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Fusion of Pedigreed Preferential Relations as Beliefs

by

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Abstract

Belief fusion, instead of AGM belief revision, was first proposed to solve the problem of inconsistency, that arose from repetitive application of the operation when agents' knowledge were amalgamated. However in the theory, all the sources must be totally ordered and thus applicable area is quite restrictive. In this paper, the author realizes the belief fusion of multiple agents for partially ordered sources. When the author considers such a partial ranking over sources, there is no need to restrict that each agent has total preorders over possible worlds. The preferential model allows each agent to have strict partial orders over possible worlds. Especially, such an order is called a preferential relation, that prescribes a world is more plausible than the other. The author introduces various representation of beliefs of agents, that is, belief states, generalized belief states, preferential relations, and generalized preferential relations, and formalizes an operation which combines multiple beliefs of agents. In addition, the author shows that this operation can properly include the ordinary belief fusion.

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Chapter 1

Introduction

The problem of knowledge representation is one of the main theme in artificial intelligence and epistemology. What is belief? How do we formalize epistemic attitude? In mathematical logic and philosophy, Hintikka's formalization is well-known [19]. Using modal logic, he supposed that KD45 modal operator had the properties of belief. However, Gärdenfors [14] did not only study the static properties of belief, but also the dynamic properties of belief. He started the study of *belief change* or *belief revision*. Given an epistemic input, a belief change operator lets an agent's belief be changed. When an agent changes her belief, she must decide whether information is given up. Thus she has a preference about information. It follows that if you consider the dynamic structure of a belief, you will define a belief to be an ordering over a language or a set of possible worlds.

Such studies was purchased by many issues from the point of Gärdenfors's view [3, 14]. However, the main theme of these studies has concerned with the belief of single agent. Therefore, when we consider the dynamic structure of multi-agents' beliefs, we can not use the single agent revision. Given a situation in which each agent has a belief, how do we decide a belief which all agents should have?

If we do not assume anything else, we must consider whether all agents believe it (unanimity), whether some ones believe it (respecting minority), or whether most of agents believe it (decision by majority), and decide a preference which all agents should believe. However, if we want to judge the information about technology, medicine, etc., then we should ask experts, for example, scientists, doctors, etc., because they are more reliable than novices. Therefore, when we can assume a credibility over information sources, we should decide a belief which all agents should have, considering the credibility.

Belief *fusion*, instead of AGM belief revision [3, 14], was first proposed by Maynard-Reid II and Shoham [29], to solve the problem about the multi-agent case. Suppose each agent has a total preorder on possible worlds [23], based on the semantic work (cf. [18, 20]). It represents the priority over possible worlds, and each agent considers that a world is more plausible than the other world with her preorder. Maynard-Reid II and Shoham called it *belief state*, and they considered that each agent has *informational source*, and each source has a belief state. Also they assumed the strict total order over the sources as the credibility of sources, and considered that if a source has higher credibility than others, then her opinion dominates other's opinion.

However, the total credibility ranking is a very strong constraint for the application in the multi-agent case. If two sources has same credibility or are not comparable, then we cannot formalize such a case. In [28], using the modular and transitive relation, called by *generalized belief state*, Maynard-Reid II proposed the model with the total preorder over the sources. However, he did not propose the model with the partial order. Therefore, my purpose is to expand the model for the strict total order to the model for the strict partial order, and to expand the model for the model for the total preorder, to the model for the partial preorder.

Describing the model for the strict partial order, in Suzuki and Tojo [44], I used *preferential model* as the representation for agent's belief. It is famous for the research of nonmonotonic reasoning [42, 22, 31], and allows each agent to have the *strict partial orders* over possible worlds. Especially, I called such an ordering *preferential relation*, which represents plausibility over possible worlds. It is different with a belief state and a generalized belief state at the point that the equivalence sets of possible worlds are not total. Therefore, it allows us to formalize more various type of beliefs than the previous two. In this paper, I will show that Maynard-Reid II's approach with belief states and generalized belief states also can be used for the partially preordered sources. Moreover, I will study coherency of belief aggregation for various types of belief.

1.1 the problem on the pedigreed belief fusion

Considering the aggregation of beliefs, I encounter the problem about the refinement operator. I call *the refinement* of an agent's belief state by the other's belief state, the process in which a belief state with higher credibility determines a belief state that two agents should believe commonly. Now suppose that $\olessthan$ is a refinement operator. I regard '$\preceq_A \olessthan \preceq_B$' as the result of the refinement of $\preceq_A$; by $\preceq_B$ where $\preceq_A$ is the order of Agent A, that is more reliable than that of Agent B, and the result of '$\olessthan$' is the refined order. However, the iterated aggregation $(\preceq_A \olessthan \preceq_C) \olessthan \preceq_B$ is spurious in case $\preceq_C$ is more reliable than both of $\preceq_A$ and $\preceq_B$.

In order to solve this problem, Maynard-Reid II and Shoham introduced the *pedigreed belief state*. They considered that each belief state had a *pedigree*, that is, each belief state was regarded as an informational source which provided information to an agent who aggregated all sources' information. Each source was assigned credibility, which was determined by the strict total order over sources, and the agent determined a belief state, which should be owned by all sources commonly, observing the credibility ranking and applying the refinement operator in order. Thus the *belief fusion* was defined by the union of two pedigreed belief states.

However, I do not want to consider that the strict total order is a unique credibility ranking for the application of such an operator. Let me consider the following piece of detective story.

