The connectives carry traditional names:

$*$: multiplicative conjunction (times),

$\sqcup$: disjunction (or),

$\sqcap$: additive conjunction (and) and

$\multimap$: linear implication.

Formulas are inductively defined by

the propositional variables and constants are formulas and

if A and B are formulas, then $(A * B), (A \sqcup B), (A \sqcap B)$ and $(A \multimap B)$ are formulas.

We shall use $A \equiv B$ as an abbreviation for $(A \multimap B) \sqcap (B \multimap A)$, and denote the set of all formulas by Φ.

5.1.2 Sequents

A *sequent* of linear logic without exponential is an expression of the form

$$\Gamma \Rightarrow C,$$

where Γ is a finite sequence of formulas and C is a formula. Both Γ and C may be empty. In the sequel, capital Greek letters will denote finite (possibly empty) sequences of formulas.

5.1.3 Axioms (initial sequents) and Rules

The calculus is given by specifying axioms and rules as following.

Definition 5.1.1 (axioms and rules of inference) The axioms of linear logic without exponential are the instances of the following four axiom-schemes:

$$A \Rightarrow A,$$

$$\Rightarrow 1,$$

$$\Gamma \Rightarrow \top,$$

$$\Gamma, \bot, \Delta \Rightarrow A.$$

The rules of inference of linear logic without exponential are the following structural rules:

$$\frac{\Gamma, \Delta \Rightarrow A}{\Gamma, 1, \Delta \Rightarrow A} \text{ (1 - weakening)},$$

$$\frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, B, A, \Delta \Rightarrow C} \text{ (exchange)},$$

$$\frac{\Gamma \Rightarrow A \quad \Delta, A, \Sigma \Rightarrow C}{\Delta, \Gamma, \Sigma \Rightarrow C} \text{ (cut)},$$

and the following logical rules:

$$\frac{\Gamma, A, \Delta \Rightarrow C \quad \Gamma, B, \Delta \Rightarrow C}{\Gamma, A \sqcup B, \Delta \Rightarrow C} (\sqcup \Rightarrow),$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \sqcup B} (\Rightarrow \sqcup 1), \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \sqcup B} (\Rightarrow \sqcup 2),$$

$$\frac{\Gamma, A, \Delta \Rightarrow C}{\Gamma, A \sqcap B, \Delta \Rightarrow C} (\sqcap 1 \Rightarrow), \qquad \frac{\Gamma, B, \Delta \Rightarrow C}{\Gamma, A \sqcap B, \Delta \Rightarrow C} (\sqcap 2 \Rightarrow),$$

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \sqcap B}\ (\Rightarrow \sqcap),$$

$$\frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, A * B, \Delta \Rightarrow C}\ (* \Rightarrow), \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A * B}\ (\Rightarrow *),$$

$$\frac{\Gamma \Rightarrow A \quad \Delta, B, \Sigma \Rightarrow C}{\Delta, A \multimap B, \Gamma, \Sigma \Rightarrow C}\ (\multimap \Rightarrow), \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \multimap B}\ (\Rightarrow \multimap).$$

Note that in the absence of contraction and weakening in linear logic,

$$\frac{\Gamma, A, A \Rightarrow B}{\Gamma, A \Rightarrow B}\ (\text{contraction}), \qquad \frac{\Gamma \Rightarrow B}{\Gamma, A \Rightarrow B}\ (\text{weakening}),$$

the choice of rules for conjunction leads to distinct connective $\sqcap$ and $*$, and the distributivity of $\sqcap$ over $\sqcup$ becomes not to hold in general.

Linear logic may be viewed as an example of a "resource-conscious" logic, where the formulas represent types of resource, and resources cannot be used ad libitum.

Example 5.1.2 The following two simple examples show why linear logic is called resource-conscious logic.

- Asserting a sequent

  $$A, A \Rightarrow B$$

  means something like: we use two data (resources) of type A to obtain one datum of type B.

- The following logical rules:

  $$\frac{A \Rightarrow B \quad A \Rightarrow C}{A, A \Rightarrow B * C}\ (\Rightarrow *)$$

  means that we can derive lower sequent $A, A \Rightarrow B * C$ from upper sequents $A \Rightarrow B$ and $A \Rightarrow C$, i.e., if we use one datum of type A to obtain one datum of type B, and we use one datum of type A to obtain one datum of type C, respectively, then we use two data of type A to obtain one datum of type B and one datum of type C.

5.1.4 Examples of Proofs

For understanding of the rules described above, we give the following examples.

Example 5.1.3 We can derive the rule

$$\frac{\Gamma \Rightarrow A \multimap B}{\Gamma, A \Rightarrow B}$$

from $(\multimap \Rightarrow)$ as follows:

$$\frac{\Gamma \Rightarrow A \multimap B \qquad \dfrac{A \Rightarrow A \quad B \Rightarrow B}{A \multimap B, A \Rightarrow B}\ (\multimap \Rightarrow)}{\Gamma, A \Rightarrow B}\ (\text{cut}) .$$

Example 5.1.4 We can derive that $(A * B) \Rightarrow C \equiv A \Rightarrow B \multimap C$, i.e., if $(A * B) \Rightarrow C$, then $A \Rightarrow B \multimap C$ and if $A \Rightarrow B \multimap C$, then $(A * B) \Rightarrow C$.

- First we show that if $(A * B) \Rightarrow C$, then $A \Rightarrow B \multimap C$:

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{A, B \Rightarrow A * B}\,(\Rightarrow *) \qquad A * B \Rightarrow C}{A, B \Rightarrow C}\,(\text{cut})}{A \Rightarrow B \multimap C}\,(\Rightarrow \multimap) \quad .$$

- Next we show that if $A \Rightarrow B \multimap C$, then $(A * B) \Rightarrow C$:

$$\dfrac{\dfrac{A \Rightarrow B \multimap C \qquad \dfrac{B \Rightarrow B \quad C \Rightarrow C}{B \multimap C, B \Rightarrow C}\,(\multimap \Rightarrow)}{A, B \Rightarrow C}\,(\text{cut})}{A * B \Rightarrow C}\,(* \Rightarrow) \quad .$$

Example 5.1.5 We can derive that $(A \sqcup B) * C \equiv (A * C) \sqcup (B * C)$ as follows:

- For $(A \sqcup B) * C \Rightarrow (A * C) \sqcup (B * C)$,

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad C \Rightarrow C}{A, C \Rightarrow A * C}\,(\Rightarrow *)}{A, C \Rightarrow A * C \sqcup B * C}\,(\Rightarrow \sqcup 1) \qquad \dfrac{\dfrac{B \Rightarrow B \quad C \Rightarrow C}{B, C \Rightarrow B * C}\,(\Rightarrow *)}{B, C \Rightarrow A * C \sqcup B * C}\,(\Rightarrow \sqcup 2)}{A \sqcup B, C \Rightarrow A * C \sqcup B * C}\,(\sqcup \Rightarrow)}{(A \sqcup B) * C \Rightarrow A * C \sqcup B * C}\,(* \Rightarrow) \quad .$$

- For $(A * C) \sqcup (B * C) \Rightarrow (A \sqcup B) * C$,

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A \sqcup B}\,(\Rightarrow \sqcup 1) \quad C \Rightarrow C}{A, C \Rightarrow (A \sqcup B) * C}\,(\Rightarrow *)}{A * C \Rightarrow (A \sqcup B) * C}\,(* \Rightarrow) \qquad \dfrac{\dfrac{\dfrac{B \Rightarrow B}{B \Rightarrow A \sqcup B}\,(\Rightarrow \sqcup 2) \quad C \Rightarrow C}{B, C \Rightarrow (A \sqcup B) * C}\,(\Rightarrow *)}{B * C \Rightarrow (A \sqcup B) * C}\,(* \Rightarrow)}{A * C \sqcup B * C \Rightarrow (A \sqcup B) * C}\,(\sqcup \Rightarrow) \quad .$$

Remark 5.1.6 *Although the distributivity of $\sqcap$ over $\sqcup$ does not hold in general, the distributivity of $\sqcup$ over $*$ holds in linear logic.*

Remark 5.1.7 *With the constraction and weakening rules, we can prove $A * B \equiv A \sqcap B$ as follows:*

- *For $A * B \Rightarrow A \sqcap B$, we have*

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, B \Rightarrow A}\,(\textit{weakening}) \qquad \dfrac{B \Rightarrow B}{A, B \Rightarrow B}\,(\textit{weakening})}{A, B \Rightarrow A \sqcap B}\,(\Rightarrow \sqcap)}{A * B \Rightarrow A \sqcap B}\,(* \Rightarrow) \quad .$$

- *For $A \sqcap B \Rightarrow A * B$, we have*

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \sqcap B \Rightarrow A}\,(\sqcap \Rightarrow) \quad \dfrac{B \Rightarrow B}{A \sqcap B \Rightarrow B}\,(\sqcap \Rightarrow)}{A \sqcap B, A \sqcap B \Rightarrow A * B}\,(\Rightarrow *)}{A \sqcap B \Rightarrow A * B}\,(constraction) \quad .$$

Remark 5.1.8 *The distributivity of $\sqcap$ over $\sqcup$ does not hold in general. Although the following proof shows that $(A \sqcap B) \sqcup (A \sqcap C) \Rightarrow A \sqcap (B \sqcup C)$ is derivable in linear logic,*

$$\dfrac{\dfrac{A \Rightarrow A}{A \sqcap B \Rightarrow A}\,(\sqcap 1 \Rightarrow) \quad \dfrac{\dfrac{B \Rightarrow B}{B \Rightarrow B \sqcup C}\,(\Rightarrow \sqcup 1)}{A \sqcap B \Rightarrow B \sqcup C}\,(\sqcap 2 \Rightarrow)}{A \sqcap B \Rightarrow A \sqcap (B \sqcup C)}\,(\Rightarrow \sqcap) \qquad (a),$$

$$\dfrac{\dfrac{A \Rightarrow A}{A \sqcap C \Rightarrow A}\,(\sqcap 1 \Rightarrow) \quad \dfrac{\dfrac{C \Rightarrow C}{C \Rightarrow B \sqcup C}\,(\Rightarrow \sqcup 2)}{A \sqcap C \Rightarrow B \sqcup C}\,(\sqcap 2 \Rightarrow)}{A \sqcap C \Rightarrow A \sqcap (B \sqcup C)}\,(\Rightarrow \sqcap) \qquad (b),$$

and from (a) and (b), we can get

$$\dfrac{A \sqcap B \Rightarrow A \sqcap (B \sqcup C) \quad A \sqcap C \Rightarrow A \sqcap (B \sqcup C)}{(A \sqcap B) \sqcup (A \sqcap C) \Rightarrow A \sqcap (B \sqcup C)}\,(\sqcup \Rightarrow),$$

we cannot prove the sequent $A \sqcap (B \sqcup C) \Rightarrow (A \sqcap B) \sqcup (A \sqcap C)$.

With the contraction and weakening rules, we can prove this as follows:

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, B \Rightarrow A}\,(weakening) \quad \dfrac{B \Rightarrow B}{A, B \Rightarrow B}\,(weakening)}{A, B \Rightarrow A \sqcap B}\,(\Rightarrow \sqcap)}{A, B \Rightarrow (A \sqcap B) \sqcup (A \sqcap C)}\,(\Rightarrow \sqcup 1) \qquad (a),$$

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, C \Rightarrow A}\,(weakening) \quad \dfrac{C \Rightarrow C}{A, C \Rightarrow C}\,(weakening)}{A, C \Rightarrow A \sqcap C}\,(\Rightarrow \sqcap)}{A, C \Rightarrow (A \sqcap B) \sqcup (A \sqcap C)}\,(\Rightarrow \sqcup 2) \qquad (b),$$

and from (a) and (b), we can get

$$\dfrac{\dfrac{\dfrac{A, B \Rightarrow (A \sqcap B) \sqcup (A \sqcap C) \quad A, C \Rightarrow (A \sqcap B) \sqcup (A \sqcap C)}{A, B \sqcup C \Rightarrow (A \sqcap B) \sqcup (A \sqcap C)}\,(\sqcup \Rightarrow)}{A \sqcap (B \sqcup C), A \sqcap (B \sqcup C) \Rightarrow (A \sqcap B) \sqcup (A \sqcap C)}\,(\sqcap 1 \text{ and } \sqcap 2 \Rightarrow)}{A \sqcap (B \sqcup C) \Rightarrow (A \sqcap B) \sqcup (A \sqcap C)}\,(contraction) \quad .$$

5.1.5 Relation between Linear Logic and Petri Nets

We explain the relation between linear logic and Petri nets, and we show that linear logic is useful in the theory of parallelism and resource conscious of Petri nets. For understanding, we give the following simple examples.

Example 5.1.9 Consider the following nets, net-1 and net-2.

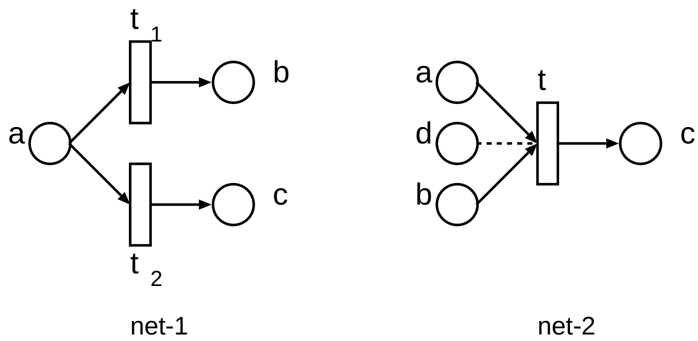

Figure 5.1: Relation between linear logic and Petri net

We show the theory of parallelism with net-1, and the theory of resource conscious with net-2.

1. Consider net-1 with $P = \{a, b, c\}$ and $T = \{t_1, t_2\}$. Pre-multiset ${}^\bullet(t_1)$ and ${}^\bullet(t_2)$ are $[a]$, and post-multiset $(t_1)^\bullet$ and $(t_2)^\bullet$ are $[b]$ and $[c]$ respectively. We consider two cases, one is the net which has one token, and the other is the net which has two tokens, in the place a, respectively.

   (a) Suppose that there is one token in a. A firing of transition t_1 changes the marking from $[a]$ to $[b]$, and a firing of transition t_2 changes the marking from $[a]$ to $[c]$. So the firings of transitions t_1 or t_2 transforms one token in a into b or into c, but not both.

   (b) Suppose that there are two tokens in a. A firing of transition t_1 changes the marking from $[a, a]$ to $[a, b]$, and a firing of transition t_2 changes the marking from $[a, a]$ to $[a, c]$. So the firings of transitions t_1 and t_2 transform two tokens in a into b and into c both.

   In linear logic, we can represent these as following:

$$\frac{a \Rightarrow b \quad a \Rightarrow c}{a \Rightarrow b \sqcap c}\ (\Rightarrow \sqcap) \qquad\qquad (a),$$

$$\frac{a \Rightarrow b \quad a \Rightarrow c}{a * a \Rightarrow b * c}\ (* \Rightarrow, \Rightarrow *) \qquad\qquad (b).$$

2. Consider net-2 with $P = \{a, b, c, d\}$ and $T = \{t\}$. We consider two cases, one is the net which has not an arc, and the other is the net which has an arc, from the place d to the transition t.

   Suppose that there is one token in a, b and d.

   (a) Since there is no arc from the place d to the transition t, Pre-multiset ${}^\bullet(t)$ is $[a, b]$, and post-multiset $(t)^\bullet$ is $[c]$. A firing of transition t changes the marking from $[a, b]$ to $[c]$.

(b) Since there is an arc from the place d to the transition t, Pre-multiset ${}^{\bullet}(t)$ is $[a, d, b]$, and post-multiset $(t)^{\bullet}$ is $[c]$. A firing of transition t changes the marking from $[a, d, b]$ to $[c]$.

In linear logic, we can represent these as following:

$$\frac{a, d, b \Rightarrow c}{a \sqcap d, b \Rightarrow c} \; (\sqcap \Rightarrow) \qquad\qquad (a),$$

$$\frac{a, d, b \Rightarrow c}{a * d, b \Rightarrow c} \; (* \Rightarrow) \qquad\qquad (b).$$

Parallelism and Resource-consciousness of linear logic seem useful to research the connection between linear logic and the theory of Petri nets.

5.2 Semantics

5.2.1 Valuation on Quantale

Definition 5.2.1 (valuation) A *valuation* v on a quantale $\mathbf{Q} = \langle Q, \vee, \bullet, 1\rangle$ is a mapping of Φ into Q satisfying the following conditions: for every $A, B \in \Phi$,

1. $v(A \sqcap B) = v(A) \wedge v(B)$,
2. $v(A \sqcup B) = v(A) \vee v(B)$,
3. $v(A * B) = v(A) \bullet v(B)$,
4. $v(A \multimap B) = v(A) {-\!\bullet} v(B)$,
5. $v(\top) = \top$,
6. $v(\bot) = \bot$,
7. $v(1) = 1$.

5.2.2 Validity

Definition 5.2.2 (valid) A formula A is said to be

1. *true* in a valuation v on a quantale $\mathbf{Q}$ if
$$1 \sqsubseteq v(A)$$
holds, which will be denoted by
$$\mathbf{Q}, v \models A;$$
2. *valid* with respect to a class $\mathcal{Q}$ of quantales if for each quantale $\mathbf{Q} \in \mathcal{Q}$ and each valuation v on $\mathbf{Q}$,
$$\mathbf{Q}, v \models A$$
holds, which will be denoted by
$$\mathcal{Q} \models A;$$

and a sequent $\Gamma \Rightarrow A$ is said to be *valid* with respect to $\mathcal{Q}$ if and only if

$$\mathcal{Q} \models \Gamma^* \multimap A,$$

where Γ^* is a formula defined by $\emptyset^* := 1$ and $(\Gamma, A)^* := \Gamma^* * A$.

5.2.3 Soundness

The soundness theorem can be shown as usual (see e.g. [47]).

Theorem 5.2.3 (soundness) *If a sequent $\Gamma \Rightarrow A$ is provable in linear logic, then it is valid with respect to the class of all quantales.*

Proof. Soundness is proved by a straightforward induction on hight of proof.

- Initial sequents are valid,
- for the rules of inference (structural rules and logical rules), if upper sequent(s) is valid, then lower sequent is valid.

We show that initial sequents, structural rules and logical rules are valid. For structural rules and logical rules, we show that if upper sequent(s) is valid, then lower sequent is valid.

1. We show that initial sequents are valid.

   (a) $\models A \Rightarrow A$.
   We show that $\models A \multimap A$. By the definition, it suffices to show that

   $$1 \sqsubseteq v(A \multimap A).$$

   Since $v(A) \sqsubseteq v(A)$, $1 \bullet v(A) \sqsubseteq v(A)$, and hence $1 \sqsubseteq v(A) \multimap\!\!\bullet\, v(A)$. Therefore $1 \sqsubseteq v(A \multimap A)$.

