Lexicographical Separation in Finite-Dimensional Vector Spaces and its Applications

Kiyoshi Ikeda
School of Information Science,
Japan Advanced Institute of Science and Technology

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Abstract

The aim of the thesis is to prove a lexicographical separation theorem and to give its applications to linear inequality systems, lexicographic expected utility, and extensive measurement.

The main result of the thesis is the following lexicographical separation theorem : Let $\mathbb{F}^n$ be the n-dimensional vector space over $\mathbb{F}$, where $\mathbb{F}$ stands for an ordered field such that $\mathbb{Q} \subseteq \mathbb{F} \subseteq \mathbb{R}$, and let P be a nonempty subset of $\mathbb{F}^n$. Suppose P is a convex cone not containing $\mathbf{0}$, and also suppose its complement $\mathbb{F}^n \setminus P$ is a convex cone in $\mathbb{F}^n$. Then, there exist real-valued linear functions $g_1, \ldots, g_n$ on $\mathbb{F}^n$ such that $x \in P$ if and only if $(g_1(x), \ldots, g_n(x)) >_L (0, \ldots, 0)$ for all $x \in \mathbb{F}^n$, where $<_L$ (or $>_L$) denotes the *lexicographic order* on $\mathbb{R}^n$, that is, given $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ with $x \neq y$, we have $x <_L y$ if $x_k < y_k$ for $k = \min\{ i \mid x_i \neq y_i \}$. This means that, in any finite-dimensional vector space over $\mathbb{F}$, a convex cone P and its convex complement $\mathbb{F}^n \setminus P$ can be separated by a set of linear functions and a lexicographic order. Moreover, we show that the first function g_1 is unique up to a positive scalar multiple. In case $\mathbb{F} = \mathbb{R}$, equivalent versions of this theorem was proved by Hausner and Wendel, Klee, Martínez-Legaz and Singer ; so that the above theorem is a generalization of their theorems, considering an ordered field $\mathbb{F}$ other than $\mathbb{R}$.

We give a proof of the lexicographical separation theorem from our original standpoint, using an *infinitesimal* ε, that is, $0 < \varepsilon$ and $\varepsilon < 1/k$ for all positive integer k. The proof given in this thesis makes use of the fact that the lexicographic order on $\mathbb{R}^n$ can be described by a polynomial ring whose variable is an infinitesimal: $(a_0, a_1, \ldots, a_n) <_L (b_0, b_1, \ldots, b_n)$ if and only if $a_0 + a_1\varepsilon + \cdots + a_n\varepsilon^n < b_0 + b_1\varepsilon + \cdots + b_n\varepsilon^n$. Although such a description is known in the literature, we gave a new role to an infinitesimal in this thesis : (i) We used an infinitesimal not only for the description of a lexicographic order but also as a useful tool of proving the lexicographical separation theorem. (ii) We adopted an infinitesimal as a solution to infinite systems of linear inequalities (as will be seen below) .

As one of the applications of the lexicographical separation theorem, we obtain a generalization of the well-known "theorem of the alternatives," which gives a necessary and sufficient condition for the existence of solutions to linear inequality systems. Let P be an nonempty subset of $\mathbb{R}^n$. Then, $\mathbf{0}$ is not contained in the convex hull of P if and only if the inequality system "$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n > 0$ for all $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in P$" has solutions $x_1, x_2, \ldots, x_n$ in $\mathbb{R}[\varepsilon]_n$, where $\mathbb{R}[\varepsilon]_n = \{ r_1 + r_2\varepsilon + \cdots + r_n\varepsilon^{n-1} \mid r_1, \cdots, r_n \in \mathbb{R} \}$. We provide several examples of infinite systems of linear inequalities, showing that it is not unreasonable to obtain such an infinitesimal ε in our solutions. As another application, we also obtain a generalization of Farkas' lemma for lexicographical inequality systems. Further, we applied these results to game theory, giving a generalization of von Neumann's minimax theorem for semi-infinite games.

As other application of the lexicographical separation theorem, we presented two kinds of lexicographic utility representations : one is about lexicographic expected utility, and the other is about lexicographic extensive utility. The lexicographic expected utility representation given in this thesis is a modification of Hausner's lexicographic expected utility theory, by omitting the existence of irrational-valued probabilities : we restrict our attention to $\mathbb{F}$-valued probabilities, where $\mathbb{F}$ stands for an ordered field such that $\mathbb{Q} \subseteq \mathbb{F} \subseteq \mathbb{R}$, and show that lexicographic expected utility theory can be founded on the domain of $\mathbb{F}$-valued lotteries. On the other hand, the lexicographical extensive utility representation given in this thesis is a modification of classical Hahn's embedding theorem : we establish a scheme of conditions which is necessary and sufficient for the existence of extensive utilities on indivisible items.

Key Words: **separation theorem, lexicographic orders, linear inequality systems, lexicographic utility theory**