Example 1.1 *A criminal is said to be one of the four: P, Q, R, and S. Two inspectors A and B had common information from an identification (s_1), but A had other information from an old man (s_2), and B had other information from a child (s_3). They uttered as follows:*

s_1 *"there are fingerprints of them except for S at the crime scene."*

s_2 *"Q bought the weapon, but R was not at the crime scene."*

s_3 *"P remained at the neighborhood of the scene."*

The investigation headquarter wants to amalgamate all these information, considering the reliability of each source. s_1 is more credible than s_2 and s_3, but s_2 is incomparable with s_3. Who should the police investigate first?

I cannot directly apply the belief fusion to this case because sources s_i $(i = 1, 2, 3)$ are only *partially* ordered. That is to say, belief fusion does not show the process in which the agents aggregate the beliefs from sources, when I can not compare the credibility of sources.

I give another example to show that the *totality* of the ranking of sources is too strong to hypothesize.

Example 1.2 *Suppose that two TV productions make a TV program cooperatively. Directors A and B belong to one production, and A has higher rank than B. Directors C and D belong to another production, and C has higher rank than D. It is already determined that the master of TV program is one of P, Q, and R. However, they do not select who is one, and they have different opinions.*

A. "R is more suitable than P, but I don't know whether Q is more suitable than P and R."

B. "R is more suitable than Q, and P is more suitable than R."

C. "R is more suitable than Q, but I don't know whether P is more suitable than R and R."

D. "Q is more suitable than P, and P is more suitable than R."

How do we solve the confliction?

In this example, I consider the ranking of directors as the credibility ordering over sources s_i $(i = 1, 2, 3, 4)$, and each opinion is regarded as ordering over possible worlds.

By [29, 28], the problem is considered as the opposition of four directors. However, I will consider that the problem arises from the opposition of a *chain* of A and B, and another chain of C and D. Because I know the comparability about A and B, and the one about C and D, I can regard each chain as a total order. Therefore, I can consider the following strategy:

- For each chain, we apply the refinement operation, and induce a belief which all sources in the chain should have commonly.

- If we calculates induced beliefs from all chains, then we take the information which all induced beliefs have, and we regard it as the induced belief which all sources should have in common.

Using such a strategy, I deal with the partiality of the credibility of sources. Note that if I use belief states or generalized belief states, then I cannot use this strategy, because the result of this process may not be a belief state nor a generalized belief state.Thus the principle of unanimity is not applicable to (generalized) belief states. Therefore, I use preferential relations, and also I can use transitive relation over possible worlds. I will call it *generalized preferential relations.*

By the way, Example 1.1 also can be represented by the model of [28]. However, Examle 1.2 can not be represented by the model, because in [28], the sources are totally preordered. How do I use belief states and generalized belief states for the case of partially ordered sources? I can consider the following strategy:

- For each chain, I apply the refinement operation, and induce a belief which all sources in the chain should have commonly.
- If I calculates induced beliefs from all chains, then we take the information which some induced beliefs have, and I regard it as the induced belief which all sources should have in common.

That is to say, the principle of respecting minority is applicable.

1.2 Belief Revision

Belief fusion is a variation of belief revision. Therefore, I briefly introduce the studies of belief revision in this section. This is the study about changes in the beliefs of minds and in the data of datadases. This subject grew out of two research areas. one of these is computer science. Since the beginning of the research, programmers have constructed databases and procedures by which they can be updated. Doyle [10] developed *Truth Maintenance System* (TMS), and then Fagin, Ullman and Vardi [11] introducced the notion of *database priorities.* The second of the research is philosophy. In the twentieth century, philosophers of science have discussed the mechanisms by which scientific theories develop, and they have proposed criteria of rationality for revisions of probability assignements. In 70's, the formal framework of rational belief change was provided by Levi [25].

In 1985, Alchourrón, Gärdenfors, and Makinson (AGM) published the very influential paper of this field [3]. They constructed the operations which incorporate the epistemic input with eliminating the inconsistent knowledge, and the postulates which it should satisfy. They call such operations *revisions.* The most important point of this paper is that they proved the representation theorem about the constructions and the postulates. Gärdenfors's motivation was to prove that whether a scientific explanation (or conditional) was accepted or not was decided by the epistemic circumstances. He considered that conditionals accepted by *Ramsey test* [37][1] were analyzed by belief revision operators. However, Ramsey test by belief revision was not comparable with the famous axiomatic system of conditional logic [27]. For details, see [14]. Whereas AGM only formalized the syntactical operation, Grove [18], and Katsuno and Mendelzon [20] studied the semantic versions of belief revision, and proved the representation theorem. Moreover, Katsuno and Mendelzon [21] formalized the another semantic operator with respect to the dynamic world, whereas revision is an operator with respect to the static world. They call it *update*. Grahne [17] showed that update was compatible with the axiomatic system of conditional logic.

In 90's, the various paper is concerned with the problem of the *iterated belief revision*, where the epistemic input is not a formula , but a sequence of formulae [8, 9, 12]. According to these studies, it is recognized that a *belief state*, which includes the strategy of revision operation, is more plausible representation than the AGM's *belief set*, which is a logically closed set. In semantic version of belief revision, belief states were considered as total preorder over possible worlds. Whereas old studies considered that revision operators accepted a belief state and a proposition, and returns a new proposition, studies about iterated belief revision considered that revision operators accepted a belief state and a proposition, and returns a new belief state. For details, see [40]. However, in all these studies, multi-agent case was not considered. Belief fusion is the studies about the aggregation of belief states of multi-agents [29, 28]. In this paper, I will represent an expansion of belief fusion.

1.3 Outline

In this paper, I propose a framework for the belief change in the multi-agent case. At first, I supply a preliminary for the discussion in Chapter 2.