   (b) $\models \Rightarrow 1$.
   We show that $\models 1$. It suffices to show that

   $$1 \sqsubseteq v(1).$$

   Since $1 \sqsubseteq 1$ and $v(1) = 1$, $1 \sqsubseteq v(1)$.

   (c) $\models \Gamma \Rightarrow \top$.
   We show that $\models \Gamma^* \multimap \top$. It suffices to show that

   $$1 \sqsubseteq v(\Gamma^* \multimap \top).$$

   Since $\top$ is the greatest element, $v(\Gamma^*) \sqsubseteq \top$. Therefore $1 \bullet v(\Gamma^*) \sqsubseteq v(\top)$, and hence $1 \sqsubseteq v(\Gamma^*) \multimap\!\!\bullet\, v(\top)$, and so $1 \sqsubseteq v(\Gamma^* \multimap \top)$.

   (d) $\models \Gamma, \bot, \Delta \Rightarrow A$.
   We show that $\models \Gamma^* * \bot * \Delta^* \multimap A$. It suffices to show that

   $$1 \sqsubseteq v(\Gamma^* * \bot * \Delta^* \multimap A).$$

   Since $\bot$ is the least element, $v(\bot) \sqsubseteq v(\Gamma^*) \bullet v(\Delta^*) \multimap\!\!\bullet\, v(A)$. Then $v(\bot) \bullet v(\Gamma^*) \bullet v(\Delta^*) \sqsubseteq v(A)$, and hence $v(\Gamma^*) \bullet v(\bot) \bullet v(\Delta^*) \sqsubseteq v(A)$. Therefore $1 \sqsubseteq (v(\Gamma^*) \bullet v(\bot) \bullet v(\Delta^*)) \multimap\!\!\bullet\, v(A)$, and so $1 \sqsubseteq v(\Gamma^* * \bot * \Delta^*) \multimap\!\!\bullet\, v(A)$. Thus $1 \sqsubseteq v(\Gamma^* * \bot * \Delta^* \multimap A)$.

2. We show that structural rules are valid.

   (a) 1-weakening.

   - Since $\models \Gamma, \Delta \Rightarrow A$, $\models \Gamma^* * \Delta^* \multimap A$. Then $1 \sqsubseteq v(\Gamma^* * \Delta^* \multimap A)$, and hence $1 \sqsubseteq v(\Gamma^* * \Delta^*) \multimap\!\!\bullet\, v(A)$. Therefore $1 \sqsubseteq v(\Gamma^*) \bullet v(\Delta^*) \multimap\!\!\bullet\, v(A)$, and so

   $$v(\Gamma^*) \bullet v(\Delta^*) \sqsubseteq v(A).$$

   - We show that if $\models \Gamma, \Delta \Rightarrow A$, then $\models \Gamma, 1, \Delta \Rightarrow A$. By the definition, it suffices to show that

   $$\models \Gamma^* * 1 * \Delta^* \multimap A.$$

   Since $v(\Gamma^*) \bullet v(\Delta^*) \sqsubseteq v(A)$, $v(\Gamma^*) \bullet v(1) \bullet v(\Delta^*) \sqsubseteq v(A)$, and hence $1 \sqsubseteq v(\Gamma^*) \bullet v(1) \bullet v(\Delta^*) \multimap\!\!\bullet\, v(A)$. Thus $1 \sqsubseteq v(\Gamma^* * 1 * \Delta^* \multimap A)$, and so $\models \Gamma^* * 1 * \Delta^* \multimap A$.

   (b) exchange.

   - Since $\models \Gamma, A, B, \Delta \Rightarrow C$, $\models \Gamma^* * A * B * \Delta^* \multimap C$. Then $1 \sqsubseteq v(\Gamma^* * A * B * \Delta^* \multimap C)$, and hence $1 \sqsubseteq v(\Gamma^*) \bullet v(A) \bullet v(B) \bullet v(\Delta^*) \multimap\!\!\bullet\, v(C)$. Thus

   $$v(\Gamma^*) \bullet v(A) \bullet v(B) \bullet v(\Delta^*) \sqsubseteq v(C).$$

   - We show that if $\models \Gamma, A, B, \Delta \Rightarrow C$, then $\models \Gamma, B, A, \Delta \Rightarrow C$. By the definition, it suffices to show that

   $$\models \Gamma^* * B * A * \Delta^* \multimap C.$$

   Since $v(\Gamma^*) \bullet v(A) \bullet v(B) \bullet v(\Delta^*) \sqsubseteq v(C)$, $v(\Gamma^*) \bullet v(B) \bullet v(A) \bullet v(\Delta^*) \sqsubseteq v(C)$, and hence $1 \sqsubseteq v(\Gamma^*) \bullet v(B) \bullet v(A) \bullet v(\Delta^*) \multimap\!\!\bullet\, v(C)$. Therefore $1 \sqsubseteq v(\Gamma^* * A * b * \Delta^* \multimap C)$, and hence $\models \Gamma^* * B * A * \Delta^* \multimap C$.

   (c) cut.

   - Since $\models \Gamma \Rightarrow A$, $\models \Gamma^* \multimap A$. Then $1 \sqsubseteq v(\Gamma^* \multimap A)$, and hence $1 \sqsubseteq v(\Gamma^*) \multimap\!\!\bullet\, v(A)$. Thus

   $$v(\Gamma^*) \sqsubseteq v(A) \cdots (1).$$

   - Since $\models \Delta, A, \Sigma \Rightarrow C$, $\models \Delta^* * A * \Sigma^* \multimap C$. Then $1 \sqsubseteq v(\Delta^* * A * \Sigma^* \multimap C)$, and hence $1 \sqsubseteq v(\Delta^*) \bullet v(A) \bullet v(\Sigma^*) \multimap\!\!\bullet\, v(C)$. Therefore $v(\Delta^*) \bullet v(A) \bullet v(\Sigma^*) \sqsubseteq v(C)$, and so

   $$v(A) \sqsubseteq v(\Delta^*) \bullet v(\Sigma^*) \multimap\!\!\bullet\, v(C) \cdots (2).$$

   - We show that if $\models \Gamma \Rightarrow A$ and $\models \Delta, A, \Sigma \Rightarrow C$, then $\models \Delta, \Gamma, \Sigma \Rightarrow C$. By the definition, it suffices to show that

   $$\models \Delta^* * \Gamma^* * \Sigma^* \multimap C.$$

   Since $v(\Gamma^*) \sqsubseteq v(\Delta^*) \bullet v(\Sigma^*) \multimap\!\!\bullet\, v(C)$ from (1) and (2),

   $$v(\Delta^*) \bullet v(\Gamma^*) \bullet v(\Sigma^*) \sqsubseteq v(C).$$

   Therefore $1 \sqsubseteq v(\Delta^*) \bullet v(\Gamma^*) \bullet v(\Sigma^*) \multimap\!\!\bullet\, v(C)$, and hence $1 \sqsubseteq v(\Delta^* * \Gamma^* * \Sigma^*) \multimap\!\!\bullet\, v(C)$. Thus $1 \sqsubseteq v(\Delta^* * \Gamma^* * \Sigma^* \multimap C)$, and so $\models \Delta^* * \Gamma^* * \Sigma^* \multimap C$.

3. We show that logical rules are valid.

   (a) $\sqcup \Rightarrow$.

   - Since $\models \Gamma, A, \Delta \Rightarrow C$, $\models \Gamma^* * A * \Delta^* \multimap C$. Then $1 \sqsubseteq v(\Gamma^* * A * \Delta^* \multimap C)$, and hence $1 \sqsubseteq v(\Gamma^*) \bullet v(A) \bullet v(\Delta^*) \multimap\!\!\bullet\, v(C)$. Therefore $v(\Gamma^*) \bullet v(A) \bullet v(\Delta^*) \sqsubseteq v(C)$, and so
$$v(A) \sqsubseteq v(\Gamma^*) \bullet v(\Delta^*) \multimap\!\!\bullet\, v(C) \cdots (1).$$
   - Since $\models \Gamma, B, \Delta \Rightarrow C$, $\models \Gamma^* * B * \Delta^* \multimap C$. Then $1 \sqsubseteq v(\Gamma^* * B * \Delta^* \multimap C)$, and hence $1 \sqsubseteq v(\Gamma^*) \bullet v(B) \bullet v(\Delta^*) \multimap\!\!\bullet\, v(C)$. Therefore $v(\Gamma^*) \bullet v(B) \bullet v(\Delta^*) \sqsubseteq v(C)$, and so
$$v(B) \sqsubseteq v(\Gamma^*) \bullet v(\Delta^*) \multimap\!\!\bullet\, v(C) \cdots (2).$$
   - We show that if $\models \Gamma, A, \Delta \Rightarrow C$ and $\models \Gamma, B, \Delta \Rightarrow C$, then $\models \Gamma, A \sqcup B, \Delta \Rightarrow C$. By the definition, it suffices to show that
$$\models \Gamma^* * A \sqcup B * \Delta^* \multimap C.$$
     Since $v(A) \vee v(B) \sqsubseteq v(\Gamma^*) \bullet v(\Delta^*) \multimap\!\!\bullet\, v(C)$ from (1) and (2),
$$v(\Gamma^*) \bullet v(A \sqcup B) \bullet v(\Delta^*) \sqsubseteq v(C).$$
     Therefore $1 \sqsubseteq v(\Gamma^*) \bullet v(A \sqcup B) \bullet v(\Delta^*) \multimap\!\!\bullet\, v(C)$, and hence $1 \sqsubseteq v(\Gamma^* * (A \sqcup B) * \Delta^*) \multimap\!\!\bullet\, v(C)$. Thus $1 \sqsubseteq v(\Gamma^* * (A \sqcup B) * \Delta^* \multimap C)$, and so $\models \Gamma^* * (A \sqcup B) * \Delta^* \multimap C$.

   (b) $\Rightarrow \sqcup 1$.

   - Since $\models \Gamma \Rightarrow A$, $v(\Gamma^*) \sqsubseteq v(A)$ by cut $\cdots (1)$.
   - We show that if $\models \Gamma \Rightarrow A$, then $\models \Gamma \Rightarrow A \sqcup B$. By the definition, it suffices to show that
$$\models \Gamma^* \multimap (A \sqcup B).$$
     Since $v(\Gamma^*) \sqsubseteq v(A)$, $v(\Gamma^*) \sqsubseteq v(A) \vee v(B)$. Therefore $v(\Gamma^*) \sqsubseteq v(A \sqcup B)$, and hence $1 \sqsubseteq v(\Gamma^*) \multimap\!\!\bullet\, v(A \sqcup B)$. Thus $1 \sqsubseteq v(\Gamma^* \multimap A \sqcup B)$, and so $\models \Gamma^* \multimap (A \sqcup B)$.

   (c) The proof is similar for ($\Rightarrow \sqcup 2$).

   (d) $\sqcap 1 \Rightarrow$.

   - Since $\models \Gamma, A, \Delta \Rightarrow C$, $v(A) \sqsubseteq v(\Gamma^*) \bullet v(\Delta^*) \multimap\!\!\bullet\, v(C)$ by cut $\cdots (2)$.
   - We show that if $\models \Gamma, A, \Delta \Rightarrow C$, then $\models \Gamma, A \sqcap B, \Delta \Rightarrow C$. By the definition, it suffices to show that
$$\models \Gamma^* * (A \sqcap B) * \Delta^* \multimap C.$$
     Since $v(A) \sqsubseteq v(\Gamma^*) \bullet v(\Delta^*) \multimap\!\!\bullet\, v(C)$, $v(A) \wedge v(B) \sqsubseteq v(\Gamma^*) \bullet v(\Delta^*) \multimap\!\!\bullet\, v(C)$, and then
$$v(A \sqcap B) \sqsubseteq v(\Gamma^*) \bullet v(\Delta^*) \multimap\!\!\bullet\, v(C).$$
     Therefore $v(\Gamma^*) \bullet v(A \sqcap B) \bullet v(\Delta^*) \sqsubseteq v(C)$, and hence $1 \sqsubseteq v(\Gamma^* * (A \sqcap B) * \Delta^*) \multimap\!\!\bullet\, v(C)$. Thus $1 \sqsubseteq v(\Gamma^* * (A \sqcap B) * \Delta^* \multimap C)$, and so $\models \Gamma^* * (A \sqcap B) * \Delta^* \multimap C$.

   (e) The proof is similar for $\sqcap 2 \Rightarrow$.

(f) $\Rightarrow \sqcap$.

- By cut $\cdots$ (1), since $\models \Gamma \Rightarrow A$, $v(\Gamma^*) \sqsubseteq v(A) \cdots (1)$ and
- since $\models \Gamma \Rightarrow B$, $v(\Gamma^*) \sqsubseteq v(B) \cdots (2)$.
- We show that if $\models \Gamma \Rightarrow A$ and $\models \Gamma \Rightarrow B$, then $\models \Gamma \Rightarrow A \sqcap B$. By the definition, it suffices to show that

$$\models \Gamma^* \multimap (A \sqcap B).$$

Since $v(\Gamma^*) \sqsubseteq v(A) \wedge v(B)$ from (1) and (2),

$$v(\Gamma^*) \sqsubseteq v(A \sqcap B).$$

Therefore $1 \sqsubseteq v(\Gamma^*) \multimap\!\bullet\, v(A \sqcap B)$, and so $1 \sqsubseteq v(\Gamma^* \multimap (A \sqcap B))$. Thus $\models \Gamma^* \multimap (A \sqcap B)$.

(g) $* \Rightarrow$.

- Since $\models \Gamma, A, B, \Delta \Rightarrow C$, $\models \Gamma^* * A * B * \Delta^* \multimap C$. Therefore $1 \sqsubseteq v(\Gamma^* * A * B * \Delta^*) \multimap\!\bullet\, v(C)$, and hence $1 \sqsubseteq v(\Gamma^*) \bullet v(A) \bullet v(B) \bullet v(\Delta^*) \multimap\!\bullet\, v(C)$. Thus

$$v(\Gamma^*) \bullet v(A) \bullet v(B) \bullet v(\Delta^*) \sqsubseteq v(C).$$

- We show that if $\models \Gamma, A, B, \Delta \Rightarrow C$, then $\models \Gamma, A * B, \Delta \Rightarrow C$. By the definition, it suffices to show that

$$\models \Gamma^* * (A * B) * \Delta^* \multimap C.$$

Since $v(\Gamma^*) \bullet v(A) \bullet v(B) \bullet v(\Delta^*) \sqsubseteq v(C)$, $v(\Gamma^*) \bullet v(A * B) \bullet v(\Delta^*) \sqsubseteq v(C)$. Therefore $1 \sqsubseteq v(\Gamma^*) \bullet v(A * B) \bullet v(\Delta^*) \multimap\!\bullet\, v(C)$, and so $1 \sqsubseteq v(\Gamma^* * (A * B) * \Delta^* \multimap C)$. Thus $\models \Gamma^* * (A * B) * \Delta^* \multimap C$.

(h) $\Rightarrow *$.

- By cut $\cdots$ (1), since $\models \Gamma \Rightarrow A$, $v(\Gamma^*) \sqsubseteq v(A)$ and
- since $\models \Delta \Rightarrow B$, $v(\Delta^*) \leq v(B)$.
- We show that if $\models \Gamma \Rightarrow A$ and $\models \Delta \Rightarrow B$, then $\models \Gamma, \Delta \Rightarrow A * B$. By the definition, it suffices to show that

$$\models \Gamma^* * \Delta^* \multimap A * B.$$

Since $v(\Gamma^*) \bullet v(\Delta^*) \sqsubseteq v(A) \bullet v(B)$, $1 \sqsubseteq v(\Gamma^*) \bullet v(\Delta^*) \multimap\!\bullet\, v(A) \bullet v(B)$. Therefore $1 \sqsubseteq v(\Gamma^* * \Delta^*) \multimap\!\bullet\, v(A * B)$, and so $1 \sqsubseteq v(\Gamma^* * \Delta^* \multimap A * B)$. Thus $\models \Gamma^* * \Delta^* \multimap A * B$.

(i) $\multimap \Rightarrow$.

First we show that if $a \sqsubseteq a'$ and $b \sqsubseteq b'$, then $a' \multimap\!\bullet\, b \sqsubseteq a \multimap\!\bullet\, b'$. Since $a' \multimap\!\bullet\, b \sqsubseteq a' \multimap\!\bullet\, b$, $a' \bullet (a' \multimap\!\bullet\, b) \sqsubseteq b$, and since $a \sqsubseteq a'$ and $b \sqsubseteq b'$,

$$\begin{aligned} a \bullet (a' \multimap\!\bullet\, b) &\sqsubseteq a' \bullet (a' \multimap\!\bullet\, b) \\ &\sqsubseteq b \\ &\sqsubseteq b', \end{aligned}$$

and hence

$$a' \multimap\!\bullet\, b \sqsubseteq a \multimap\!\bullet\, b'.$$

- Since $\models \Gamma \Rightarrow A$, $v(\Gamma^*) \sqsubseteq v(A)$ by cut $\cdots(1)$.
- Since $\models \Delta, B, \Sigma \Rightarrow C$, $v(B) \sqsubseteq v(\Delta^*) \bullet v(\Sigma^*) \multimap\!\!\bullet\, v(C)$ by cut $\cdots(2)$.
- We show that if $\models \Gamma \Rightarrow A$ and $\models \Delta, B, \Sigma \Rightarrow C$, then $\models \Delta, A \multimap B, \Gamma, \Sigma \Rightarrow C$. By the definition, it suffices to show that

  $$\models \Delta^* * (A \multimap B) * \Gamma^* * \Sigma^* \multimap C.$$

  Since $v(A) \multimap\!\!\bullet\, v(B) \sqsubseteq v(\Gamma^*) \multimap\!\!\bullet\, (v(\Delta^*) \bullet v(\Sigma^*) \multimap\!\!\bullet\, v(C))$ by the results above, $(v(A) \multimap\!\!\bullet\, v(B)) \bullet v(\Gamma^*) \sqsubseteq v(\Delta^*) \bullet v(\Sigma^*) \multimap\!\!\bullet\, v(C)$. Then $(v(A) \multimap\!\!\bullet\, v(B)) \bullet v(\Gamma^*) \bullet v(\Delta^*) \bullet v(\Sigma^*) \sqsubseteq v(C)$, and so

  $$v(\Delta^*) \bullet (v(A) \multimap\!\!\bullet\, v(B)) \bullet v(\Gamma^*) \bullet v(\Sigma^*) \sqsubseteq v(C).$$

  Therefore $1 \sqsubseteq (v(\Delta^*) \bullet (v(A) \multimap\!\!\bullet\, v(B)) \bullet v(\Gamma^*) \bullet v(\Sigma^*)) \multimap\!\!\bullet\, v(C)$, and hence $1 \sqsubseteq v(\Delta^*) \bullet v(A \multimap B) \bullet v(\Gamma^*) \bullet v(\Sigma^*) \multimap\!\!\bullet\, v(C)$. Thus $1 \sqsubseteq v(\Delta^* * (A \multimap B) * \Gamma^* * \Sigma^* \multimap C)$, and so $\models \Delta^* * (A \multimap B) * \Gamma^* * \Sigma^* \multimap C$.