I introduce the refinement operator for preferential relations in Chapter 3; although the same operator was mentioned rather easily in [29], my definition of the refinement includes various problems, and thus I spare one chapter for the explanation. That is to say, I will discuss the following points in this chapter:

- The tentative definition of the refinement operator of preferential relations and the problem about the definition.
- The revised definitions of the previous one and the various proposition.
- The definition of the refinement operator of generalized preferential relations

In Chapter 4, I construct the aggregation operator. This operator is shown by [28] with generalized belief states. In [29, 44], this operation was regarded as the process which constructs the induced relation. Therefore, I can rewrite the process of [29, 44] with the aggregation. In this chapter, I formalize the various aggregations as follows:

[1]Accept a conditional 'If A, then C' in an epistemic state K iff the minimal change of K needed to accept A also requires accepting C.

- Various operations for constructing an aggregator of the beliefs given a set of equally ranked sources.

- Various operations for constructing an aggregator of the beliefs given a set of total strictly ranked sources.

- Various operations for constructing an aggregator of the beliefs given a set of totally preordered sources.

- Various operations for constructing an aggregator of the beliefs given a set of partially preordered sources.

In Chapter 5, I describe fusion operators of pre-aggregated belief states, generalized belief state, preferential relations and generalized preferential relations. Therefore, the following points will be discussed:

- introducing the *pedigreed belief state*, the *pedigreed generalized belief state*, the *pedigreed preferential relation* and the *pedigreed generalized preferential relation*.

- the fusion operators with the pedigreed belief states, the pedigreed generalized belief states, the pedigreed preferential relations and the pedigreed generalized preferential relations.

In Chapter 6, I show that the aggregation operators have various somethings in common. Especially, I can show the following points:

- the propositions about the relation between the operation for belief state and preferential relation.

- the propositions about the relation between the operation for belief state and generalized preferential relation.

Finally in Chapter 7, I summarize our contribution and discuss various issues of my formalization. If the proof of a proposition or lemma is very short, I will write it under the proposition or lemma. Otherwise, I will show the proof in Appendix A.

Chapter 2

Formal Preliminaries

As the previous study [44], my research is related with *possible world semantics*. I denote the non-empty set of worlds as $\mathcal{W}$. I will assume that $\mathcal{W}$ is finite. This assumption is not necessary, but this restriction will let the discussion be simple.

Syntactically, I assume a language $\mathcal{L}$. A world w is an interpretation over $\mathcal{L}$. In addition, I can define such satisfaction relation $\models$ that, for a world $w \in \mathcal{W}$ and a sentence $p \in \mathcal{L}$, $w \models p$ iff p is evaluated to be true in w. Given a sentence p, I denote the set $\{w \in \mathcal{W} | w \models p\}$ as $|p|$. Let $p \models q$ dente $\forall w \in |p|$, $w \models q$. We use $\leq$ as an arbitrary relation, but it usually means an order. If (x, y) satisfies $\leq$, I denote $x \leq y$ or $(x, y) \in \leq$, interchangeably. Otherwise, I denote $x \not\leq y$ or $(x, y) \notin \leq$. I will write the set of all the relations $2^{\mathcal{W} \times \mathcal{W}}$ as $\mathcal{R}$.

In this paper, I will use various restrictions over relations. I define some of them which will be useful for the discussion. I will call arbitrary relations simply relations except the special case.

Definition 2.1 *Suppose that $\leq$ is a relation over a set Ω, i.e., $\leq \subseteq \Omega \times \Omega$. The relation $\leq$ is:*

1. serial *iff for all $x \in \Omega$ there exists $y \in \Omega$ such that $x \leq y$.*
2. reflexive *iff $x \leq x$ for all $x \in \Omega$. It is* irreflexive *iff $x \not\leq x$ for all $x \in \Omega$.*
3. symmetric *iff $x \leq y \Rightarrow y \leq x$ for all $x, y \in \Omega$. It is* asymmetric *iff $x \leq y \Rightarrow y \not\leq x$ for all $x, y \in \Omega$. It is* antisymmetric *iff $x \leq y \wedge y \leq x \Rightarrow x = y$ for all $x, y \in \Omega$.*
4. Euclidean *iff $x \leq y \wedge x \leq z \Rightarrow y \leq z$ for all $x, y, z \in \Omega$.*
5. *the* strict version *of a relation $\leq'$ over Ω iff $x \leq y \Leftrightarrow x \leq' y \wedge y \not\leq' x$ for all $x, y \in \Omega$.*
6. total *iff $x \leq y \vee y \leq x$ for all $x, y \in \Omega$. It is* partial *iff it is not total.*
7. connected *iff $x \leq y$ for all $x, y \in \Omega$. It is* disconnected *iff $x \not\leq y$ for all $x, y \in \Omega$.*
8. modular *iff $x \leq y \Rightarrow x \leq z \vee z \leq y$ for all $x, y, z \in \Omega$.*
9. transitive *iff $x \leq y \wedge y \leq x \Rightarrow x \leq z$ for all $x, y, z \in \Omega$.*

10. *the* transitive closure *of a relation* $\leq'$ *over* Ω *iff* $x \leq y$ *implies* $\exists w_0, ..., w_n \in \Omega . x = w_0 \leq' \cdots \leq' w_n = y$. *for some integer* n, *for* $x, y \in \Omega$. [1]

11. acyclic *iff* $\forall w_0, ..., w_n \in \Omega . w_0 \leq \cdots \leq w_n$ *implies* $w_n \not\leq w_0$ *for all integers* n.[2] *It is cyclic iff it is not acyclic.*

Definition 2.2 *Suppose that* $\leq$ *is a relation over a set* Ω, *i.e.,* $\leq \subseteq \Omega \times \Omega$. *The relation* $\leq$ *is:*

1. *a* partial order *iff it is refliexive, anti-symmetric, and transitive.*

2. *a* partial preorder *iff it is reflexive and transitive. It is a* total preorder *iff it is also total. It is* total order *iff it is also anti-symmetric.*

3. *a* strict partial order *iff it is the strict version of a partial order. It is a* strict total order *iff it is the strict version of a total order. It is a* strict total preorder *iff it is the strict version of a total preorder.*

4. *an* equivalence relation *iff it is reflexive, symmetric, and transitive.*

I can show the following proposition.