(j) $\Rightarrow \multimap$.

- Since $\models \Gamma, A \Rightarrow B$, $\models \Gamma^* * A \multimap B$. Then $1 \sqsubseteq v(\Gamma^* * A \multimap B)$, and hence $1 \sqsubseteq v(\Gamma^* * A) \multimap\!\!\bullet\, v(B)$. Therefore $1 \sqsubseteq v(\Gamma^*) \bullet v(A) \multimap\!\!\bullet\, v(B)$, and so

  $$v(\Gamma^*) \bullet v(A) \sqsubseteq v(B).$$

- We show that if $\models \Gamma, A \Rightarrow B$, then $\models \Gamma \Rightarrow A \multimap B$. By the definition, it suffices to show that

  $$\models \Gamma^* \multimap (A \multimap B).$$

  Since $v(\Gamma^*) \bullet v(A) \sqsubseteq v(B)$, $v(\Gamma^*) \sqsubseteq v(A) \multimap\!\!\bullet\, v(B)$. Therefore

  $$1 \sqsubseteq v(\Gamma^*) \multimap\!\!\bullet\, (v(A) \multimap\!\!\bullet\, v(B)),$$

  and hence $1 \sqsubseteq v(\Gamma^*) \multimap\!\!\bullet\, v(A \multimap B)$. Thus $1 \sqsubseteq v(\Gamma^* \multimap (A \multimap B))$, and so $\models \Gamma^* \multimap (A \multimap B)$.

■

5.3 Completeness

First we discuss the classes of quantales. Then we show how to construct the quantales from Petri nets, and prove completeness of linear logic without exponential for our quantales.

Let $\mathcal{Q}_0$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be the classes of quantales defined by

$$\mathcal{Q}_0 := \{\mathbf{P}(\mathbf{X}_{\mathrm{N}}) \mid \mathrm{N} \text{ is a Petri net}\},$$

$$\mathcal{Q}_1 := \{C_1(\mathbf{P}(\mathbf{X}_{\mathrm{N}})) \mid \mathrm{N} \text{ is a Petri net}\},$$

$$\mathcal{Q}_2 := \{C_2(\mathbf{P}(\mathbf{X}_{\mathrm{N}})) \mid \mathrm{N} \text{ is a Petri net}\}.$$

Then $\mathbf{P}(\mathbf{X}_{\mathrm{N}})$, $C_1(\mathbf{P}(\mathbf{X}_{\mathrm{N}}))$ and $C_2(\mathbf{P}(\mathbf{X}_{\mathrm{N}}))$ are quantales obtained from the preordered commutative monoid $\mathbf{X}_{\mathrm{N}}$ using closure operations C_1 and C_2.

We say that linear logic without exponential is *complete* for a class $\mathcal{Q}$ of quantales, if $\Gamma \Rightarrow A$ is provable in linear logic without exponential whenever $\Gamma \Rightarrow A$ is valid with respect to $\mathcal{Q}$.

$\mathcal{Q}_1$ is the class of quantales used in Engberg and Winskel [15]. In any quantale in $\mathcal{Q}_0$ or $\mathcal{Q}_1$, the distributivity is always valid: in fact, in $\mathcal{Q}_0$ and $\mathcal{Q}_1$, the lattice operations meet and join correspond to usual set operations intersection and union, respectively, and hence the distributivity automatically holds. As mentioned in Chapter 1, any logic which is complete for $\mathcal{Q}_0$ or $\mathcal{Q}_1$, must have the distributivity as a theorem. Here we consider the class $\mathcal{Q}_2$, and prove completeness for $\mathcal{Q}_2$ in which the distributivity is not always valid.

In order to prove completeness, we have constructed quantales from a Petri net as follows (see Figure 5.2):

First we construct a Petri net $\mathrm{N} = \langle P, T, {}^\bullet(-), (-)^\bullet \rangle$. For constructing N, we take formulas as places and sequents (provability) as transitions. Then from N, we construct a preordered commutative monoid $\mathbf{X}_{\mathrm{N}} = \langle \mathcal{M}, \rhd, +, [\,] \rangle$, and from the powerset $\mathbf{P}(\mathbf{X}_{\mathrm{N}}) = \langle \mathcal{P}(\mathcal{M}), \cup, \bullet, \{[\,]\} \rangle$ of the preordered commutative monoid, we construct a quantale $\mathbf{Q}_2 = \langle C_2(\mathcal{P}(\mathcal{M})), \cup_{C_2}, \bullet_{C_2}, C_2(\{[\,]\}) \rangle$ by closure operation C_2. Finally we prove completeness using the quantales.

$\boxed{\text{Petri net} \quad \mathrm{N}}$

$\Downarrow$ *Proposition* 3.2.4

$\boxed{\text{preordered commutative monoid} \quad \mathbf{X}_{\mathrm{N}}}$

$\Downarrow$ *Proposition* 4.1.9

$\boxed{\text{quantale} \quad \mathbf{P}(\mathbf{X}_{\mathrm{N}})}$

$\Downarrow$ *Proposition* 4.2.11

$\boxed{\text{quantale by closure operation } C_2 \quad \mathbf{Q}_2}$

Figure 5.2: The construction of the quantales $\mathbf{Q}_2$.

In the sequel, we shall prove completeness for $\mathcal{Q}_2$

Theorem 5.3.1 (completeness) *If a sequent $\Gamma \Rightarrow A$ is valid in $\mathcal{Q}_2$, then it is provable in linear logic without exponential.*

Proof. First we construct the Petri net $\mathrm{N} = \langle P, T, {}^\bullet(-), (-)^\bullet \rangle$ as follows:

1. $P := \Phi$ (the set of all formulas),
2. $T := \{(\Gamma, \Delta) \mid \Gamma \Rightarrow \Delta^* \text{ is provable in linear logic without exponential}\}$ (in intuitionistic linear logic, the formulas on right hand side of the sequent are restricted at most one formula occurrence, so the formulas in Δ are connected with $*$),
3. for each $t = (\Gamma, \Delta) \in T$,
   (a) ${}^\bullet t := [\Gamma]$,
   (b) $t^\bullet := [\Delta]$.

Then we define a mapping v of Φ into the quantale $\mathbf{Q}_2$ by

$$v(C) := C_2(\{[C]\}).$$

Note that in the preordered commutative monoid $\mathbf{X}_{\mathrm{N}} = \langle \mathcal{M}, \rhd, +, [\,] \rangle$, for a sequence Γ, since $[\Gamma]$ is a multiset consisting of places in Γ,

$$[\Gamma] \rhd [\Delta] \text{ if and only if } \Gamma \Rightarrow \Delta^* \text{ is provable}$$

in linear logic without exponential, and hence

$$C_2(\{[\Gamma]\}) \subseteq C_2(\{[\Delta]\}) \text{ if and only if } \Gamma \Rightarrow \Delta^* \text{ is provable}$$

in linear logic without exponential.

We can show by induction on the complexity of C that v is a valuation on $\mathbf{Q}_2$.

Case 1 . $C \equiv A \sqcap B$. By the definition of v, it suffices to show that

$$C_2(\{[A \sqcap B]\}) = C_2(\{[A]\}) \cap C_2(\{[B]\}).$$

Suppose that $[\Gamma] \in C_2(\{[A \sqcap B]\})$. Then $[\Gamma] \rhd [A \sqcap B]$, and since $A \sqcap B \Rightarrow A$ is provable in linear logic without exponential, $[A \sqcap B] \rhd [A]$. Therefore $[\Gamma] \rhd [A]$, and so $[\Gamma] \in C_2(\{[A]\})$. Similarly we have $[\Gamma] \in C_2(\{[B]\})$. Thus $[\Gamma] \in C_2(\{[A]\}) \cap C_2(\{[B]\})$. Conversely suppose that $[\Gamma] \in C_2(\{[A]\}) \cap C_2(\{[B]\})$. Then $\Gamma \Rightarrow A$ and $\Gamma \Rightarrow B$ are provable in linear logic without exponential, and hence we have

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \sqcap B}\ (\Rightarrow \sqcap).$$

Therefore $[\Gamma] \in C_2(\{[A \sqcap B]\})$.

Case 2 . $C \equiv A \sqcup B$. Since

$$\begin{aligned} v(A) \cup_{C_2} v(B) &= C_2(\{[A]\}) \cup_{C_2} C_2(\{[B]\}) \\ &= C_2(C_2(\{[A]\}) \cup C_2(\{[B]\})) \\ &= C_2(\{[A]\} \cup \{[B]\}), \end{aligned}$$

it suffices to show that

$$C_2(\{[A \sqcup B]\}) = C_2(\{[A]\} \cup \{[B]\}).$$

Suppose that $[\Gamma] \in C_2(\{[A \sqcup B]\})$ and $[\Delta] \in (\{[A]\} \cup \{[B]\})^{\rightarrow}$. Then since $[A] \rhd [\Delta]$ and $[B] \rhd [\Delta]$, we see that $\Gamma \Rightarrow A \sqcup B$, $A \Rightarrow \Delta^*$ and $B \Rightarrow \Delta^*$ are provable in linear logic without exponential, and hence

$$\dfrac{\Gamma \Rightarrow A \sqcup B \qquad \dfrac{A \Rightarrow \Delta^* \quad B \Rightarrow \Delta^*}{A \sqcup B \Rightarrow \Delta^*}(\sqcup \Rightarrow)}{\Gamma \Rightarrow \Delta^*}(\text{cut}) \quad .$$

Thus $[\Gamma] \rhd [\Delta]$. Since $[\Delta]$ is arbitrary, we have $[\Gamma] \in C_2(\{[A]\} \cup \{[B]\})$. Conversely suppose that $[\Gamma] \in C_2(\{[A]\} \cup \{[B]\})$. Since $A \Rightarrow A \sqcup B$ and $B \Rightarrow A \sqcup B$ are provable in linear logic without exponential, we have $[A] \rhd [A \sqcup B]$ and $[B] \rhd [A \sqcup B]$, and hence $[A \sqcup B] \in (\{[A]\} \cup \{[B]\})^{\rightarrow}$. Therefore $[\Gamma] \rhd [A \sqcup B]$, and so $[\Gamma] \in C_2(\{[A \sqcup B]\})$.

Case 3 . $C \equiv A * B$. Since

$$\begin{aligned} v(A) \bullet_{C_2} v(B) &= C_2(\{[A]\}) \bullet_{C_2} C_2(\{[B]\}) \\ &= C_2(C_2(\{[A]\}) \bullet C_2(\{[B]\})) \\ &= C_2(\{[A]\} \bullet \{[B]\}) \\ &= C_2(\{[A] + [B]\}) \\ &= C_2(\{[A, B]\}), \end{aligned}$$

it suffices to show that

$$C_2(\{[A * B]\}) = C_2(\{[A, B]\}).$$

Suppose that $[\Gamma] \in C_2(\{[A * B]\})$. Then $\Gamma \Rightarrow A * B$ is provable in linear logic without exponential, and hence

$$\dfrac{\Gamma \Rightarrow A * B \qquad \dfrac{\dfrac{\dfrac{\Rightarrow 1 \quad A \Rightarrow A}{A \Rightarrow 1 * A}(\Rightarrow *) \qquad B \Rightarrow B}{A, B \Rightarrow (1 * A) * B}(\Rightarrow *)}{A * B \Rightarrow (A, B)^*}(* \Rightarrow)}{\Gamma \Rightarrow (A, B)^*}(\text{cut}) \quad .$$

Thus $[\Gamma] \in C_2(\{[A, B]\})$. Conversely suppose that $[\Gamma] \in C_2(\{[A, B]\})$. Then $[\Gamma] \rhd [A, B]$. Since $A, B \Rightarrow A * B$ is provable in linear logic without exponential, we have $[A, B] \rhd [A * B]$ and hence $[\Gamma] \rhd [A * B]$. Therefore $[\Gamma] \in C_2(\{[A * B]\})$.

Case 4 . $C \equiv A \multimap B$. We show that

$$C_2(\{[A \multimap B]\}) = C_2(\{[A]\}) \multimap\!\bullet\, C_2(\{[B]\}).$$

Suppose that $[\Gamma] \in C_2(\{[A \multimap B]\})$ and $[\Delta] \in C_2(\{[A]\})$. Then $\Gamma \Rightarrow A \multimap B$ and $\Delta \Rightarrow A$ are provable in linear logic without exponential, and hence

$$\dfrac{\Delta \Rightarrow A \quad \dfrac{\Gamma \Rightarrow A \multimap B \quad \dfrac{A \Rightarrow A \quad B \Rightarrow B}{A \multimap B, A \Rightarrow B}(\multimap \Rightarrow)}{\Gamma, A \Rightarrow B}(\text{cut})}{\Gamma, \Delta \Rightarrow B}(\text{cut}).$$

Therefore $[\Gamma, \Delta] \in C_2(\{[B]\})$. Since $[\Gamma, \Delta] = [\Gamma] + [\Delta]$ and $[\Delta]$ is arbitrary, we have $[\Gamma] \in C_2(\{[A]\}) \multimap\!\bullet\, C_2(\{[B]\})$ by Remark 4.1.10 and 4.2.6. Conversely suppose that $[\Gamma] \in C_2(\{[A]\}) \multimap\!\bullet\, C_2(\{[B]\})$. Then $[\Gamma] + [A] \in C_2(\{[B]\})$ as $[A] \in C_2(\{[A]\})$. Hence $[\Gamma] + [A] \rhd [B]$, and so $[\Gamma, A] \rhd [B]$. Thus $\Gamma, A \Rightarrow B$, and hence

$$\dfrac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \multimap B}(\Rightarrow \multimap).$$

Therefore $[\Gamma] \in C_2(\{[A \multimap B]\})$.

Case 5 . $C \equiv \top$. For any $[\Gamma] \in \mathcal{M}$, since $\Gamma \Rightarrow \top$ is provable in linear logic without exponential, we have $[\Gamma] \in C_2(\{[\top]\})$. Hence

$$\mathcal{M} \subseteq C_2(\{[\top]\}) \subseteq \mathcal{M}.$$

Case 6 . $C \equiv \bot$. It suffices to show that

$$C_2(\{[\bot]\}) = C_2\emptyset.$$

Suppose that $[\Gamma] \in C_2(\{[\bot]\})$ and $[\Delta] \in \emptyset^{\rightarrow} = \mathcal{M}$. Then $\Gamma \Rightarrow \bot$ is provable in linear logic without exponential, and since $\bot \Rightarrow \Delta^*$ is the axiom-scheme of linear logic without exponential, we have

$$\dfrac{\Gamma \Rightarrow \bot \quad \bot \Rightarrow \Delta^*}{\Gamma \Rightarrow \Delta^*}(\text{cut}).$$

Thus $[\Gamma] \rhd [\Delta]$, and hence $[\Gamma] \in C_2\emptyset$. Conversely suppose that $[\Gamma] \in C_2\emptyset$ and $[\Delta] \in \{[\bot]\}^{\rightarrow}$. Then since $[\Gamma] \in (\emptyset^{\rightarrow})^{\leftarrow}$ and $[\bot] \in \mathcal{M} = \emptyset^{\rightarrow}$, $\Gamma \Rightarrow \bot$ is provable in linear logic without exponential. Therefore we have $\Gamma \Rightarrow \Delta^*$, and hence $[\Gamma] \in C_2(\{[\bot]\})$.

Case 7 . $C \equiv 1$. Since $v(1) = C_2(\{[1]\})$, it suffices to show that

$$C_2(\{[1]\}) = C_2(\{[\,]\}).$$

Suppose that $[\Gamma] \in C_2(\{[1]\})$. Then $[\Gamma] \rhd [1] = [\emptyset^*]$, and hence

$$[\Gamma] \in C_2(\{[\,]\}).$$

Conversely suppose that $[\Gamma] \in C_2(\{[\,]\})$. Then $[\Gamma] \rhd [\emptyset^*] = [1]$, and hence

$$[\Gamma] \in C_2(\{[1]\}).$$

Finally we prove that $\Rightarrow A$ is provable in linear logic without exponential whenever $1 \sqsubseteq v(A)$. To this end, suppose that $1 \sqsubseteq v(A)$. Then $C_2(\{[\,]\}) \subseteq C_2(\{[A]\})$, and hence $[\,]\rhd[A]$ in the original preordered commutative monoid $\mathbf{X}_{\mathrm{N}}$. Thus $\Rightarrow A$ is provable in linear logic without exponential. Therefore if $\Gamma^* \multimap A$ is true in $\mathbf{Q}_2$ with v, then

$$\dfrac{\Rightarrow \Gamma^* \multimap A \qquad \dfrac{\Gamma \Rightarrow \Gamma^* \quad A \Rightarrow A}{\Gamma^* \multimap A, \Gamma \Rightarrow A}\,(\multimap\Rightarrow)}{\Gamma \Rightarrow A}\,(\mathrm{cut}),$$

and hence $\Gamma \Rightarrow A$ is provable in linear logic without exponential. ■

The semantics looks like traditional, so called phase space semantics for linear logic and also like traditional Routley-Meyer semantics for relevant logics (see e.g. [10, 11, 12, 33, 44]). Note that these quantales do generate a phase space in the traditional manner, and that this result can then be viewed as a tighter completeness result.

Chapter 6

Intuitionistic Linear Logic

In this chapter, we will extend linear logic without exponential to linear logic with exponential. In linear logic, exponential ! is added to compensate the absence of the rules of weakening and contraction. For example, $!A$ indicates that we may extract as many data of type A as we like, i.e., a datum of type $!A$ is a finite collection of data of type A.

First, we discuss intuitionistic linear logic with exponential (its syntax and semantics), and then prove soundness theorem for quantales with exponential. Then we extend the construction of the quantales to those with exponential. And we also prove completeness for quantales with exponential generated from Petri nets by using similar construction in Chapter 5.

In this thesis, we follow notation of linear logic in [47]. Therefore we have kept Girard's symbols $1, \top, !$, and replaced $\otimes, \oplus, \&$ by $*, \sqcup, \sqcap$ respectively, and interchanged $\perp$ and 0. For technical background on linear logic, see [19, 21, 22].

6.1 Syntax

6.1.1 Formulas

The *language* of linear logic has an alphabet consisting of

propositional variables: $a, b, c, \ldots$,

propositional constants: $1, \top, \perp$,

connectives: $*, \sqcup, \sqcap, \multimap$,

an unary connective: ! and

auxiliary symbols: (,).

The connectives carry traditional names:

$*$: multiplicative conjunction (times),

$\sqcup$: disjunction (or),

$\sqcap$: additive conjunction (and),

$\multimap$: linear implication and

!: exponential (storage or of course).

Note that storage operator is superficially similar to the modal operators $\Box, \Diamond$ in the usual modal logics. The role of ! is to introduce weakening and contraction in a controlled way for individual formulas. By inspection of the rules we see immediately that if we add weakening and contraction, then we may interpret ! by identity, i.e., $!A := A$ validates all the rules.