Proposition 2.1

1. *The transitive closure of a relation is transitive.*

2. *The transitive closure of a modular relation is modular.*

3. *The transitive closure of a cyclic relation is not irreflexive.*

4. *A relation is a strict partial order iff it is irreflexive and transitive.*

5. *A relation is a total order iff it is a partial order and total.*

6. *If a relation is a strict total order, then it is a strict partial order and modular.*

7. *If* $\leq^+$ *is the transitive closure of a modular relation* $\leq$ *over set* Ω, *then* $x \leq^+$ *and* $x \not\leq y$ *imply* $y \leq^+ x$ *for* $x, y \in \Omega$.

8. *If a relation is the strict version of some relation, then it is irreflexive.*

9. *If a relation is a strict version of a transitive relation, then it is acyclic and transitive.*

[1] The definition of transitive closure depends on the assumption that Ω is infinite. It is not sufficient if Ω can be infinite.

[2] The definition of acyclicity also depends on the assumption that Ω is finite and is not sufficient and is not sufficient if Ω can be infinite.

Proof. See Appendix A. □

Given a relation over a set of alternatives and a subset of these alternatives, I often want to select the set of best elements with respect to the relation. I define the set of best elements to be the set's *choice set*:

Definition 2.3 *If $\leq$ is a relation over a finite set Γ, $<$ is its strict version, and $X \subseteq \Omega$, then the* choice set of X *with respect to $\leq$ is*

$$min(X, \leq) = \{x \in X : \forall x' \in X . x' \not< x\}.$$

A *choice function* is a function which accepts every subset X and returns a non-empty subset of X:

Definition 2.4 *A Choice function over a finite set Ω is a function $f : 2^{\Omega} \backslash \emptyset \rightarrow 2^{\Omega} \backslash \emptyset$ such that $f(X) \subseteq X$ for every $X \subseteq \Omega$.*

Now, every acyclic relation defines a choice function, one which assigns to each subset its choice set:

Proposition 2.2 *Given a relation $\leq$ over a finite set Ω, the choice set operation min defines a choice function iff $\leq$ is acyclic.*

Proof. See [41]. □

If a relation is cyclic, elements involved in a cycle are said to be in a *conflict* because I cannot order them:

Definition 2.5 *Given a relation $\leq$ over a finite set Ω, x and y are in a conflict w.r.t. $<$ iff there exist $w_0, ..., w_n, z_0, ..., z_m \in \Omega$ such that $x = w_0 < \cdots < w_n = y = z_0 < \cdots < z_m = x$, where all $x, y \in \Omega$.*

Chapter 3

Refinement

3.1 Introduction

Formalizing the examples in Chapter 1, I should consider two following requirements.

1. An appropriate knowledge representation not only for individual beliefs, but for collective beliefs as well that accommodates conflicting opinions.

2. A process for constructing an agent's belief state by aggregating the information from informant sources, accounting for the relative credibility of these sources.

However, before discussing collective beliefs, I will prepare the knowledge representation for individual beliefs, and the operation which accepts two individual beliefs and generates one individual belief. For the knowledge representation, I will introduce *preferential relation* and *generalized preferential relation*. They represent opinions on the relative likelihood of worlds. Preferential relation is used in the semantic works of the nonmonotonic reasoning community [42, 22, 31], but generalized preferential relation is my innovation for dealing with conflicting information. I will declare the strict version of a generalized preferential relation to be a preferential relation. Therefore, I can consider that a generalized preferential relation also can be viewed as the semantics for nonmonotonic reasoning. The connection of the two representations is an analogy of the connection of belief states and generalized belief states [28], because generalized belief states are not different from the belief states much. As we already discussed, using preferential relations and generalized preferential relations, the equivalence sets of possible worlds are not total.

For the aggregating process, I introduce an operation which combines two preferential relations (or generalized preferential relations) of agents, where one of them is more reliable than another one, in the similar way of the refinement of belief states(or generalized preferential relations), naming *refinement of preferential relations* (or *refinement of generalized preferential relations*). Note again that preferential relations are strict partial orders whereas belief states are total preorders. Naturally, the operation would become more complicated than the operation of belief states.

3.2 Representation of individual beliefs

I begin with the problem on knowledge representation for individual beliefs. My representation is owed to the representation developed in the nonmonotonic reasoning community. In addition, my formalization is related with Maynard-reid II and Shoham's formalization, and they use the representation developed in the belief revision community. It follows that I review the research about nonmonotonic reasoning and the relation between belief revision and it. Then I will consider whether my representation is plausible.

3.2.1 Nonmonotonic reasoning

Gabbay [13] is the first researcher who examined the important properties which, even in the absence of monotonicity, may still sometimes be satisfied. He noticed that even if monotonicity was not satisfied by a consequence relation, an important property might still be satisfied by it. He called it *cumulativity.* Makinson [30] also studied about cumulativity deeply.

In addition to the two studies, various researchers tried to list up the property for nonmonotonic reasoning. Now the following properties are regarded as usual and useful properties [7, 31].

Definition 3.1 *A binary relation* $\mid\sim$ *on* $\mathcal{L} \times \mathcal{L}$ *is a* rational consequence relation *if for all* $p, q, r \in \mathcal{L}$*, it satifies*

$p \mid\sim p$ *REF (Reflexivity)*

$$\frac{p \mid\sim q, p \dashv\models r}{r \mid\sim q} \quad LLE \quad \textit{(Left Logical Equivalence)}$$

$$\frac{p \mid\sim q, q \models r}{p \mid\sim r} \quad RWE \quad \textit{(Right Weakening)}$$

$$\frac{p \mid\sim q, p \mid\sim r}{p \mid\sim q \wedge r} \quad AND$$

$$\frac{p \mid\sim q, r \mid\sim q}{p \vee r \mid\sim q} \quad OR$$

$$\frac{p \mid\sim q, p \wedge q \mid\sim r}{p \mid\sim r} \quad CUT$$

$$\frac{p \mid\sim q, p \mid\sim r}{p \wedge q \mid\sim r} \quad CMO \quad \textit{(Cautious Monotonicity)}$$

$$\frac{p \mid\sim q, p \not\mid\sim \neg r}{p \wedge r \mid\sim q} \quad RMO \quad \textit{(Rational Monotonicity).}$$

A binary relation $\mid\sim$ *on* $\mathcal{L} \times \mathcal{L}$ *is called a* preferential consequence relation *if it satisfies all the above except, possibly, RMO. A binary relation* $\mid\sim$ *on* $\mathcal{L} \times \mathcal{L}$ *is called a* cumulative consequence relation *if it satisfies all the above except, possibly, OR and RMO.*

$p \mid\sim q$ means "If p, then naturally q." The combined rule of CUT and CMO are also called cumulativity. REFs show that premise must be included in the conclusion. LLE shows that the conclusion can not be affected by the syntax of premise. RW and AND show that a classical consequence relation is included in the relations. OR show that if a conclusion can be derived from both premises, then it also can be derived from the disjunction of the premises. Cumulativity showed that adding or deleting medium lemmas should not affect the result of reasoning. RMO showed that if the negation of a formula is not derived, then adding it does not affect the result of reasoning. In fact, CUT is not necessary for rational and preferential consequence relation.

Proposition 3.1 *The rules REF,LLE,RW,AND,and OR imply CUT.*

Proof. See [22]. □

Kraus, Lehmann, and Magidor [22] showed that the rules were able to be used to criticize, through the failure of one or more of them, the main alternative approaches of circumscription [33], autoepistemic logic [34, 35], and default reasoning [38]. For details, see [4]. Makinson [31] extended the consequence relation to the *consequence operation*, dealing with the infinite set of formulae, and he classified the approaches of defeasible inheritance net [45], default reasoning and its kin, maxiconsistent set [36], epsilon/delta approach [1, 2], and preferential model.

Preferential model is the semantics which is proposed by Shoham [42, 43] for a generalization of circumscription [33]. Let me define *anonymous* preferential (or rational) relations and preferential (or ranked) models.

Definition 3.2 *A* (anonymous) preferential relation $\leq^p$ *(over* $\mathcal{W}$*) is a strict partial order over* $\mathcal{W}$*. A* preferential model *is a triplet* $\mathcal{M}^p = (\mathcal{W}, \models, \leq^p)$.

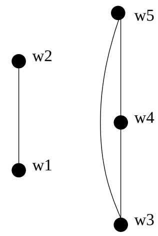
Definition 3.3 *A* (anonymous) rational relation $\leq^r$ *(over* $\mathcal{W}$*) is a strict total order over* $\mathcal{W}$*. A* ranked model *is a triplet* $\mathcal{M}^r = (\mathcal{W}, \models, \leq^r)$.

When (w_a, w_b) is neither $w_a \leq^p w_b$ nor $w_b \leq^p w_a$, I denote it by $w_a \sim^p w_b$. I also denote the set of preferential relations by $\mathcal{P}$. I interpret $w_a \leq^p w_b$ to mean "there is reason to consider w_a as more plausible than w_b."

I can define a nonmonotonic consequence relation with a preferential (or ranked) model.

Definition 3.4 *Given a triplet* $\mathcal{M} = (\mathcal{W}, \models, \leq)$*, the consequence relation defined by* $\mathcal{M}$ *will be denoted by* $\mid\!\sim_{\mathcal{M}}$ *and is defined by* $p \mid\!\sim_{\mathcal{M}} q$ *iff for any* $w \in min(|p|, \leq)$*,* $w \models q$.

I show some example for the interpretation. Suppose that an agent has an preferential relation $\leq^p$ in the left side of Figure 3.1. In this figure, she regards w_1 as more plausible world than w_2, but she thinks that w_3, w_4, w_5 are not comparable with w_1. Each possible world $w_1, ..., w_5$ is an interpretation of the propositions ϕ, ψ, and χ, which is at the right side of Figure 3.1. For example, in w_1, ϕ and ψ are true, but χ is false. In this model, the agent considers that $\phi \mid\!\sim \psi$, $\neg\psi \mid\!\sim \phi$, $\chi \mid\!\sim \phi$ are true, but $\psi \mid\!\sim \phi$, $\chi \mid\!\sim \psi$, $\neg\chi \mid\!\sim \phi$ are false. For example, w_1 and w_4 are all worlds in $min(|\phi|, \leq^p)$, and ψ is true for w_1 and w_4. It



	ϕ	ψ	χ
w_1	1	1	0
w_2	1	0	1
w_3	0	1	0
w_4	1	1	1
w_5	1	0	0

Figure 3.1: The example of the interpretation for a consequence relation.

follows that $\phi \mid\sim \psi$ is true in this model. However, w_3 is some world in $min(|\psi|, \leq^p)$, but ϕ is false in w_3. It follows that $\psi \mid\sim \phi$ is false.