Formulas are inductively defined by

the propositional variables and constants are formulas,

if A and B are formulas, then $(A * B), (A \sqcup B), (A \sqcap B)$ and $(A \multimap B)$ are formulas and

if A is formula, then $!A$ is formula.

We shall use $A \equiv B$ as an abbreviation for $(A \multimap B) \sqcap (B \multimap A)$, and denote the set of all formulas by Φ.

6.1.2 Sequents

A *sequent* of linear logic is an expression of the form

$$\Gamma \Rightarrow C,$$

where Γ is a finite sequence of formulas and C is a formula. Both Γ and C may be empty. In the sequel, capital Greek letters will denote finite (possibly empty) sequences of formulas.

6.1.3 Axioms (initial sequents) and Rules

Definition 6.1.1 (axioms and rules of inference) The axioms of linear logic are the instances of the following four axiom-schemes:

$$A \Rightarrow A,$$

$$\Rightarrow 1,$$

$$\Gamma \Rightarrow \top,$$

$$\Gamma, \bot, \Delta \Rightarrow A.$$

The rules of inference of linear logic are the following structural rules:

$$\frac{\Gamma, \Delta \Rightarrow A}{\Gamma, 1, \Delta \Rightarrow A} \text{(1 - weakening)},$$

$$\frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, B, A, \Delta \Rightarrow C} \text{(exchange)},$$

$$\frac{\Gamma \Rightarrow A \quad \Delta, A, \Sigma \Rightarrow C}{\Delta, \Gamma, \Sigma \Rightarrow C} \text{ (cut)},$$

and the following logical rules:

$$\frac{\Gamma, A, \Delta \Rightarrow C \quad \Gamma, B, \Delta \Rightarrow C}{\Gamma, A \sqcup B, \Delta \Rightarrow C} \ (\sqcup \Rightarrow),$$

$$\frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \sqcup B} \ (\Rightarrow \sqcup 1), \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \sqcup B} \ (\Rightarrow \sqcup 2),$$

$$\frac{\Gamma, A, \Delta \Rightarrow C}{\Gamma, A \sqcap B, \Delta \Rightarrow C} \ (\sqcap 1 \Rightarrow), \qquad \frac{\Gamma, B, \Delta \Rightarrow C}{\Gamma, A \sqcap B, \Delta \Rightarrow C} \ (\sqcap 2 \Rightarrow),$$

$$\frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \sqcap B} \ (\Rightarrow \sqcap),$$

$$\frac{\Gamma, A, B, \Delta \Rightarrow C}{\Gamma, A * B, \Delta \Rightarrow C} \ (* \Rightarrow), \qquad \frac{\Gamma \Rightarrow A \quad \Delta \Rightarrow B}{\Gamma, \Delta \Rightarrow A * B} \ (\Rightarrow *),$$

$$\frac{\Gamma \Rightarrow A \quad \Delta, B, \Sigma \Rightarrow C}{\Delta, A \multimap B, \Gamma, \Sigma \Rightarrow C} \ (\multimap \Rightarrow), \qquad \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \multimap B} \ (\Rightarrow \multimap),$$

and the following exponential rules:

$$\frac{\Gamma, A \Rightarrow B}{\Gamma, !A \Rightarrow B} \ (! \Rightarrow),$$

$$\frac{\Gamma \Rightarrow B}{\Gamma, !A \Rightarrow B} \ (! \text{ - weakening}),$$

$$\frac{\Gamma, !A, !A \Rightarrow B}{\Gamma, !A \Rightarrow B} \ (! \text{ - contraction}),$$

$$\frac{!\Gamma \Rightarrow B}{!\Gamma \Rightarrow !B} \ (\Rightarrow !),$$

where $!\Gamma$ is a shorthand for $!A_1, \ldots, !A_n$ where $\Gamma = A_1, \ldots, A_n$.

Definition 6.1.2 The addition of free storage to linear logic corresponds to a rule of *free storage*

$$\frac{B \Rightarrow A \quad B \Rightarrow 1 \quad B \Rightarrow B * B}{B \Rightarrow !A}\ (\mathrm{FS}).$$

The three exponential rules ($! \Rightarrow$, ! - weakening and ! - contraction) and FS may also be formulated as follows (see Remark 6.1.7):

$$!A \Rightarrow A \cdots (1),$$

$$!A \Rightarrow 1 \cdots (2),$$

$$!A \Rightarrow !A * !A \cdots (3),$$

$$\frac{B \Rightarrow A \sqcap 1 \sqcap B * B}{B \Rightarrow !A}\ (\mathrm{FS'}).$$

In this thesis, we call linear logic with FS “FS-linear logic”.

Given a proposition A, the assertion of $!A$ has the possibility of being instantiated by the proposition A, the unit 1 or $!A*!A$, and thus of arbitrarily many assertions of $!A$.

6.1.4 Examples of Proofs

For understanding of the rules described above, we give the following examples.

Example 6.1.3 We can derive that $!A \Rightarrow A*!A$ and $!A \Rightarrow !(A * A)$.

- For $!A \Rightarrow A*!A$, we have

$$\cfrac{\cfrac{\cfrac{A \Rightarrow A \quad !A \Rightarrow !A}{A, !A \Rightarrow A*!A}\ (\Rightarrow *)}{!A, !A \Rightarrow A*!A}\ (! \Rightarrow)}{!A \Rightarrow A*!A}\ (!\ \text{- contruction}).$$

- For $!A \Rightarrow !(A * A)$, we have

$$\cfrac{\cfrac{\cfrac{\cfrac{A \Rightarrow A \quad A \Rightarrow A}{A, A \Rightarrow A * A}\ (\Rightarrow *)}{!A, !A \Rightarrow A * A}\ (! \Rightarrow)}{!A \Rightarrow A * A}\ (!\ \text{- contruction})}{!A \Rightarrow !(A * A)}\ (\Rightarrow !).$$

Example 6.1.4 We can derive that $!(A \sqcap B) \equiv !A*!B$.

- For $!(A \sqcap B) \Rightarrow !A*!B$, we have

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A \Rightarrow A}{A \sqcap B \Rightarrow A}\ (\sqcap 1 \Rightarrow)}{!(A \sqcap B) \Rightarrow A}\ (! \Rightarrow)}{!(A \sqcap B) \Rightarrow !A}\ (\Rightarrow !) \quad \cfrac{\cfrac{\cfrac{B \Rightarrow B}{A \sqcap B \Rightarrow B}\ (\sqcap 2 \Rightarrow)}{!(A \sqcap B) \Rightarrow B}\ (! \Rightarrow)}{!(A \sqcap B) \Rightarrow !B}\ (\Rightarrow !)}{!(A \sqcap B), !(A \sqcap B) \Rightarrow !A*!B}\ (\Rightarrow *)}{!(A \sqcap B) \Rightarrow !A*!B}\ (!\ \text{- contraction}).$$

- For $!A*!B \Rightarrow !(A \sqcap B)$, we have

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{!A \Rightarrow A}(! \Rightarrow)}{!A, !B \Rightarrow A}(! \text{ - weakening}) \quad \dfrac{\dfrac{B \Rightarrow B}{!B \Rightarrow B}(! \Rightarrow)}{!A, !B \Rightarrow B}(! \text{ - weakening})}{!A, !B \Rightarrow A \sqcap B}(\Rightarrow \sqcap)}{!A, !B \Rightarrow !(A \sqcap B)}(\Rightarrow !)}{!A*!B \Rightarrow !(A \sqcap B)}(* \Rightarrow) \quad .$$

Example 6.1.5 We can derive that $!(A \sqcap B) \Rightarrow !A \sqcap !B$ and $!(!A \sqcap !B) \equiv !(A \sqcap B)$.

- First we show that $!(A \sqcap B) \Rightarrow !A \sqcap !B$.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \sqcap B \Rightarrow A}(\sqcap 1 \Rightarrow)}{!(A \sqcap B) \Rightarrow A}(! \Rightarrow)}{!(A \sqcap B) \Rightarrow !A}(\Rightarrow !) \quad \dfrac{\dfrac{\dfrac{B \Rightarrow B}{A \sqcap B \Rightarrow B}(\sqcap 2 \Rightarrow)}{!(A \sqcap B) \Rightarrow B}(! \Rightarrow)}{!(A \sqcap B) \Rightarrow !B}(\Rightarrow !)}{!(A \sqcap B) \Rightarrow !A \sqcap !B}(\Rightarrow \sqcap) .$$

- Next we show that $!(!A \sqcap !B) \equiv !(A \sqcap B)$.

  - For $!(!A \sqcap !B) \Rightarrow !(A \sqcap B)$, we have

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{!A \Rightarrow A}(! \Rightarrow)}{!A \sqcap !B \Rightarrow A}(\sqcap 1 \Rightarrow) \quad \dfrac{\dfrac{B \Rightarrow B}{!B \Rightarrow B}(! \Rightarrow)}{!A \sqcap !B \Rightarrow B}(\sqcap 2 \Rightarrow)}{!A \sqcap !B \Rightarrow A \sqcap B}(\Rightarrow \sqcap)}{!(!A \sqcap !B) \Rightarrow A \sqcap B}(! \Rightarrow)}{!(!A \sqcap !B) \Rightarrow !(A \sqcap B)}(\Rightarrow !) \quad .$$

  - For $!(A \sqcap B) \Rightarrow !(!A \sqcap !B)$, we have

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \sqcap B \Rightarrow A}(\sqcap 1 \Rightarrow)}{!(A \sqcap B) \Rightarrow A}(! \Rightarrow)}{!(A \sqcap B) \Rightarrow !A}(\Rightarrow !) \quad \dfrac{\dfrac{\dfrac{B \Rightarrow B}{A \sqcap B \Rightarrow B}(\sqcap 2 \Rightarrow)}{!(A \sqcap B) \Rightarrow B}(! \Rightarrow)}{!(A \sqcap B) \Rightarrow !B}(\Rightarrow !)}{!(A \sqcap B) \Rightarrow !A \sqcap !B}(\Rightarrow \sqcap)}{!(A \sqcap B) \Rightarrow !(!A \sqcap !B)}(\Rightarrow !) \quad .$$

Example 6.1.6 We can derive that $1 \equiv !\top$.

- For $1 \Rightarrow !\top$, we have

$$\dfrac{\dfrac{\Rightarrow \top}{\Rightarrow !\top}(\Rightarrow !)}{1 \Rightarrow !\top}(1 \text{ - weakening}) .$$

- For $!\top \Rightarrow 1$, we have

$$\dfrac{\Rightarrow 1}{!\top \Rightarrow 1}(! \text{ - weakening}) .$$

Therefore if there is $!$, then 1 is expressed by $\top$.

Remark 6.1.7 The rules of (1), (2) and (3) of Definition 6.1.2 are derivable from the above original rules as follows:

$$\frac{A \Rightarrow A}{!A \Rightarrow A}\,(!\ \Rightarrow),$$

$$\frac{\Rightarrow 1}{!A \Rightarrow 1}\,(!\text{ - weakening}),$$

$$\dfrac{\dfrac{!A \Rightarrow !A \quad !A \Rightarrow !A}{!A, !A \Rightarrow !A * !A}\,(\Rightarrow *)}{!A \Rightarrow !A * !A}\,(!\text{ - contraction}).$$

Proposition 6.1.8 *The rule of (1), (2) and (3) of Definition 6.1.2 are interderivable with the following single rule:*

$$!A \Rightarrow A \sqcap 1 \sqcap (!A * !A).$$

Proof.

$$\dfrac{\dfrac{!A \Rightarrow A \quad !A \Rightarrow 1}{!A \Rightarrow A \sqcap 1}\,(\Rightarrow \sqcap) \qquad !A \Rightarrow !A * !A}{!A \Rightarrow A \sqcap 1 \sqcap (!A * !A)}\,(\Rightarrow \sqcap).$$

And if $!A \Rightarrow A \sqcap 1 \sqcap (!A * !A)$, then $!A \Rightarrow A$, $!A \Rightarrow 1$ and $!A \Rightarrow !A * !A$ immediately. ■

Proposition 6.1.9 *Using the rule of (FS), we can derive the following rule:*

$$A \sqcap 1 \sqcap (!A * !A) \Rightarrow !A.$$

Proof. Define $B := A \sqcap 1 \sqcap (!A * !A)$.

$$\dfrac{\dfrac{B \Rightarrow !A * !A \qquad \dfrac{\dfrac{\dfrac{\dfrac{!A \Rightarrow A \quad !A \Rightarrow 1}{!A \Rightarrow A \sqcap 1}\,(\Rightarrow \sqcap) \qquad !A \Rightarrow !A * !A}{!A \Rightarrow B}\,(\Rightarrow \sqcap) \qquad !A \Rightarrow B}{!A, !A \Rightarrow B * B}\,(\Rightarrow *)}{!A * !A \Rightarrow B * B}\,(* \Rightarrow)}{B \Rightarrow A \quad B \Rightarrow 1 \quad B \Rightarrow B * B}\,(\text{cut})}{B \Rightarrow !A}\,(2).$$

■

Therefore in FS-linear logic,

$$!A \equiv A \sqcap 1 \sqcap (!A * !A).$$

6.1.5 Relation between the exponential and Petri Nets

We explain the relation between exponential ! and Petri nets, and we show that exponential is useful to represent a place which can supply arbitrary many but finite tokens by firing transitions. For understanding, we give the following simple example.

Example 6.1.10 Consider the following net, with $P = \{!a, a\}$ and $T = \{t_1, t_2, t_3\}$. Pre-multisets $^\bullet(t_1)$, $^\bullet(t_2)$ and $^\bullet(t_3)$ are $[!a]$, and post-multisets $(t_1)^\bullet$, $(t_2)^\bullet$ and $(t_3)^\bullet$ are $[!a, !a]$, $[\,]$ and $[a]$, respectively.

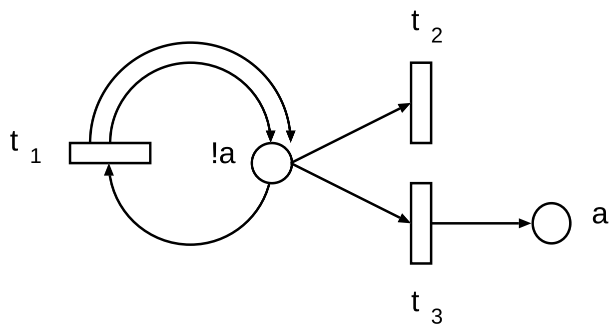

Figure 6.1: Relation between the exponential ! and Petri net.

Suppose that there is one token in $!a$.

1. A firing of transition t_1 changes the marking from $[!a]$ to $[!a, !a]$,
2. a firing of transition t_2 changes the marking from $[!a]$ to $[\,]$ and
3. a firing of transition t_3 changes the marking from $[!a]$ to $[a]$.

The firing of the transition t_1 transforms one token in $!a$ into two tokens in $!a$. Then we can supply arbitrary many but finite tokens by firing transition t_1. By the firing of the transition t_2, the token in $!a$ vanishes. The firing of the transition t_3 transforms one token in $!a$ into one token in a.

In linear logic, we can represent these as following:

1. $!a \Rightarrow !a * !a$,
2. $!a \Rightarrow 1$,
3. $!a \Rightarrow a$.

6.2 Semantics

6.2.1 Valuation on Quantale

Definition 6.2.1 (valuation) A *valuation* v on a quantale $\mathbf{Q} = \langle Q, \vee, \bullet, 1\rangle$ is a mapping of Φ into Q satisfying the following conditions: for every $A, B \in \Phi$,

1. $v(A \sqcap B) = v(A) \wedge v(B)$,
2. $v(A \sqcup B) = v(A) \vee v(B)$,
3. $v(A * B) = v(A) \bullet v(B)$,
4. $v(A \multimap B) = v(A) \multimap\!\!\bullet\, v(B)$,

5. $v(\top) = \top$,

6. $v(\bot) = \bot$,

7. $v(1) = 1$,

8. $v(!A) = !v(A)$.

6.2.2 Validity

Definition 6.2.2 (valid) A formula A is said to be

1. *true* in a valuation v on a quantale $\mathbf{Q}$ if
$$1 \sqsubseteq v(A)$$
holds, which will be denoted by
$$\mathbf{Q}, v \models A;$$

2. *valid* with respect to a class $\mathcal{Q}$ of quantales if for each quantale $\mathbf{Q} \in \mathcal{Q}$ and each valuation v on $\mathbf{Q}$,
$$\mathbf{Q}, v \models A$$
holds, which will be denoted by
$$\mathcal{Q} \models A;$$

and a sequent $\Gamma \Rightarrow A$ is said to be *valid* with respect to $\mathcal{Q}$ if and only if
$$\mathcal{Q} \models \Gamma^* \multimap A,$$
where Γ^* is a formula defined by $\emptyset^* := 1$ and $(\Gamma, A)^* := \Gamma^* * A$.

6.2.3 Soundness

The soundness theorem can be shown as usual (see e.g. [47]).

Theorem 6.2.3 (soundness) *If a sequent $\Gamma \Rightarrow A$ is provable in linear logic, then it is valid with respect to the class of all quantales with exponential.*

Proof. Soundness is proved by a straightforward induction on hight of proof.

- Initial sequents are valid,
- for the rules of inference (structural rules and logical rules), if upper sequent(s) is valid, then lower sequent is valid.

We show that initial sequents, structural rules and logical rules are valid. For structural rules and logical rules, we show that if upper sequent(s) is valid, then lower sequent is valid. In this section we show only four structural rules for exponential which are added to linear logic without exponential in Chapter 5.

We show that structural rules are valid.

(a) $! \Rightarrow$.

- Since $\models \Gamma, A \Rightarrow B$, $\models \Gamma^* * A \multimap B$. Then $1 \sqsubseteq v(\Gamma^* * A \multimap B)$, and hence $1 \sqsubseteq v(\Gamma^* * A) \multimap\!\!\bullet\, v(B)$. Therefore $1 \sqsubseteq v(\Gamma^*) \bullet v(A) \multimap\!\!\bullet\, v(B)$, and so $v(\Gamma^*) \bullet v(A) \sqsubseteq v(B)$. Thus
$$v(A) \sqsubseteq v(\Gamma^*) \multimap\!\!\bullet\, v(B).$$
- We show that if $\models \Gamma, A \Rightarrow B$, then $\models \Gamma, !A \Rightarrow B$. By the definition, it suffices to show that
$$\models \Gamma^* * !A \multimap B.$$
Since $v(A) \sqsubseteq v(\Gamma^*) \multimap\!\!\bullet\, v(B)$, $v(!A) \sqsubseteq v(\Gamma^*) \multimap\!\!\bullet\, v(B)$ by $!v(A) \sqsubseteq v(A)$. Therefore
$$\begin{aligned} v(\Gamma^*) \bullet !v(A) &= v(\Gamma^*) \bullet v(!A) \\ &\sqsubseteq v(B), \end{aligned}$$
and hence $1 \sqsubseteq v(\Gamma^*) \bullet v(!A) \multimap\!\!\bullet\, v(B)$. Thus $1 \sqsubseteq v(\Gamma^* * !A \multimap B)$, and so $\models \Gamma^* * !A \multimap B$.