Kraus, Lehmann, and Magidor showed the following representation theorems. [1]

Proposition 3.2 *A consequence relation is a preferential consequence relation iff it is defined by some preferential model.*

Proof. See [22]. □

Proposition 3.3 *A consequence relation is a rational consequence relation iff it is defined by some ranked model.*

Proof. See [26]. □

However, Makinson [31] considered that RMO was unsuitable for the application, because all the approaches of default reasoning and its kin, maxiconsistent set, epsilon/delta approach might not satisfy this rule. In addition, the ranked relation cannot be used to formalize the examples in Chapter 1. Therefore, the ranked relation is not useful for my purpose. I will concentrate on the preferential relation and its extension.

Whereas preferential relations are used for the study of nonmonotonic reasoning, *belief states* are used for the study of belief revision.

Definition 3.5 *A* (anonymous) belief state $\leq^b$ *(over $\mathcal{W}$) is a total preorder over $\mathcal{W}$.*

When (w_a, w_b) is both $w_a \leq^b w_b$ and $w_b \leq^b w_a$, we denote it by $w_a \infty^b w_b$. We also denote the set of belief states by $\mathcal{B}$.

I can not only interpret the minimal worlds of a belief state as actual belief, which is believed now, but also interpret a belief state as conditional belief. That is to say, a belief state represents what would be believed if other conditions were the case. According to this criteria, the conditional belief "if p then q" holds, when for any $w \in min(|p|, \leq^b)$, $w \models q$, as well as the nonmonotonic reasoning. I write $Bel^b(p?q)$ when the conditional

[1] Note that I restrict $\mathcal{W}$ to be finite, and then it is not necessary to assume that a relation is *stoppered* [31].

belief holds. b denotes the belief state $\leq^b$. Therefore, given a preferential relation $\leq^p$, I write $Bel^p(p?q)$ as the abbreviation that $p \mid\sim q$ is defined by $\mathcal{M}^p = (\mathcal{W}, \models, \leq^p)$. If neither the belief $p?q$ nor its negation hold in the belief state $\leq^b$, it is said to be *agnostic* with respect to $p?q$ in the belief state $\leq^b$, written $Agn^b(p?q)$. In the same way, if neither the belief $p?q$ nor its negation hold in the preferential relation $\leq^p$, it is said to be *agnostic* with respect to $p?q$ in the preferential relation $\leq^p$, written $Agn^p(p?q)$. Note that $Agn^b(p?q)$ implies $Agn^p(p?q)$.

From the above discussion, I can consider the semantic version of belief revision. Given a belief state $\leq^b$, let K be $min(\mathcal{W}, \leq^b)$. Then I can define the AGM belief revision in the semantic works [18, 20].

Definition 3.6 *Given a belief state $\leq^b$. A revision operator $*$ defined by $\leq^b$ is such that $K * p = min(|p|, \leq^b)$.*

It is already proved by [18, 20] that a belief revision operator is defined by some belief state iff it satisfies the famous *AGM postulates* [3].

Makinson and Gärdenfors [32, 15] studied the relation between nonmonotonic reasoning and belief revision. Comparing these studies with my notation, the translation between the two is performed by the following condition.

$$p \mid\sim_K q \text{ iff for all } w \in K * p, w \models q$$

They indicated the correspondence between many of AGM postulates and many rules of rational consequence relation. For details, see [16].

Maynard-Reid II [28] also discussed the relation between the aggregation operation of belief states and aggregation operation of social choice functions [5, 41]. However, I concentrate on the relation between the aggregation operation of belief states and the aggregation operation of preferential relation. Besides, see Rott [39] about the relation about the properties of belief revision, nonmonotonic reasoning, and choice function.

3.2.2 Generalized preferential relations

Maynard-Reid II [28] defined *generalized belief states* for representing conflict information and solving the problem on the aggregation operation of social choice functions.

Definition 3.7 *A* generalized belief state $\leq^{gb}$ *is a modular, transitive relation over $\mathcal{W}$.*

When (w_a, w_b) is neither $w_a \leq^{gb} w_b$ nor $w_b \leq^{gb} w_a$, I denote it by $w_a \sim^{gb} w_b$ and I call the relationship *agnosticism*. When (w_a, w_b) is both $w_a \leq^{gb} w_b$ and $w_b \leq^{gb} w_a$, I denote it by $w_a \infty^{gb} w_b$ and I call the relationship *conflict*. (Note Definition 2.5. In fact, I can easily show that ∞^{gb} represents conflict in the sence of the definition.) I also denote the set of generalized belief states by $\mathcal{GB}$. I interpret $w_a \leq^{gb} w_b$ to mean "there is reason to consider w_a as strictly more likely than w_b."

The following proposition is given by [28].

Proposition 3.4 *$\leq^{gb} \in \mathcal{GB}$ iff there is a partition $\mathbf{W}$ of $\mathcal{W}$ and a total order $\leq$ over $\mathbf{W}$ such that:*

1. *Every $W \in \mathbf{W}$ is either fully connected ($w \leq^{gb} w'$ for all $w, w' \in W$) or fully disconnected ($w \not\leq^{gb} w'$ for all $w, w' \in W$).*
2. *For every $W, W' \in \mathbf{W}$, $w \in W$, and $w' \in W'$, $W \leq W'$ iff $w \leq^{gb} w'$.*

Proof. See [28]. □

Note that we also can show the following proposition.

Proposition 3.5 *$\leq^{b} \in \mathcal{B}$ iff there is a partition $\mathbf{W}$ of $\mathcal{W}$ and a total order $\leq$ over $\mathbf{W}$ such that:*

1. *Every $W \in \mathbf{W}$ is fully connected ($w \leq^{b} w'$ for all $w, w' \in W$).*
2. *For every $W, W' \in \mathbf{W}$, $w \in W$, and $w' \in W'$, if $W \neq W'$ then $W \leq W'$ iff $w \leq^{gb} w'$.*

Proof. See Appendix A. □

Thus, generalized belief states are not much different from the strict versions of total preorders. Besides, there is a straight translation between strict versions of total preorders and the total preorders. Therefore, generalized belief states are not much different change from the belief states. Let $\mathcal{B}_<$ be the set of strict total preoreder over $\mathcal{W}$.

Proposition 3.6 *There is a bijection from $\mathcal{B}$ to $\mathcal{B}_<$.*

Proof. See Appendix A. □

From the proposition, I can understand that belief states have no difference between agnostic and conflict information.

As belief states, I can define generalize preferential relations.

Definition 3.8 *A* generalized preferential relation *$\leq^{gp}$ is a transitive relation over $\mathcal{W}$.*

Following the above discussion, when (w_a, w_b) is neither $w_a \leq^{gp} w_b$ nor $w_b \leq^{gp} w_a$, I denote it by $w_a \sim^{gp} w_b$. When (w_a, w_b) is both $w_a \leq^{gp} w_b$ and $w_b \leq^{gp} w_a$, I denote it by $w_a \infty^{gp} w_b$. I also denote the set of generalized preferential relations by $\mathcal{GP}$. I interpret $w_a \leq^{gp} w_b$ to mean "there is a reason to consider w_a as strictly more plausible than w_b."

Now I can distinguish between agnostic and conflicting conditional belief with a generalized beliet state (or a generalized preferential relation). A generalized beliet state $\leq^{gb}$

(or a generalized preferential relation $\leq^{gp}$) is agnostic about conditional belief $p?q$ (i.e., $Agn^{gb}(p?q)$ (or $Agn^{gp}(p?q)$)) if for some $w, w' \in min(|p|, \leq^{gb})$ (or $min(|p|, \leq^{gp})$) which is fully disconnected, $w \models q$ and $w' \models \neg q$. A generalized beliet state $\leq^{gb}$ (or a generalized preferential relation $\leq^{gp}$) is conflict about conditional belief $p?q$ (i.e., $Con^{gb}(p?q)$ (or $Con^{gp}(p?q)$)) if for some $w, w' \in min(|p|, \leq^{gb})$ (or $min(|p|, \leq^{gp})$) which is fully connected, $w \models q$ and $w' \models \neg q$.

I can show the following property:

Proposition 3.7 *Let $\leq^{gb}$ be a generalized belief state (or $\leq^{gb}$ be a generalized preferential relation).*

1. *∞^{gb} (or ∞^{gp}) is a symmetric and transitive relation.*
2. *$\sim^{gb}$ is a symmetric and transitive relation.*

Proof. See Appendix A. □

Now I compare the power of the class of various representations. Maynard-Reid II already showed the following properties [28].

Proposition 3.8

1. *$\mathcal{B} \subset \mathcal{GB}$, and is the set of reflexive relations in $\mathcal{B}$.*
2. *$\mathcal{B}_< \subset \mathcal{GB}$, and is the set of irreflexive relations in $\mathcal{B}$.*

Proof. See [28]. □

I also can show the following properties.

Proposition 3.9

$\mathcal{P} \subset \mathcal{GP}$.

$\mathcal{GB} \subset \mathcal{GP}$.

$\mathcal{P} \cap \mathcal{GB} = \mathcal{B}_<$.

Proof. See Appendix A. □

Finally, I will show that Example 1.1 and 1.2 in Chapter 1 can be formalized by the preferential relations.

Example 3.1 *See Example 1.1. Let P, Q, R, and S be possible worlds representing that the P, Q, R, and S are criminals, respectively. The preferential relations for the three sources are shown in Figure 3.2.*

Example 3.2 *See Example 1.2. Let P, Q, and R be possible worlds representing that the P, Q, and R are musters, respectively. The preferential relations for the four sources are shown in Figure 3.3.*

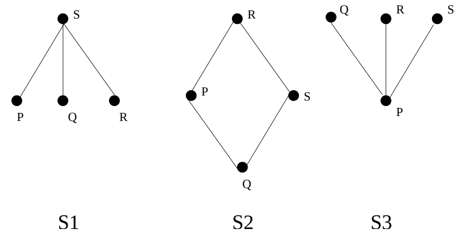


Figure 3.2: The preferential relations of s_1,s_2, and s_3 in Example 3.1.

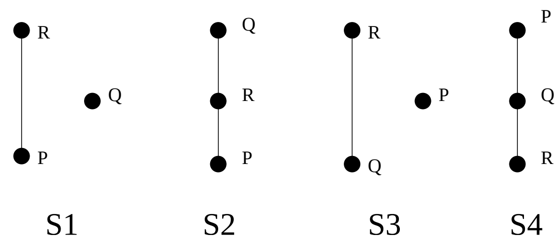


Figure 3.3: The preferential relations of s_1,s_2, s_3, and s_4 in Example 3.2.