(b) $!$ – *weakening*.

- Since $\models \Gamma \Rightarrow B$, $\models \Gamma^* \multimap B$. Then $1 \sqsubseteq v(\Gamma^* \multimap B)$, and hence
$$1 \sqsubseteq v(\Gamma^*) \multimap\!\!\bullet\, v(B).$$
- We show that if $\models \Gamma \Rightarrow B$, then $\models \Gamma, !A \Rightarrow B$. By the definition, it suffices to show that
$$\models \Gamma^* * !A \multimap B.$$
Since $1 \sqsubseteq v(\Gamma^*) \multimap\!\!\bullet\, v(B)$ and $!v(A) \sqsubseteq 1$, $!v(A) \sqsubseteq v(\Gamma^*) \multimap\!\!\bullet\, v(B)$, and then $!v(A) = v(!A) \sqsubseteq v(\Gamma^*) \multimap\!\!\bullet\, v(B)$. Therefore $v(\Gamma^*) \bullet v(!A) \sqsubseteq v(B)$, and hence $1 \sqsubseteq v(\Gamma^*) \bullet v(!A) \multimap\!\!\bullet\, v(B)$. Thus $1 \sqsubseteq v(\Gamma^* * !A \multimap B)$, and so $\models \Gamma^* * !A \multimap B$.

(c) $!$ – *contraction*.

- Since $\models \Gamma, !A, !A \Rightarrow B$, $\models \Gamma^* * !A * !A \multimap B$. Then $1 \sqsubseteq v(\Gamma^* * !A * !A \multimap B)$, and hence $1 \sqsubseteq v(\Gamma^*) \bullet v(!A) \bullet v(!A) \multimap\!\!\bullet\, v(B)$. Therefore $v(\Gamma^*) \bullet v(!A) \bullet v(!A) \sqsubseteq v(B)$, and so $v(!A) \bullet v(!A) \sqsubseteq v(\Gamma^*) \multimap\!\!\bullet\, v(B)$. Thus
$$!v(A) \bullet !v(A) \sqsubseteq v(\Gamma^*) \multimap\!\!\bullet\, v(B).$$
- We show that if $\models \Gamma, !A, !A \Rightarrow B$, then $\models \Gamma, !A \Rightarrow B$. By the definition, it suffices to show that
$$\models \Gamma^* * !A \multimap B.$$
Since $!v(A) \bullet !v(A) \sqsubseteq v(\Gamma^*) \multimap\!\!\bullet\, v(B)$ and $!v(A) \sqsubseteq !v(A) \bullet !v(A)$,
$$!v(A) \sqsubseteq v(\Gamma^*) \multimap\!\!\bullet\, v(B),$$
and then $v(!A) \sqsubseteq v(\Gamma^*) \multimap\!\!\bullet\, v(B)$. Therefore $v(\Gamma^*) \bullet v(!A) \sqsubseteq v(B)$, and hence $1 \sqsubseteq v(\Gamma^*) \bullet v(!A) \multimap\!\!\bullet\, v(B)$. Thus $1 \sqsubseteq v(\Gamma^* * !A \multimap B)$, and so $\models \Gamma^* * !A \multimap B$.

(d) $\Rightarrow !$.

- Since $\models !\Gamma \Rightarrow B$, $\models (!\Gamma)^* \multimap B$. Then $1 \sqsubseteq v((!\Gamma)^* \multimap B)$, and hence $1 \sqsubseteq v((!\Gamma)^*) \multimap\!\!\bullet\, v(B)$. Thus
$$v((!\Gamma)^*) \sqsubseteq v(B).$$
- We show that if $\models !\Gamma \Rightarrow B$, then $\models !\Gamma \Rightarrow !B$. By the definition, it suffices to show that
$$\models (!\Gamma)^* \multimap !B.$$
Suppose that $\models (!\Gamma)^* \multimap !B$. Then $1 \sqsubseteq v((!\Gamma)^* \multimap !B)$, and hence $1 \sqsubseteq v((!\Gamma)^*) \multimap\!\!\bullet\, v(!B)$. Therefore we show that
$$v((!\Gamma)^*) \sqsubseteq v(!B).$$
Since $v((!\Gamma)^*) \sqsubseteq v(B)$ and if $v(A) \sqsubseteq v(B)$, then $!v(A) \sqsubseteq !v(B)$, $!v((!\Gamma)^*) \sqsubseteq !v(B)$. Therefor $v(!(!\Gamma)^*) \sqsubseteq v(!B)$, and hence $v((!!\Gamma)^*) \sqsubseteq v(!B)$. Since $v((!\Gamma)^*) \sqsubseteq v((!!\Gamma)^*)$, $v((!\Gamma)^*) \sqsubseteq v(!B)$. (Note that $!\Gamma$ is a shorthand for $!A_1, \ldots, !A_n$ where $\Gamma = A_1, \ldots, A_n$.)

■

Theorem 6.2.4 (soundness) *If a sequent $\Gamma \Rightarrow A$ is provable in FS-linear logic, then it is valid with respect to the class of all FS-quantales with exponential.*

Proof. We only show that additional one rule, FS, is valid.

FS.

(a) Suppose that $B \Rightarrow A$, $B \Rightarrow 1$ and $B \Rightarrow B * B$ are true. Then $\models B \Rightarrow A$, $\models B \Rightarrow 1$ and $\models B \Rightarrow B * B$, and hence $\models B \multimap A$, $\models B \multimap 1$ and $\models B \multimap B * B$. Then $1 \sqsubseteq v(B \multimap A)$, $1 \sqsubseteq v(B \multimap 1)$ and $1 \sqsubseteq v(B \multimap B * B)$, and hence $1 \sqsubseteq v(B) \multimap\!\!\bullet\, v(A)$, $1 \sqsubseteq v(B) \multimap\!\!\bullet\, v(1)$ and $1 \sqsubseteq v(B) \multimap\!\!\bullet\, v(B) \bullet v(B)$. Therefore
$$v(B) \sqsubseteq v(A),$$
$$v(B) \sqsubseteq v(1) = 1,$$
$$v(B) \sqsubseteq v(B) \bullet v(B).$$

(b) We show that if $\models B \Rightarrow A$, $\models B \Rightarrow 1$ and $\models B \Rightarrow B * B$, then $\models B \Rightarrow !A$. By the definition, it suffices to show that
$$\models B \multimap !A.$$
By Definition 4.3.7, on the FS-quantale with exponential, if $v(B) \sqsubseteq v(A)$, $v(B) \sqsubseteq 1$ and $v(B) \sqsubseteq v(B) \bullet v(B)$, then $v(B) \sqsubseteq !v(A) = v(!A)$. Therefore $1 \sqsubseteq v(B) \multimap\!\!\bullet\, v(!A)$, and so $1 \sqsubseteq v(B \multimap !A)$. Thus $B \Rightarrow !A$ is true.

■

6.3 Completeness

First we discuss the classes of quantales. Then we show how to construct the quantales with exponential from Petri nets, and prove completeness of linear logic for our quantales.

Let $\mathcal{Q}_0$, $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_2^!$ be the classes of quantales defined by

$$\mathcal{Q}_0 := \{\mathbf{P}(\mathbf{X}_{\mathrm{N}}) \mid \mathrm{N} \text{ is a Petri net}\},$$

$$\mathcal{Q}_1 := \{C_1(\mathbf{P}(\mathbf{X}_{\mathrm{N}})) \mid \mathrm{N} \text{ is a Petri net}\},$$

$$\mathcal{Q}_2 := \{C_2(\mathbf{P}(\mathbf{X}_{\mathrm{N}})) \mid \mathrm{N} \text{ is a Petri net}\},$$

$$\mathcal{Q}_2^! := \{C_2(\mathbf{P}(\mathbf{X}_{\mathrm{N}}))^! \mid \mathrm{N} \text{ is a Petri net}\}.$$

Then $\mathbf{P}(\mathbf{X}_{\mathrm{N}})$, $C_1(\mathbf{P}(\mathbf{X}_{\mathrm{N}}))$, $C_2(\mathbf{P}(\mathbf{X}_{\mathrm{N}}))$ and $C_2(\mathbf{P}(\mathbf{X}_{\mathrm{N}}))^!$ are quantales obtained from the preordered commutative monoid $\mathbf{X}_{\mathrm{N}}$ using closure operations C_1 and C_2.

We say that linear logic is *complete* for a class $\mathcal{Q}$ of quantales, if $\Gamma \Rightarrow A$ is provable in linear logic whenever $\Gamma \Rightarrow A$ is valid with respect to $\mathcal{Q}$.

$\mathcal{Q}_1$ is the class of quantales used in Engberg and Winskel [15]. In any quantale in $\mathcal{Q}_0$ or $\mathcal{Q}_1$, the distributivity is always valid: in fact, in $\mathcal{Q}_0$ and $\mathcal{Q}_1$, the lattice operations meet and join correspond to usual set operations intersection and union, respectively, and hence the distributivity automatically holds. As mentioned in Chapter 1, any logic which is complete for $\mathcal{Q}_0$ or $\mathcal{Q}_1$, must have the distributivity as a theorem. In Chapter 5, we have considered the class $\mathcal{Q}_2$, and have proved completeness for $\mathcal{Q}_2$ in which the distributivity is not always valid. Here we consider the class $\mathcal{Q}_2^!$, and prove completeness for $\mathcal{Q}_2^!$ in which the Petri nets have the place with exponential !.

In order to prove completeness, we have constructed quantales with exponential from a Petri net as follows (see Figure 6.2):

First we construct a Petri net $\mathrm{N} = \langle P, T, {}^\bullet(-), (-)^\bullet\rangle$. For constructing N, we take formulas as places and sequents (provability) as transitions. Then from N, we construct a preordered commutative monoid $\mathbf{X}_{\mathrm{N}} = \langle \mathcal{M}, \rhd, +, [\,]\rangle$, and from the powerset $\mathbf{P}(\mathbf{X}_{\mathrm{N}}) = \langle \mathcal{P}(\mathcal{M}), \cup, \bullet, \{[\,]\}\rangle$ of the preordered commutative monoid, we construct a quantale $\mathbf{Q}_2 = \langle C_2(\mathcal{P}(\mathcal{M})), \cup_{C_2}, \bullet_{C_2}, C_2(\{[\,]\})\rangle$ by closure operation C_2. Finally we construct a quantale $\mathbf{Q}_2^! = \langle C_2(\mathcal{P}(\mathcal{M})), \cup_{C_2}, \bullet_{C_2}, C_2(\{[\,]\}), !\rangle$ or $\mathbf{Q}_F^! = \langle C_2(\mathcal{P}(\mathcal{M})), \cup_{C_2}, \bullet_{C_2}, C_2(\{[\,]\}), !, \mathrm{F}\rangle$ from $\mathbf{Q}_2$.

$$\boxed{\text{Petri net} \quad \mathrm{N}}$$

$$\Downarrow \qquad\qquad \textit{Proposition}\ 3.2.4$$

$$\boxed{\text{preordered commutative monoid} \quad \mathbf{X}_{\mathrm{N}}}$$

$$\Downarrow \qquad\qquad \textit{Proposition}\ 4.1.9$$

$$\boxed{\text{quantale} \quad \mathbf{P}(\mathbf{X}_{\mathrm{N}})}$$

$$\Downarrow \qquad\qquad \textit{Proposition}\ 4.2.11$$

$\boxed{\text{quantale by closure operation } C_2 \quad \mathbf{Q}_2}$

$$\Downarrow \qquad\qquad \textit{Proposition}\ 4.3.6\ \textit{or}\ 4.3.8$$

$\boxed{\text{quantale } \mathbf{Q}_2 \text{ with exponential} \quad \mathbf{Q}_2^! \quad \text{or} \quad \mathbf{Q}_F^!}$

Figure 6.2: The construction of the quantales with exponential $\mathbf{Q}_2^!$ or $\mathbf{Q}_F^!$.

The Petri net $\mathrm{N} = \langle P, T, {}^\bullet(-), (-)^\bullet \rangle$ is constructed as follows:

1. $P := \Phi$ (the set of all formulas),
2. $T := \{(\Gamma, \Delta) \mid \Gamma \Rightarrow \Delta^* \text{ is provable in linear logic}\}$
(in intuitionistic linear logic, the formulas on right hand side of the sequent are restricted at most one formula occurrence, so the formulas in Δ are connected with $*$),
3. for each $t = (\Gamma, \Delta) \in T$,
   (a) ${}^\bullet t := [\Gamma]$,
   (b) $t^\bullet := [\Delta]$.

Then we define a mapping v of Φ into the quantale $\mathbf{Q}_2$ by

$$v(C) := C_2(\{[C]\}).$$

Next we construct the FS-quantale with exponential $\mathbf{Q}_2^!$ or the quantale with exponential $\mathbf{Q}_F^!$.

Note that in the preordered commutative monoid $\mathbf{X}_\mathrm{N} = \langle \mathcal{M}, \rhd, +, [\,] \rangle$, for a sequence Γ, since $[\Gamma]$ is a multiset consisting of places in Γ,

$$[\Gamma] \rhd [\Delta] \text{ if and only if } \Gamma \Rightarrow \Delta^* \text{ is provable}$$

in linear logic, and hence

$$C_2(\{[\Gamma]\}) \subseteq C_2(\{[\Delta]\}) \text{ if and only if } \Gamma \Rightarrow \Delta^* \text{ is provable}$$

in linear logic.

We could not prove completeness for $\mathbf{Q}_2^!$, but we could for $\mathbf{Q}_F^!$. In the sequel, we shall prove completeness for $\mathbf{Q}_F^!$. First we explain why we could not prove completeness using $\mathbf{Q}_2^!$, and then we show how to prove completeness using $\mathbf{Q}_F^!$.

6.3.1 Completeness for FS-Linear Logic

We had shown in Chapter 4, as an interpretation of $!a$, for an element a of a quantale, we require an element x such that

$$x \sqsubseteq a \wedge 1 \wedge (x \bullet x).$$

Although this will not in general characterize a unique value of the quantale, since from FS it follows that any x satisfying $x \sqsubseteq a \wedge 1 \wedge (x \bullet x)$ should be below $!a$, $!a$ should be the greatest fixed point of the monotone mapping

$$x \longrightarrow a \wedge 1 \wedge (x \bullet x)$$

in the complete lattice given together with the quantale.

Note that by definition, a quantale with exponential $\mathbf{Q}^! = \langle Q, \vee, \bullet, 1, ! \rangle$ is an FS-quantale with exponential if

$$b \sqsubseteq a, b \sqsubseteq 1 \text{ and } b \sqsubseteq b \bullet b \text{ implies } b \sqsubseteq !a.$$

Moreover by proposition, let $\mathbf{Q} = \langle Q, \vee, \bullet, 1 \rangle$ be a quantale, and for each $a \in Q$, define an operator ! on Q by

$$!a := \bigvee \{x \in Q | x \sqsubseteq a \wedge 1 \wedge (x \bullet x)\},$$

then $\mathbf{Q}^! = \langle Q, \vee, \bullet, 1, ! \rangle$ is an FS-quantale with exponential.

Proposition 6.3.1 *Let*

$$\mathbf{Q}_2 = \langle C_2(\mathcal{P}(\mathcal{M})), \cup_{C_2}, \bullet_{C_2}, C_2(\{[\,]\}) \rangle$$

be a quantale. For each $A \in C_2(\mathcal{P}(\mathcal{M}))$, define an operator ! on $C_2(\mathcal{P}(\mathcal{M}))$ by

$$!A := \bigcup_{C_2} \{X \in C_2(\mathcal{P}(\mathcal{M})) | X \subseteq A \cap C_2(\{[\,]\}) \cap (X \bullet_{C_2} X)\}.$$

Then

$$\mathbf{Q}_2^! = \langle C_2(\mathcal{P}(\mathcal{M})), \cup_{C_2}, \bullet_{C_2}, C_2(\{[\,]\}), ! \rangle$$

is an FS-quantale with exponential.

Proof. It is immediate from the definition of exponential. ■

Remark 6.3.2 Using Lemma 4.3.3, we can show that

$$v((!\Gamma)^*) \leq !v((!\Gamma)^*)$$

by induction on the length of Γ.

Proof. $!v(A) \sqsubseteq v(!A)$ from definition, then $!!v(A) \sqsubseteq !v(!A)$ by 4 of Lemma 4.3.4, and hence

$$\begin{aligned} !!v(A) &= !v(A) \text{ (by 1 of Lemma 4.3.3)} \\ &= v(!A) \\ &\sqsubseteq !v(!A). \end{aligned}$$

Therefore $v(!A) \sqsubseteq !v(!A)$, and so $v((!\Gamma)^*) \sqsubseteq !v((!\Gamma)^*)$. ■

Let $\mathcal{Q}_2^!$ be the class of FS-quantales with exponential defined by

$$\mathcal{Q}_2^! := \{\mathbf{Q}_2^! \mid \mathbf{Q}_2 \in \mathcal{Q}_2, !A := \bigcup_{C_2}\{X|X = A \cap C_2(\{[\,]\}) \cap (X \bullet_{C_2} X)\},$$
$$A, X \in C_2(\mathcal{P}(\mathcal{M}))\},$$

where $\mathcal{Q}_2 := \{C_2(\mathbf{P}(\mathbf{X}_\mathrm{N})) \mid \mathrm{N}$ is a Petri net$\}$, and $C_2(\mathbf{P}(\mathbf{X}_\mathrm{N}))$ are commutative quantales defined from a preordered commutative monoid $\mathbf{X}_\mathrm{N}$ and closure operations C_2.

We say that FS-linear logic is *complete* for a class $\mathcal{Q}_2^!$ of commutative quantales, if $\Gamma \Rightarrow A$ is provable in FS-linear logic whenever $\Gamma \Rightarrow A$ is valid with respect to $\mathcal{Q}_2^!$.

Here we consider the class $\mathcal{Q}_2^!$, and prove completeness for $\mathcal{Q}_2^!$, since the distributivity is not always valid in FS-quantales with exponential from $\mathcal{Q}_2^!$.

In the sequel, we shall prove completeness for $\mathcal{Q}_2^!$.

Theorem 6.3.3 (completeness) *If a sequent $\Gamma \Rightarrow A$ is valid in $\mathcal{Q}_2^!$, then it is provable in FS-linear logic.*

Proof.

We can show by induction on the complexity of C that v is a valuation on $\mathbf{Q}_2^!$.

Since the completeness proof of the quantales automatically extends to the FS-quantales with exponential, it suffices to show that

$$v(!A) = !v(A).$$

Note that, by the definition of v,

- $v(!A) = C_2(\{[!A]\})$,
- $!v(A) = !C_2(\{[A]\})$
  $= \bigcup_{C_2}\{C_2X|C_2X = C_2(\{[A]\}) \cap C_2(\{[\,]\}) \cap (C_2X \bullet_{C_2} C_2X)\}$,

  where C_2X denotes the set of C_2-closed elements of a set X.

Here it suffices to show that

$$C_2(\{[!A]\} = !C_2(\{[A]\}).$$

We could prove $C_2(\{[!A]\} \subseteq !C_2(\{[A]\})$, but we could not the inverse.

1. First we show $C_2(\{[!A]\}) \subseteq !C_2(\{[A]\})$.

   Suppose that $[\Gamma] \in C_2(\{[!A]\})$. By the definition of exponential !, since

   $$!C_2(\{[A]\}) = \bigcup_{C_2}\{C_2X|C_2X = C_2(\{[A]\}) \cap C_2(\{[\,]\}) \cap (C_2X \bullet_{C_2} C_2X)\},$$

   $[\Gamma] \in !C_2(\{[A]\})$ if and only if there exists C_2X such that

   $$C_2(\{[A]\}) \cap C_2(\{[\,]\}) \cap (C_2X \bullet_{C_2} C_2X) = C_2X \text{ and } [\Gamma] \in C_2X.$$

   Then it suffices to show that

   $$C_2(\{[!A]\}) = C_2(\{[A]\}) \cap C_2(\{[\,]\}) \cap (C_2(\{[!A]\}) \bullet_{C_2} C_2(\{[!A]\})).$$

- For
$$C_2(\{[!A]\}) \subseteq C_2(\{[A]\}) \cap C_2(\{[\,]\}) \cap (C_2(\{[!A]\}) \bullet_{C_2} C_2(\{[!A]\})),$$
suppose that $[\Delta] \in C_2(\{[!A]\})$. Then since $[\Delta] \rhd [!A]$, we see that $\Delta \Rightarrow !A$ is provable in FS-linear logic. Since $!A \Rightarrow A \sqcap 1 \sqcap (!A * !A)$ is provable in FS-linear logic, we have
$$\frac{\Delta \Rightarrow !A \quad !A \Rightarrow A \sqcap 1 \sqcap (!A * !A)}{\Delta \Rightarrow A \sqcap 1 \sqcap (!A * !A)} \text{(cut)}.$$
Therefore
$$[\Delta] \rhd A[\sqcap 1 \sqcap (!A * !A)],$$
and hence
$$[\Delta] \in C_2(\{[A \sqcap 1 \sqcap (!A * !A)]\}).$$
Thus
$$[\Delta] \in C_2(\{[A]\}) \cap C_2(\{[\,]\}) \cap (C_2(\{[!A]\}) \bullet_{C_2} C_2(\{[!A]\})).$$
(Remark $C_2(\{[!A]\}) \bullet_{C_2} C_2(\{[!A]\}) = C_2(\{[!A * !A]\})$. Because
$$\begin{aligned} C_2(\{[!A]\}) \bullet_{C_2} C_2(\{[!A]\}) &= C_2(C_2(\{[!A]\}) \bullet C_2(\{[!A]\})) \\ &= C_2(\{[!A]\} \bullet \{[!A]\}) \\ &= C_2(\{[!A] + [!A]\}) \\ &= C_2(\{[!A, !A]\}), \end{aligned}$$
and $C_2(\{[!A * !A]\}) = C_2(\{[!A, !A]\})$.)
- For
$$C_2(\{[A]\}) \cap C_2(\{[\,]\}) \cap (C_2(\{[!A]\}) \bullet_{C_2} C_2(\{[!A]\})) \subseteq C_2(\{[!A]\}),$$
suppose that
$$[\Delta] \in C_2(\{[A]\}) \cap C_2(\{[\,]\}) \cap (C_2(\{[!A]\}) \bullet_{C_2} C_2(\{[!A]\})).$$
Then since $[\Delta] \rhd [A \sqcap 1 \sqcap (!A * !A)]$, we see that
$$\Delta \Rightarrow A \sqcap 1 \sqcap (!A * !A)$$
is provable in FS-linear logic. Since $A \sqcap 1 \sqcap (!A * !A) \Rightarrow !A$ is provable in FS-linear logic, we have
$$\frac{\Delta \Rightarrow A \sqcap 1 \sqcap (!A * !A) \quad A \sqcap 1 \sqcap (!A * !A) \Rightarrow !A}{\Delta \Rightarrow !A} \text{(cut)}.$$
Therefore $[\Delta] \rhd [!A]$, and hence $[\Delta] \in C_2(\{[!A]\})$.

2. Next we show $!C_2(\{[A]\}) \subseteq C_2(\{[!A]\})$.

   Since $!C_2(\{[A]\}) = \bigcup_{C_2}\{C_2X | C_2X = C_2(\{[A]\}) \cap C_2(\{[\,]\}) \cap (C_2X \bullet_{C_2} C_2X)\}$, it suffices to show that if
$$C_2X = C_2(\{[A]\}) \cap C_2(\{[\,]\}) \cap (C_2X \bullet_{C_2} C_2X),$$
then
$$C_2X \subseteq C_2(\{[!A]\}).$$
We could not prove this.

■

Remark 6.3.4 Consider the following rule:

$$\frac{B \Rightarrow A^n \text{ for all n}}{B \Rightarrow !A} \text{(FS'')}$$

where $A^0 := 1, A^{n+1} := A * A^n$.

By the addition of FS" rule to linear logic, we can derive FS as follows:

$$\dfrac{B \Rightarrow A \quad B \Rightarrow 1 \quad \dfrac{B \Rightarrow B * B \quad \dfrac{\dfrac{B \Rightarrow A \quad B \Rightarrow A}{B, B \Rightarrow A * A}(\Rightarrow *)}{B * B \Rightarrow A * A}(* \Rightarrow)}{B \Rightarrow A * A}(\text{cut})}{B \Rightarrow !A}(\text{FS''}) \quad .$$

6.3.2 Completeness for Linear Logic

We will show how to prove completeness of linear logic for the class of quantales with exponential.

First we show the construction of our quantale with exponential.

Proposition 6.3.5 *Let* $\mathbf{Q}_2 = \langle C_2(\mathcal{P}(\mathcal{M})), \cup_{C_2}, \bullet_{C_2}, C_2(\{[\,]\}) \rangle$ *be a quantale. Define* $\mathrm{F} \subseteq C_2(\mathcal{P}(\mathcal{M}))$ *by*

$$\mathrm{F} := \{C_2(\{[!A]\}) \mid A \in \Phi\}.$$

Then

$$\mathbf{Q}^!_{\mathrm{F}} = \langle C_2(\mathcal{P}(\mathcal{M})), \cup_{C_2}, \bullet_{C_2}, C_2(\{[\,]\}), !, \mathrm{F} \rangle$$

is a quantale with exponential.

Proof. We can prove that F defined as above satisfies the conditions 1 to 3 of Definition 4.3.7.

1. If $X, Y \in F$, then $X \bullet Y \in F$. It suffices to show that for $C_2(\{[!A]\}) \in \mathrm{F}$ and $C_2(\{[!B]\}) \in \mathrm{F}$,

   $$C_2(\{[!A]\}) \bullet_{C_2} C_2(\{[!B]\}) \in \mathrm{F}.$$

   Since

   $$\begin{aligned} C_2(\{[!A]\}) \bullet_{C_2} C_2(\{[!B]\}) &= C_2(C_2(\{[!A]\}) \bullet C_2(\{[!B]\})) \\ &= C_2(\{[!A]\} \bullet \{[!B]\}) \\ &= C_2(\{[!A] + [!B]\}) \\ &= C_2(\{[!A, !B]\}) \\ &= C_2(\{[!A * !B]\}), \end{aligned}$$

   we show

   $$C_2(\{[!A * !B]\}) \in \mathrm{F}.$$

   By Example 6.1.4, $!A * !B \equiv !(A \sqcap B)$, and hence

   $$C_2(\{[!A * !B]\}) = C_2(\{[!(A \sqcap B)]\}) \in \mathrm{F}.$$

2. $X \bullet X = X$ fir all $X \in F$. It suffices to show that

$$C_2(\{[!A]\}) \bullet_{C_2} C_2(\{[!A]\}) = C_2(\{[!A]\}).$$

Since $C_2(\{[!A]\}) \bullet_{C_2} C_2(\{[!A]\}) = C_2(\{[!A*!A]\})$, we show

$$C_2(\{[!A*!A]\}) = C_2(\{[!A]\}).$$

Since $!A*!A \equiv !(A \sqcap A) \equiv !A$,

$$\begin{array}{rcl} C_2(\{[!A*!A]\}) & = & C_2(\{[!(A \sqcap A)]\}) \\ & = & C_2(\{[!A]\}) \\ & \in & \mathrm{F}. \end{array}$$

3. $1 \in F$ and $X \sqsubseteq 1$ for all $X \in F$. It suffices to show that

$$C_2(\{[\,]\}) \in \mathrm{F}$$

and

$$C_2(\{[!A]\}) \subseteq C_2(\{[\,]\})$$

for all $C_2(\{[!A]\}) \in \mathrm{F}$. Note that

$$\begin{array}{rcl} C_2(\{[\,]\}) & = & C_2(\{[1]\}) \\ & = & C_2(\{[!1]\}) \\ & \in & \mathrm{F}. \end{array}$$

Therefore $C_2(\{[\,]\}) \in \mathrm{F}$. For $C_2(\{[!A]\}) \subseteq C_2(\{[\,]\})$, suppose that $[\Gamma] \in C_2(\{[!A]\})$. Then $[\Gamma] \rhd [!A]$, and hence $\Gamma \Rightarrow !A$ is provable in linear logic. Since $!A \Rightarrow 1$ is also provable in linear logic, we have

$$\frac{\Gamma \Rightarrow !A \quad !A \Rightarrow 1}{\Gamma \Rightarrow 1} \text{(cut)},$$

and then $[\Gamma] \rhd [1]$. Therefore $[\Gamma] \in C_2(\{[\,]\})$, and so $C_2(\{[!A]\}) \subseteq C_2(\{[\,]\})$.

■

Note that F consists of C_2-closures of multisets which have a token in a place with exponential !, that is the place can supply arbitrary but finite tokens. For the FS-quantales with exponential which the definition of exponential ! is the one in [15, 16, 17], we could prove soundness but could not prove completeness. Because we could not show that v defined above was a valuation on the FS-quantales with exponential, i.e., we could prove

$$v(!A) \subseteq !v(A),$$

but could not prove the converse. So we use the other definition of exponential !. Since in the quantales with exponential of our definition, we can also show $!v(A) \subseteq v(!A)$, then we can prove completeness.

Let $\mathcal{Q}_2^!$ be the class of quantales with exponential defined by

$$\mathcal{Q}_2^! := \{\mathbf{Q}_\mathrm{F}^! \mid \mathbf{Q}_2 \in \mathcal{Q}_2, \mathrm{F} \subseteq C_2(\mathcal{P}(\mathcal{M})) \text{ with 1 to 3 of Definition 4.3.7}\},$$

where $\mathcal{Q}_2 := \{C_2(\mathbf{P}(\mathbf{X}_{\mathrm{N}})) \mid \mathrm{N}$ is a Petri net$\}$, and $C_2(\mathbf{P}(\mathbf{X}_{\mathrm{N}}))$ are commutative quantales defined from a preordered commutative monoid $\mathbf{X}_{\mathrm{N}}$ and closure operations C_2.

We say that linear logic is *complete* for the class $\mathcal{Q}_2^!$ of quantales with exponential, if $\Gamma \Rightarrow A$ is provable in linear logic whenever $\Gamma \Rightarrow A$ is valid with respect to $\mathcal{Q}_2^!$. Here we consider the class $\mathcal{Q}_2^!$, and prove completeness for $\mathcal{Q}_2^!$, since the distributivity is not always valid in quantales with exponential from $\mathcal{Q}_2^!$.

In the sequel, we shall prove completeness for $\mathcal{Q}_2^!$.

Theorem 6.3.6 (completeness) *If a sequent $\Gamma \Rightarrow A$ is valid in $\mathcal{Q}_2^!$, then it is provable in linear logic.*

Proof.

Since the completeness proof of the quantales automatically extends to the quantales with exponential, it suffices to show that

$$v(!A) = !v(A).$$

Note that, by the definition of v,

- $v(!A) = C_2(\{[!A]\})$,
- $!v(A) = !C_2(\{[A]\})$
  $= \bigcup_{C_2}\{C_2X \in \mathrm{F} \mid C_2X \subseteq C_2(\{[A]\})\}$,

  where C_2X denotes the set of C_2-closed elements in the set F defined in Proposition 6.3.5.

Here we show that

$$C_2(\{[!A]\}) = !C_2(\{[A]\}).$$

1. For $C_2(\{[!A]\}) \subseteq !C_2(\{[A]\})$, since the closure operation C_2 is order preserving and $!A \Rightarrow A$ by the definition, we see that

$$C_2(\{[!A]\}) \subseteq C_2(\{[A]\}).$$

   On the other hand $C_2(\{[!A]\}) \in \mathrm{F}$, hence

$$C_2(\{[!A]\}) \subseteq !C_2(\{[A]\}).$$

2. For $!C_2(\{[A]\}) \subseteq C_2(\{[!A]\})$, suppose that $C_2(\{[!B]\}) \in \mathrm{F}$ with $C_2(\{[!B]\}) \subseteq C_2(\{[A]\})$. Then $[!B] \rhd [A]$, and hence $!B \Rightarrow A$ is provable in linear logic. Therefore $!B \Rightarrow !A$, and so $[!B] \rhd [!A]$. Thus

$$C_2(\{[!B]\}) \subseteq C_2(\{[!A]\}),$$

   hence

$$\begin{aligned} !C_2(\{[A]\}) &= \bigcup_{C_2}\{C_2X \in \mathrm{F} \mid C_2X \subseteq C_2(\{[A]\})\} \\ &\subseteq C_2(\{[!A]\}). \end{aligned}$$

Finally we prove that $\Rightarrow A$ is provable in linear logic whenever $1 \sqsubseteq v(A)$. To this end, suppose that $1 \sqsubseteq v(A)$. Then $C_2(\{[\,]\}) \subseteq C_2(\{[A]\})$, and hence $[\,] \rhd [A]$ in the original preordered monoid $\mathbf{M}_{\mathrm{N}}$. Thus $\Rightarrow A$ is provable in linear logic. Therefore if $\Gamma^* \multimap A$ is true in $\mathbf{Q}^!_2$ with v, then

$$\dfrac{\Rightarrow \Gamma^* \multimap A \qquad \dfrac{\Gamma \Rightarrow \Gamma^* \quad A \Rightarrow A}{\Gamma^* \multimap A, \Gamma \Rightarrow A}\,(\multimap\Rightarrow)}{\Gamma \Rightarrow A}\,(\mathrm{cut}) \quad ,$$

and hence $\Gamma \Rightarrow A$ is provable in linear logic.

■

Chapter 7

Interpretation on Petri Nets

In this chapter, we give an impression on the meaning of the logic on the proposed Petri net model, comparing with that by [15].

First, using some examples, we consider a difference of interpretations between C_1 and C_2, and then we show that the distributivity is not always valid for the quantales constructed with C_2. We give also the interpretation of exponential ! for a Petri net using an example. Moreover we try to give a different interpretation of formulas under our closure operation.

7.1 Interpretation of C_1 and C_2

The markings forward reachable from m form the set

$$\uparrow m = \{m' \in \mathcal{M} \mid m \to m'\}.$$

Similarly, the markings downward reachable to m form the set

$$\downarrow m = \{m' \in \mathcal{M} \mid m' \to m\}.$$

In this thesis, for a marking m, we consider the set of downward reachable to m, and generally call the set the downward closure of m.

In [15, 16, 17], this notation is extended from the marking m to the set M of markings as following:

$$\downarrow M := \{m' \in \mathcal{M} \mid \exists m \in \mathcal{M}(m' \to m)\},$$

and we call this downward closure $\downarrow M$ C_1.

In this thesis, this notation is extended as following:

$$M^{\to} := \{m' \in \mathcal{M} \mid \forall m \in M(m \to m')\},$$

$$M^{\leftarrow} := \{m' \in \mathcal{M} \mid \forall m \in M(m' \to m)\},$$

and we call this downward closure $(M^{\to})^{\leftarrow}$ C_2.

In [15], C_1 is used as a closure operation when the class of quantales is defined, while the closure operation used in this thesis is C_2. First, we describe an informal difference of interpretations between C_1 and C_2. The formal definition of $\models$ and *valuation* were given in Chapter 5 and Chapter 6, but here for intuitive understanding, let us agree with $\models A \multimap B$ means that $A \multimap B$ holds in the quantales created with C_1 or C_2.

For a single marking $m \in \mathcal{M}$, it is easy to see that

$$\downarrow \{m\} = (\{m\}^{\rightarrow})^{\leftarrow},$$

and therefore

$$C_1(\{m\}) = C_2(\{m\})$$

holds. Hence, for $m, m' \in \mathcal{M}$,

$$C_i(\{m\}) \subseteq C_i(\{m'\}) \text{ if and only if } m \rhd m' \ (i = 1, 2),$$

i.e., inclusion between two C_1-closed markings in $\mathcal{M}$ coincides with the reachability relation between them, and also for two C_2-closed markings.

We will identify a marking with the formula representing it. Since

$$\models A \multimap B \text{ if and only if } v(A) \subseteq v(B),$$

the following holds for both of quantales $C_1(\mathbf{P}(\mathbf{X}_{\mathrm{N}}))$ and $C_2(\mathbf{P}(\mathbf{X}_{\mathrm{N}}))$:

$$\models m \multimap m' \text{ if and only if } m \rhd m'.$$

Hence we can read $\models m \multimap m'$ as 'm' is reachable from m'. For a subset of markings, C_1 and C_2 may give different results.

7.2 Difference between C_1 and C_2

First we give some interpretations of linear logic on the Petri net models created using C_1 and C_2.

Example 7.2.1 Consider the following net:

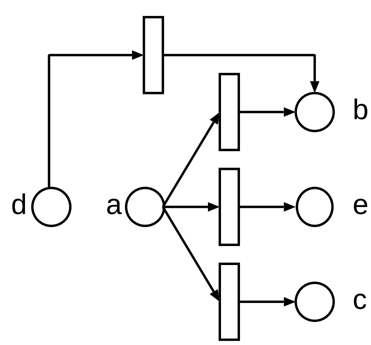

Figure 7.1: Petri net - I.

Here we have

$$\begin{aligned} C_1(\{[a]\}) &= C_2(\{[a]\}) \\ &= \{[a]\}, \end{aligned}$$

$$\begin{aligned} C_1(\{[b]\}) &= C_2(\{[b]\}) \\ &= \{[a],[b],[d]\}, \end{aligned}$$

$$\begin{aligned} C_1(\{[c]\}) &= C_2(\{[c]\}) \\ &= \{[a],[c]\}. \end{aligned}$$

In this example, with interpretations of connectives $*$, $\sqcup$ and $\sqcap$, the following holds:

1. $\models a * a \multimap a * b$ and $\models a * a \multimap b * c$. Because

$$\begin{aligned} C_1(\{[a*a]\}) &= C_1(\{[a]\}) \bullet_{C_1} C_1(\{[a]\}) \\ &= C_1(C_1(\{[a]\}) \bullet C_1(\{[a]\})) \\ &= C_1(\{[a]\} \bullet \{[a]\}) \\ &= C_1(\{m + m' \mid m \in \{[a]\}, m' \in \{[a]\}\}) \\ &= C_1(\{[a] + [a]\}) \\ &= C_1(\{[a,a]\}) \\ &= \{[a,a]\} \end{aligned}$$

and

$$\begin{aligned} C_1(\{[a*b]\}) &= C_1(\{[a]\}) \bullet_{C_1} C_1(\{[b]\}) \\ &= C_1(\{m + m' \mid m \in \{[a]\}, m' \in \{[a],[b],[d]\}\}) \\ &= C_1(\{[a]+[a],[a]+[b],[a]+[d]\}) \\ &= C_1(\{[a,a],[a,b],[a,d]\}) \\ &= \{[a,a],[a,b],[a,d]\}. \end{aligned}$$

Since $\{[a,a]\} \subseteq \{[a,a],[a,b],[a,d]\}$,

$$C_1(\{[a*a]\}) \subseteq C_1(\{[a*b]\}).$$

(Remark

$$\begin{aligned} C_2(\{[a*b]\}) &= C_2(\{[a]\}) \bullet_{C_2} C_2(\{[b]\}) \\ &= C_2(\{m + m' \mid m \in \{[a]\}, m' \in \{[a],[b],[d]\}\}) \\ &= C_2(\{[a]+[a],[a]+[b],[a]+[d]\}) \\ &= C_2(\{[a,a],[a,b],[a,d]\}) \\ &= \{[a,a],[a,b],[a,d],[b,b],[b,d],[d,d]\}. \end{aligned}$$

Since $\{[a,a]\} \subseteq \{[a,a],[a,b],[a,d],[b,b],[b,d],[d,d]\}$,

$$C_2(\{[a*a]\}) \subseteq C_2(\{[a*b]\}).)$$

We can prove $\models a * a \multimap b * c$ similarly.

2. $\models a \sqcup d \multimap b$ and $\models d \multimap a \sqcup b$. Because

$$\begin{aligned} C_1(\{[a \sqcup d]\}) &= C_1(\{[a]\}) \cup_{C_1} C_1(\{[d]\}) \\ &= C_1(C_1(\{[a]\}) \cup C_1(\{[d]\})) \\ &= C_1(\{[a]\} \cup \{[d]\}) \\ &= C_1(\{[a],[d]\}) \\ &= \{[a],[d]\} \end{aligned}$$

and $C_1(\{[b]\}) = \{[a],[b],[d]\}$. Since $\{[a],[d]\} \subseteq \{[a],[b],[d]\}$,

$$C_1(\{[a \sqcup d]\}) \subseteq C_1(\{[d]\}).$$

(Remark

$$\begin{aligned} C_2(\{[a \sqcup d]\}) &= C_2(\{[a]\}) \cup_{C_2} C_2(\{[d]\}) \\ &= C_2(C_2(\{[a]\}) \cup C_2(\{[d]\})) \\ &= C_2(\{[a]\} \cup \{[d]\}) \\ &= C_2(\{[a],[d]\}) \\ &= \{[a],[b],[d]\} \end{aligned}$$

and $C_2(\{[b]\}) = \{[a],[b],[d]\}$. Since $\{[a],[b],[d]\} \subseteq \{[a],[b],[d]\}$,

$$C_2(\{[a \sqcup d]\}) \subseteq C_2(\{[d]\}).)$$

We can prove $\models d \multimap a \sqcup b$ similarly.

3. $\models a \multimap b \sqcap c$ and $\models b \sqcap c \multimap e$. Because

$$C_1(\{[a]\}) = \{[a]\}$$

and

$$\begin{aligned} C_1(\{[b \sqcap c]\}) &= C_1(\{[b]\}) \cap C_1(\{[c]\}) \\ &= \{[a],[b],[d]\} \cap \{[a],[c]\} \\ &= \{[a]\}. \end{aligned}$$

Since $\{[a]\} \subseteq \{[a]\}$,

$$C_1(\{[a]\}) \subseteq C_1(\{[b \sqcap c]\}).$$

We can prove $\models b \sqcap c \multimap e$ similarly. Since $C_1(\{[b \sqcap c]\}) = \{[a]\}$, $C_1(\{[e]\}) = \{[a],[e]\}$ and $\{[a]\} \subseteq \{[a],[e]\}$,

$$C_1(\{[b \sqcap c]\}) \subseteq C_1(\{[e]\}).$$

Remark 7.2.2 Here we consider the meaning of $\sqcap$, 3 of the example given above. $\models a \multimap b \sqcap c$ means that b and c are reachable from a, but $\models b \sqcap c \multimap e$ does not mean that e is reachable from b or c. Note that in the interpretation of $m_1 \sqcap m_2 \multimap m_3$, it is not always true that we understand the meaning of each marking in formulas as it has. It will be described later, so here we show a case using the above example, $\models b \sqcap c \multimap e$. The rules of $\sqcap$ are the following:

$$\frac{b \Rightarrow e}{b \sqcap c \Rightarrow e}\ (\sqcap 1 \Rightarrow), \qquad \frac{c \Rightarrow e}{b \sqcap c \Rightarrow e}\ (\sqcap 2 \Rightarrow),$$

The meaning is not 'e is reachable from b or c', but 'e is reachable from every markings from which b and c are reachable'. If $x \rhd b$ and $x \rhd c$, then $x \rhd e$. This does not always mean that e is reachable from b or c.

Example 7.2.3 Consider the following net:

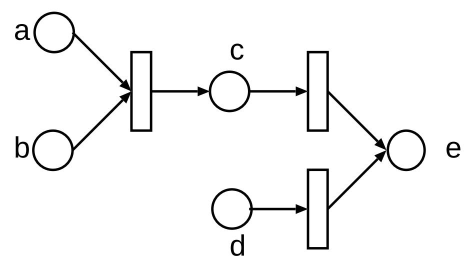

Figure 7.2: Petri net - II.

Here we have

$$\begin{aligned} C_1(\{[c]\}) &= C_2(\{[c]\}) \\ &= \{[a,b],[c]\}, \end{aligned}$$

$$\begin{aligned} C_1(\{[e]\}) &= C_2(\{[e]\}) \\ &= \{[a,b,d],[c,d],[e]\}. \end{aligned}$$

In this example, with interpretations of connectives $\multimap$, the following holds:

1. $C_1(\{[b \multimap c]\}) = C_2(\{[b \multimap c]\}) = \{[a]\}$, and then

$$\models (b \multimap c) \multimap a.$$

Because

$$\begin{aligned} C_1(\{[b \multimap c]\}) &= C_1(\{[b]\}) \multimap\!\bullet\, C_1(\{[c]\}) \\ &= \bigcup_{C_1}\{X \mid X \bullet C_1(\{[b]\}) \subseteq C_1(\{[c]\})\} \\ &= C_1(\{x \mid x + y \in C_1(\{[c]\}) \text{ for all } y \in C_1(\{[b]\})\}) \\ &= C_1(\{x \mid x + [b] \in \{[a,b],[c]\}\}). \end{aligned}$$

If $x = [a]$, then $[a] + [b] = [a,b] \in \{[a,b],[c]\}$, and hence

$$C_1(\{[b \multimap c]\}) = \{[a]\}.$$

2. $C_1([d\multimap e]) = C_2([d\multimap e]) = \{[a,b],[c]\}$, and then

$$\models (d\multimap e)\multimap a * b \text{ and } \models (d\multimap e)\multimap c.$$

Because

$$\begin{aligned}
C_1(\{[d\multimap e]\}) &= C_1(\{[d]\})\multimap\!\!\bullet C_1(\{[e]\}) \\
&= \bigcup_{C_1}\{X \mid X \bullet C_1(\{[d]\}) \subseteq C_1(\{[e]\})\} \\
&= C_1(\{x \mid x + y \in C_1(\{[e]\}) \text{ for all } y \in C_1(\{[d]\})\}) \\
&= C_1(\{x \mid x + [d] \in \{[a,b,d],[c,d],[e]\}\}).
\end{aligned}$$

If $x = [a,b]$, then

$$[a,b] + [d] = [a,b,d] \in \{[a,b,d],[c,d],[e]\},$$

and if $x = [c]$, then $[c] + [d] = [c,d] \in \{[a,b,d],[c,d],[e]\}$. Therefore

$$C_1(\{[d\multimap e]\}) = \{[a,b],[c]\}.$$

The following example shows the difference between C_1 and C_2.

Example 7.2.4 Consider the net of Figure 7.1:
Here we have

- $C_1(\{[b]\}) = C_2(\{[b]\}) = \{[a],[b],[d]\}$,
- $C_1(\{[c]\}) = C_2(\{[c]\}) = \{[a],[c]\}$,
- $C_1(\{[a],[d]\}) \neq C_2(\{[a],[d]\})$, because

$$C_1(\{[a],[d]\}) = \{[a],[d]\},$$

and since $\{[a],[d]\}^{\rightarrow} = \{[b]\}$ and $\{[b]\}^{\leftarrow} = \{[a],[b],[d]\}$, then

$$C_2(\{[a],[d]\}) = \{[a],[b],[d]\}.$$

Next we show that, while the quantale constructed using C_1 is distributive, it is not always true in the quantales using C_2.

Example 7.2.5 Consider the net of Figure 7.2:
When we construct a quantale using C_1, we have

1. $(C_1(\{[a,b]\}) \cup_{C_1} C_1(\{[d]\})) \cap C_1(\{[c]\})$

$$\begin{aligned}
&= C_1(C_1(\{[a,b]\}) \cup C_1(\{[d]\})) \cap C_1(\{[c]\}) \\
&= C_1(\{[a,b]\} \cup \{[d]\}) \cap C_1(\{[c]\}) \\
&= C_1(\{[a,b,d]\}) \cap C_1(\{[c]\}) \\
&= \{[a,b,d]\} \cap \{[a,b,c]\} \\
&= \{[a,b]\},
\end{aligned}$$

2. $(C_1(\{[a,b]\}) \cap C_1(\{[c]\})) \cup_{C_1} (C_1(\{[d]\}) \cap C_1(\{[c]\}))$

$$\begin{aligned}
&= C_1((C_1(\{[a,b]\}) \cap C_1(\{[c]\})) \cup (C_1(\{[d]\}) \cap C_1(\{[c]\}))) \\
&= C_1((\{[a,b]\} \cap \{[a,b,c]\}) \cup (\{[d]\} \cap \{[a,b,c]\})) \\
&= C_1(\{[a,b]\} \cup \emptyset) \\
&= C_1(\{[a,b]\}) \\
&= \{[a,b]\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&(C_1(\{[a,b]\}) \cup_{C_1} C_1(\{[d]\})) \cap C_1(\{[c]\}) \\
&= (C_1(\{[a,b]\}) \cap C_1(\{[c]\})) \cup_{C_1} (C_1(\{[d]\}) \cap C_1(\{[c]\})),
\end{aligned}$$

then distributivity is valid. But when we construct a quantale using C_2, we have

1. $(C_2(\{[a,b]\}) \cup_{C_2} C_2(\{[d]\})) \cap C_2(\{[c]\}) = \{[a,b],[c]\}$, because

$$\begin{aligned}
C_2(\{[a,b]\}) \cup_{C_2} C_2(\{[d]\}) &= C_2(C_2(\{[a,b]\}) \cup C_2(\{[d]\})) \\
&= C_2(\{[a,b]\} \cup \{[d]\}) \\
&= C_2(\{[a,b],[d]\}) \\
&= (\{[a,b],[d]\}^{\rightarrow})^{\leftarrow} \\
&= \{[e]\}^{\leftarrow} \\
&= \{[a,b],[c],[d],[e]\}
\end{aligned}$$

and

$$\begin{aligned}
C_2(\{[c]\}) &= (\{[c]\}^{\rightarrow})^{\leftarrow} \\
&= \{[c],[e]\}^{\leftarrow} \\
&= \{[a,b],[c]\},
\end{aligned}$$

then $(C_2(\{[a,b]\}) \cup_{C_2} C_2(\{[d]\})) \cap C_2(\{[c]\})$

$$\begin{aligned}
&= \{[a,b],[c],[d],[e]\} \cap \{[a,b],[c]\} \\
&= \{[a,b],[c]\},
\end{aligned}$$

2. but $(C_2(\{[a,b]\}) \cap C_2(\{[c]\})) \cup_{C_2} (C_2(\{[d]\}) \cap C_2(\{[c]\}))$

$$\begin{aligned}
&= C_2((C_2(\{[a,b]\}) \cap C_2(\{[c]\})) \cup (C_2(\{[d]\}) \cap C_2(\{[c]\}))) \\
&= C_2((\{[a,b]\} \cap \{[a,b],[c]\}) \cup (\{[d]\} \cap \{[a,b],[c]\})) \\
&= C_2(\{[a,b]\} \cup \emptyset) \\
&= C_2(\{[a,b]\}) \\
&= (\{[a,b]\}^{\rightarrow})^{\leftarrow} \\
&= \{[a,b],[c],[e]\}^{\leftarrow} \\
&= \{[a,b]\}.
\end{aligned}$$

Therefore

$$(C_2(\{[a,b]\}) \cup_{C_2} C_2(\{[d]\})) \cap C_2(\{[c]\})$$
$$\neq (C_2(\{[a,b]\}) \cap C_2(\{[c]\})) \cup_{C_2} (C_2(\{[d]\}) \cap C_2(\{[c]\})),$$

then distributivity is not valid.

We consider conjunction and disjunction of two markings. Let m_1, m_2, and m_3 be markings in $\mathcal{M}$. The interpretation in $C_1(\mathbf{P}(\mathbf{X}_\mathrm{N}))$ is described as follows.

- $\models m_1 \multimap m_2 \sqcup m_3$:

  m_2 or m_3 are reachable from m_1.

- $\models m_1 \multimap m_2 \sqcap m_3$:

  m_2 and m_3 are reachable from m_1.

- $\models m_1 \sqcup m_2 \multimap m_3$:

  m_3 is reachable from m_1 and m_2.

- $\models m_1 \sqcap m_2 \multimap m_3$:

  m_3 is reachable from every markings from which m_1 and m_2 are reachable, i.e., if $m \rhd m_1$ and $m \rhd m_2$ then $m \rhd m_3$. This does not always mean that m_3 is reachable from m_1 or m_2.

From the last formula, it is not always true that we understand the meaning of each marking in formulas as it has. We should read $\models m \multimap m'$ as 'if m is reachable from some marking, then m' is also reachable from it'. Using this interpretation, the last formula is read as 'if m_1 and m_2 is reachable from some marking, then m_3 is also reachable from it'.

Using the same examples, we show the interpretation in $C_2(\mathbf{P}(\mathbf{X}_\mathrm{N}))$. Only the difference between $C_1(\mathbf{P}(\mathbf{X}_\mathrm{N}))$ and $C_2(\mathbf{P}(\mathbf{X}_\mathrm{N}))$ is found in formulas with $\sqcup$, because

$$v(m_1 \sqcup m_2) = C_2(\{m_1\}) \cup_{C_2} C_2(\{m_2\})$$

is not necessarily equal to

$$C_2(\{m_1\}) \cup C_2(\{m_2\}).$$

- $\models m_1 \multimap m_2 \sqcup m_3$:

  Every marking reachable in common from m_2 and m_3 is also reachable from m_1.

- $\models m_1 \sqcup m_2 \multimap m_3$:

  m_3 is reachable from every marking reachable in common from m_1 and m_2. As a result, m_3 is reachable from m_1 and m_2, provided that there exist some markings reachable from both m_1 and m_2.

7.3 Interpretation of Exponential !

For understanding exponential, we give the following example.

Example 7.3.1 Consider the following net:

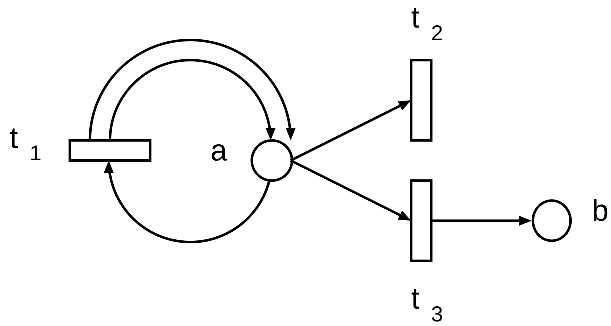

Figure 7.3: Petri net - III.

If we have a token in the place a, by the firings of transition t_1, we may extract as many tokens as we like. Because we can get two tokens from one token. So we can regard the place a as one which can supply arbitrary many but finite tokens.

When we construct a quantale using C_2, we have the followings:

1. $C_2(\{[a]\}) = \{[a], [a,a], [a,a,a], \ldots\}$, because $C_2(\{[a]\}) = (\{[a]\}^{\rightarrow})^{\leftarrow}$, and $\{[a]\}^{\rightarrow}$ is $\{[\,], [b], [a,a,a,\ldots]\}$, and then
$$\{[\,], [b]\}^{\leftarrow} = \{[a], [a,a], [a,a,a], \ldots\}.$$

2. $C_2(\{[\,]\}) = \{[\,], [a], [a,a], [a,a,a], \ldots\}$, because $C_2(\{[\,]\}) = (\{[\,]\}^{\rightarrow})^{\leftarrow}$, and $\{[\,]\}^{\rightarrow}$ is $\{[\,]\}$, and then
$$\begin{aligned}\{[\,]\}^{\leftarrow} &= \{[\,]\} \cup \{[a], [a,a], [a,a,a], \ldots\} \\ &= \{[\,], [a], [a,a], [a,a,a], \ldots\}.\end{aligned}$$

3. $C_2(\{[b]\}) = \{[b], [a], [a,a], [a,a,a], \ldots\}$, because $C_2(\{[b]\}) = (\{[b]\}^{\rightarrow})^{\leftarrow}$, and $\{[b]\}^{\rightarrow} = \{[b]\}$, and then
$$\begin{aligned}\{[b]\}^{\leftarrow} &= \{[b]\} \cup \{[a], [a,a], [a,a,a], \ldots\} \\ &= \{[b], [a], [a,a], [a,a,a], \ldots\}.\end{aligned}$$

Note that
$$C_2(\{[b]\}) = \{[b], [a], [a,a], [a,a,a], \ldots\}$$
and
$$C_2(\{[a]\}) = \{[a], [a,a], [a,a,a], \ldots\} = C_2(\{[!b]\}).$$
Therefore $C_2(\{[b]\})$ properly contains $C_2(\{[!b]\})$.

Remark 7.3.2 When we construct the quantale using C_2 from the example given above, for the set F of the quantale with exponential we have

$$\mathrm{F} = \{\{[a^n] \mid n \geq 1\}, \{[a^n] \mid n \geq 0\}\},$$

where $[a^n] := [\underbrace{a, \ldots, a}_{n}]$.

We can show that these hold Definition 4.3.7.

1. For if $x, y \in \mathrm{F}$, then $x \bullet y \in \mathrm{F}$,

$$\{[a^n] \mid n \geq 1\} \bullet \{[a^n] \mid n \geq 0\} = \{[a^n] \mid n \geq 1\} \in \mathrm{F}.$$

2. For $x \bullet x = x$ for all $x \in \mathrm{F}$,

$$\{[a^n] \mid n \geq 1\} \bullet \{[a^n] \mid n \geq 1\} = \{[a^n] \mid n \geq 1\},$$

$$\{[a^n] \mid n \geq 0\} \bullet \{[a^n] \mid n \geq 0\} = \{[a^n] \mid n \geq 0\}.$$

3. For $1 \in \mathrm{F}$ and $x \sqsubseteq 1$ for all $x \in \mathrm{F}$, $C_2(\{[1]\}) = \{[a^n] \mid n \geq 0\}$, and

$$\{[a^n] \mid n \geq 1\} \subseteq C_2(\{[1]\}),$$

$$\{[a^n] \mid n \geq 0\} \subseteq C_2(\{[1]\}).$$

7.4 Another Interpretation

We now give an intuitional meaning of each marking in formulas. By the definition of the closure operation, it is easy to see that for any $X \in \mathcal{P}(\mathcal{M})$,

$$C_2(X)^{\rightarrow} := X^{\rightarrow}$$

and

$$(C_2(X)^{\rightarrow})^{\leftarrow} = C_2(X)$$

holds. Therefore, a function

$$\mathcal{I} : C_2(\mathcal{P}(\mathcal{M})) \longrightarrow \mathcal{P}(\mathcal{M})^{\rightarrow}$$

defined by $\mathcal{I}(X) := X^{\rightarrow}$ is a bijection, and its inverse is $\mathcal{I}^{-1}(Y) := Y^{\leftarrow}$. Note that for any $X_1, X_2 \subset \mathcal{M}$,

$$C_2(X_1) \subseteq C_2(X_2) \text{ if and only if } \mathcal{I}(X_1) \supseteq \mathcal{I}(X_2).$$

We try to give the meaning of $X \in \mathcal{P}(\mathcal{M})$ by $\mathcal{I}(X)$ instead of $C_2(X)$. $\mathcal{I}(X)$ is the set of all markings reachable from every marking in X. Considering each token as a resource, $\mathcal{I}(\{m\})$, $m \in \mathcal{M}$ corresponds to *results* obtainable by providing resources m, where the results includes all markings reachable from m.

Let m_1, m_2 be markings. For the meaning of $m_1 \sqcap m_2$ and $m_1 \sqcup m_2$, we have the following.

- $C_2(\{m_1\}) \cap C_2(\{m_2\})$ is the set of all markings from which both of m_1 and m_2 are reachable. Therefore, the meaning of $m_1 \sqcap m_2$, i.e., $\mathcal{I}(C_2(\{m_1\}) \cap C_2(\{m_2\}))$, is the set of all results obtainable from every markings in $C_2(\{m_1\}) \cap C_2(\{m_2\})$. As a results, it contains both of $\mathcal{I}(\{m_1\})$ and $\mathcal{I}(\{m_2\})$.

- The meaning of $m_1 \sqcup m_2$, i.e., $\mathcal{I}(C_2(\{m_1\}) \cup_{C_2} C_2(\{m_2\}))$, is the set of all results obtainable in common by resources m_1 and by resources m_2.

Then the following interpretation is possible.

- $\models m_1 \multimap m_2$ means that

  'if one can get any result obtainable from resource m_1, then one can also get any result obtainable from m_2'. As a result, m_2 is reachable from m_1.

- $\models m_1 \sqcup m_2 \multimap m_3$ means that

  'if one can get any result obtainable in common from m_1 and m_2, then one can also get any result obtainable from m_3'. As a result, m_3 is reachable from both m_1 and m_2, provided that there exist some markings reachable from both m_1 and m_2.

- $\models m_1 \multimap m_2 \sqcup m_3$ means that

  'if one can get any result obtainable from m_1, then one can also get any result obtainable in common from m_2 and m_3'.

- $\models m_1 \sqcap m_2 \multimap m_3$ means that

  'if one can get both of any result obtainable from m_1 and any result obtainable from m_2, then one can also get any result obtainable from m_3'. This does not necessarily imply that m_3 is reachable from m_1 and m_2.

- $\models m_1 \multimap m_2 \sqcap m_3$ means that

  'if one can get any result obtainable from m_1, then one can also get both of any result obtainable from m_2 and any result obtainable from m_3'. As a result, both m_2 and m_3 are reachable from m_1.

Comparing with temporal logics, one of advantages of using linear logic in Petri nets is to describe properties on individual tokens. For example,

$$\models c * m \multimap (a \sqcup b) * m$$

means that 'any results obtainable in common from marking m with additional resource a and marking m with additional resource b is also obtainable by adding resource c'.

The difference between the interpretation shown above and that by [15] can be found in the following example:

'once place a is marked, it is possible to reach a marking where b is marked'.

On the model of [15], it is described by

$$\models a * \top \multimap b * \top.$$

On the model of this thesis, this has a different meaning:

'if one can get any result obtainable in common from any marking with a marked, then one can also get any result obtainable in common from any marking with b marked'.

Note that since $\mathcal{I}(v(a * \top))$ may be empty in most cases, it is hard to describe such *don't care* submarkings on the model of this thesis.

7.5 Relation of Linear Logic, Models and Petri Nets

Here we consider the relation of linear Logic, models (quantales with exponential !) and Petri nets.

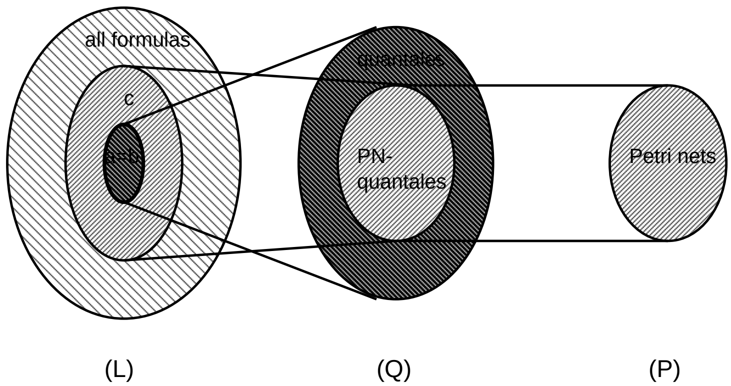

Figure 7.4: The relation of linear logic, models and Petri nets.

These mean that (L) is the set of formulas, (Q) is the set of quantales and (P) is the set of Petri nets, respectively.

- L : the set of formulas

  all formulas : the set of all formulas, i.e., Φ

  $a \subseteq \Phi$: the set of all provable formulas

  $b \subseteq \Phi$: the set of all formulas which are valid in the quantales

  $c \subseteq \Phi$: the set of all formulas which are valid in the quantales generated from Petri nets

- Q : the set of quantales

  quantales : the set of all quantales

  PN-quantales : the set of all quantales generated from Petri nets

1. relation between a and b

   soundness : $a \subseteq b$

   completeness : $b \subseteq a$

2. relation between a and c

   soundness : $a \subseteq c$

   completeness : $c \subseteq a$

Soundness and completeness of linear logic are proved for quantales [1, 5, 19, 21, 22, 39, 47]. Therefore the set of all provable formulas equals the set of all formulas which are valid in quantales, i.e., $a = b$. The problem of this thesis is how is the relation between a and c. We considered whether the set of all formulas which are valid in the quantales, i.e., the set of all provable formulas equals the set of all formulas valid in the quantales generated from Petri nets or not, and have proved $a = c$.

Chapter 8

Concluding Remarks

We have seen how to construct quantales in which the distributivity is not always valid from Petri nets, and prove completeness of linear logic for the quantales. Moreover, we extend the quantales to the quantales with exponential !. We introduce how to construct quantales with exponential from Petri nets, and prove completeness of linear logic for the quantales with exponential.

In this concluding chapter, moreover we consider the connection between classical linear logic and Petri nets.

Definition A modal classical quantale is a classical quantale with additional unary operation ? satisfying:

1. $!(a \multimap\!\!\bullet\, b) \sqsubseteq ?a \multimap\!\!\bullet\, ?b$,
2. $a \sqsubseteq ?a$,
3. $?0 = 0$,
4. $??a \sqsubseteq ?a$,
5. $0 \sqsubseteq ?a$.

Lemma In every modal classical quantale, the following holds:

1. if $a \sqsubseteq b$ then $?a \sqsubseteq ?b$,
2. if $!a \bullet b \sqsubseteq ?c$ then $!a \bullet ?b \sqsubseteq ?c$.

Definition A structure $\mathbf{Q} = \langle Q, \vee, \bullet, 1, 0 \rangle$ is a *commutative classical quantale* if

1. $\langle Q, \vee \rangle$ is a complete lattice,
2. $\langle Q, \bullet, 1 \rangle$ is a commutative monoid,
3. $(\bigvee x_i) \bullet y = \bigvee (x_i \bullet y)$ for all $x_i, y \in Q$,
4. $\sim\sim x = x$ $(\sim x := x \multimap\!\!\bullet\, 0)$ for all $x \in Q$.

We shall use simply classical quantale for commutative classical quantale with unit.

Definition Define a binary operation $-\!\bullet$ on $\mathbf{Q}$ by

$$y-\!\bullet z := \bigvee\{x \mid x \bullet y \sqsubseteq z\}.$$

Then

$$x \sqsubseteq y-\!\bullet z \text{ if and only if } x \bullet y \sqsubseteq z.$$

Definition Syntax (formulas, sequents, axioms and rules) of classical linear logic are follows: (In this section we show only formulas, axioms and rules we have to add to linear logic in Chapter 6.)

1. Formulas

   The language of classical linear logic has an alphabet consisting of

   a propositional constant: 0,
   an unary connective: ? and
   a binary connective: $\oplus$.

   The connective carry traditional name:

   $\oplus$: disjunction (par) and
   ?: modality (why not, consumption or costorage).

   Formulas are inductively defined by

   The propositional variables and constants are formulas and
   if A is formula, then $?A$ is formula.

2. Axioms and Rules

   The basic calculus is obtained from the calculus of linear logic by adding the following rules.

   (a) The adding axiom of classical linear logic is the instance of the following one axiom-scheme:

   $$\Gamma, 0 \Rightarrow \Delta.$$

   (b) The adding rules of inference of classical linear logic are the following structural rules:

   $$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, 0}\ (0\text{ - weakening}),$$

   $$\frac{\Gamma \Rightarrow \Delta, A, B, \Lambda}{\Gamma \Rightarrow \Delta, B, A, \Lambda}\ (\text{exchange}),$$

   and the following logical rules:

   $$\frac{\Gamma \Rightarrow \Delta, D}{D^{\perp}, \Gamma \Rightarrow \Delta}\ (\perp\ \Rightarrow),$$

   $$\frac{D, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, D^{\perp}}\ (\Rightarrow\ \perp),$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Pi \Rightarrow \Lambda}{A \oplus B, \Gamma, \Pi \Rightarrow \Delta, \Lambda} (\oplus \Rightarrow),$$

$$\frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \oplus B} (\Rightarrow \oplus),$$

and the following modality rules:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, ?A} (?\text{ - weakening}),$$

$$\frac{\Gamma \Rightarrow \Delta, ?A, ?A}{\Gamma \Rightarrow \Delta, ?A} (?\text{ - contraction}),$$

$$\frac{A \Rightarrow ?\Sigma}{?A \Rightarrow ?\Sigma} (? \Rightarrow),$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, ?A} (\Rightarrow ?),$$

where $?\Gamma$ is a shorthand for $?A_1, \ldots, ?A_n$ where $\Gamma = A_1, \ldots, A_n$.

For understanding of the adding rules described above, we give the following examples.

Examples of Proofs

1. We can derive that $(A * B)^\perp \equiv A^\perp \oplus B^\perp$.

   - For $(A * B)^\perp \Rightarrow A^\perp \oplus B^\perp$,

$$\cfrac{\cfrac{\cfrac{A \Rightarrow A \quad B \Rightarrow B}{A, B \Rightarrow A * B} (\Rightarrow *)}{(A * B)^\perp \Rightarrow A^\perp, B^\perp} (\perp \Rightarrow, \Rightarrow \perp)}{(A * B)^\perp \Rightarrow A^\perp \oplus B^\perp} (\Rightarrow \oplus) \quad .$$

   - For $A^\perp \oplus B^\perp \Rightarrow (A * B)^\perp$,

$$\cfrac{\cfrac{\cfrac{\cfrac{A \Rightarrow A}{A^\perp, A \Rightarrow} (\perp \Rightarrow) \quad \cfrac{B \Rightarrow B}{B^\perp, B \Rightarrow} (\perp \Rightarrow)}{A, B, A^\perp \oplus B^\perp \Rightarrow} (\oplus \Rightarrow)}{A * B, A^\perp \oplus B^\perp \Rightarrow} (* \Rightarrow)}{A^\perp \oplus B^\perp \Rightarrow (A * B)^\perp} (\Rightarrow \perp) \quad .$$

2. We can derive that $A \multimap B \equiv A^\perp \oplus B$.

   - For $A \multimap B \Rightarrow A^\perp \oplus B$,

$$\cfrac{\cfrac{\cfrac{A \Rightarrow A}{\Rightarrow A, A^\perp} (\Rightarrow \perp) \quad B \Rightarrow B}{A \multimap B \Rightarrow A^\perp, B} (\multimap \Rightarrow)}{A \multimap B \Rightarrow A^\perp \oplus B} (\Rightarrow \oplus) \quad .$$

- For $A^\perp \oplus B \Rightarrow A \multimap B$,

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, A^\perp \Rightarrow}(\perp \Rightarrow) \qquad B \Rightarrow B}{A, A^\perp \oplus B \Rightarrow B}(\oplus \Rightarrow)}{A^\perp \oplus B \Rightarrow A \multimap B}(\Rightarrow \multimap) \quad .$$

3. We can derive that $(!A)^\perp \equiv ?A^\perp$.

   - For $(!A)^\perp \Rightarrow ?A^\perp$,

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\Rightarrow A, A^\perp}(\Rightarrow \perp)}{\Rightarrow A, ?A^\perp}(\Rightarrow ?)}{\Rightarrow !A, ?A^\perp}(\Rightarrow !)}{(!A)^\perp \Rightarrow ?A^\perp}(\perp \Rightarrow) \quad .$$

   - For $?A^\perp \Rightarrow (!A)^\perp$,

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A^\perp, A \Rightarrow}(\perp \Rightarrow)}{A^\perp, !A \Rightarrow}(! \Rightarrow)}{?A^\perp, !A \Rightarrow}(? \Rightarrow)}{?A^\perp \Rightarrow (!A)^\perp}(\Rightarrow \perp) \quad .$$

4. We can derive that $A^\perp \equiv A \multimap 0$. We use this later.

   - For $A^\perp \Rightarrow A \multimap 0$,

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, A^\perp \Rightarrow}(\perp \Rightarrow)}{A, A^\perp \Rightarrow 0}(0 \text{ - weakening})}{A^\perp \Rightarrow A \multimap 0}(\Rightarrow \multimap) \quad .$$

   - For $A \multimap 0 \Rightarrow A^\perp$,

$$\dfrac{\dfrac{0 \Rightarrow \qquad A \Rightarrow A}{A, A \multimap 0 \Rightarrow}(\multimap \Rightarrow)}{A \multimap 0 \Rightarrow A^\perp}(\Rightarrow \perp) \quad .$$

5. We derive that $0 \equiv ?\perp$.

   - For $0 \Rightarrow ?\perp$, we have

$$\dfrac{0 \Rightarrow}{0 \Rightarrow ?\perp}(? \text{ - weakening}) .$$

   - For $?\perp \Rightarrow 0$, we have

$$\dfrac{\dfrac{\perp \Rightarrow}{?\perp \Rightarrow}(? \Rightarrow)}{?\perp \Rightarrow 0}(0 \text{ - weakening}) .$$

   Therefore if there is ?, then 0 is expressed by $\perp$.

Suppose that we want to express the set of all markings which cannot be reached from a marking.

For example, suppose that we want to express that in a Petri net model, a marking in a place a cannot get into a place b. How we can express this ? For understanding, we give the following example.

Example

Consider the following net:

Suppose that there are one token in b, w_1 and w_2, and we want to express that in a Petri net model of mutual exclusion, processes p_1 and p_2 cannot get into their critical regions c_1 and c_2 at the same time.

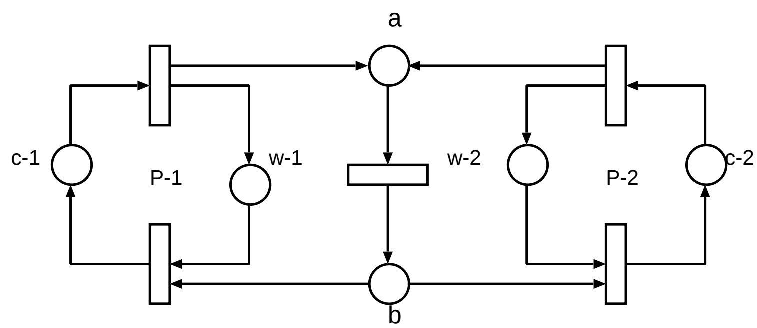

Figure 8.1: Mutual exclusion.

Where the marking of the place w_1 indicates that the first process, p_1, is working outside its critical region, c_1, and similarly for the other process, p_2. The resource corresponding to b is used to ensure mutual exclusion of the critical regions and after a process has been in its critical region it returns a resource, a, which then is prepared (transformed into b) for the next turn. The initial marking, m_0, will be

$$m_0 = [b] + [w_1] + [w_2].$$

We can now express that e.g. p_1 can enter its critical region (from the initial marking) by:

$$\models m_0 \multimap c_1 * \top.$$

However this does not ensure that no undesired tokens are present, so it is better to express it:

$$\models m_0 \multimap c_1 * w_2.$$

If the system is in a "working state" then both processes have the possibility of entering their critical section:

$$\models w_1 * (a \sqcup b) * w_2 \multimap c_1 * w_2 \sqcap w_1 * c_2.$$

The property, that when p_1 is in its critical section and p_2 is working it is possible that p_2 can later came into its critical section with p_1 working, is expressed by:

$$\models c_1 * w_2 \multimap w_1 * c_2.$$

Similar other "positive" properties can be expressed. Shortly we shall see how to express the "negative" property that both processes cannot be in their critical regions at the same time.

Suppose we want to see how to express the negative property for Petri net, for example, two processes cannot be in their critical regions at the same time. Then we consider the operation of linear negation.

Definition

Linear negation can be expressed in terms of 0 and $\multimap$ by

$$A^{\perp} := A \multimap 0.$$

Its semantics with respect to a quantale is determined by a choice for the denotation 0.

Now we can prove soundness theorem for quantales generated by Petri nets, but have not succeeded to prove completeness for quantales by using similar construction in Chapter 5 and 6. It is one of subjects of further research.

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