3.3 Refinement of Preferential Relations

3.3.1 A problem about the definition of the refinement operator

If we know that an agent (A) is more credible than another agent (B), then it is natural to consider that A's judgment dominates B's. I will define the refinement operator which accepts two preferential relations and produce another, where one agent (A) is more credible than the other (B) and A's judgment dominates B's.

At first, I consider the following tentative definition of *refinement*. This definition is an analogy of the definition of the refinement of belief states.

Definition 3.9 *Suppose* $\leq^p_A, \leq^p_B \in \mathcal{P}$. *The* tentative refinement *of* $\leq^p_A$ *by* $\leq^p_B$ *is* $\leq^p_A \widehat{\circledcirc}^p \leq^p_B = \{(w_a, w_b) : w_a \leq^p_A w_b \vee (w_a \sim^p_A w_b \wedge w_a \leq^p_B w_b)\}$.

In other words, to construct the refined relation, whenever the more credible agent prefers one world to another, I side with this preference. In case the most credible agent has no preference, I follow the ranking of the less credible agent. However, this definition has problems, because the produced relation may not be a preferential relation. At first, a produced relation may not be transitive. In Figure 3.4, $w_3 \leq^p_A w_2$ and $w_2 \leq^p_B w_1$, but $(w_3, w_1) \notin \leq^p_A \widehat{\circledcirc}^p \leq^p_B$ by the definition. Secondly, even if a produced relation would be transitively closed, the relation might not be irreflexive. See Figure 3.5 where $(w_1, w_1) \in [\leq^p_A \widehat{\circledcirc}^p \leq^p_B]^+$ is not irreflexive.

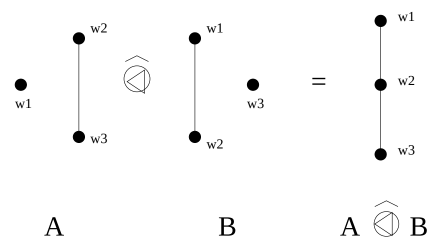


Figure 3.4: Example : the produced relation is not transitive.

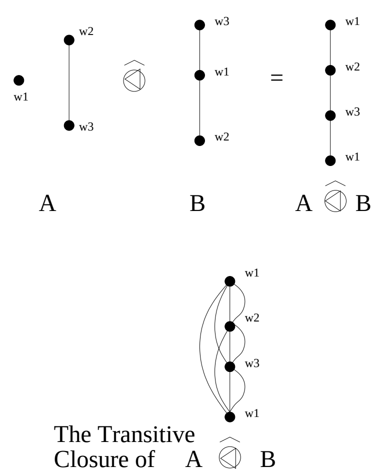


Figure 3.5: Example : the transitive closure of the produced relation is not irreflexive.

3.3.2 Revision of the refinement operator

Thus, if we want to define the refiment operator for preferential relation, then it is not sufficient that when agent A has no preference, we obey agent B's preference, and then we must consider the way to eliminate a cause of cyclicity. Therefore, I will consider the two way of the definition.

1. Avoiding cyclicity with a fixed-point equation.
2. Using strict version of the tentative refinement.

I will discuss the merit and demerit of the two ways in the following discussion. Now, I construct the first way. We will write the transitive closure of the relation $\leq$ as $\leq^{+}$. If $\mathcal{SW}$ is a subset of $\mathcal{W}$, I let $\mathcal{SW}^{+}$ be the set of transitive closures of $\mathcal{SW}$.

Definition 3.10 *Suppose* $\leq^p_A, \leq^p_B \in \mathcal{P}$. *A relation* $\leq \in R$ *is a* primitive refinement of $\leq^p_A$ by $\leq^p_B$ *iff* $\leq$ *satisfies the following equation.*

$$\begin{aligned}\leq = \leq^p_A \cup \{(w_a, w_b) : \ & w_a \sim^p_A w_b \wedge w_a \leq^p_B w_b \\ & \wedge (w_b, w_a) \notin \leq^{+}\}.\end{aligned}$$

$PRF(\leq^p_A, \leq^p_B)$ is the set of all primitive refinements of $\leq^p_A$ by $\leq^p_B$.

Proposition 3.10 *For any* $\leq^p_A, \leq^p_B \in \mathcal{P}$, *for some* $\leq \in \mathcal{R}$, $\leq \in PRF(\leq^p_A, \leq^p_B)$.

Proof. See Appendix A. □

From the proof, I can show the following corollary.

Corollary 3.1 *For any* $\leq^p_A, \leq^p_B \in \mathcal{P}$, *we can construct* $\leq \in \mathcal{R}$ *by finte steps such that* $\leq \in PRF(\leq^p_A, \leq^p_B)$.

Proposition 3.11 *For any* $\leq^p_A, \leq^p_B \in \mathcal{P}$, *if* $\leq \in PRF(\leq^p_A, \leq^p_B)$, *then* $\leq^{+}$ *is irreflexive.*

Proof. Suppose that for some $w \in \mathcal{W}$, $w \leq^{+} w$. Thus, I can show $(w, w) \in \leq^{+}$ by finite steps. However, I can not show $(w, w) \in \leq^{+}$ with only the elements of $\leq^p_A$. For if I can do so, then $(w, w) \in \leq^{p+}_A = \leq^p_A$, and it cotradicts the fact that $\leq^p_A$ is irreflexive. Therefore, I must use at least one element to show $(w, w) \in \leq^{+}$. Let the element be (w'_a, w'_b). Then I can construct the following sequence with $\leq$.

$$w, \cdots, w'_a, w'_b, \cdots, w$$

However, I can also construct the following sequence with $\leq$.

$$w'_b, \cdots, w, \cdots, w'_a$$

It contradicts $(w'_b, w'_a) \notin \leq^{+}$. Therefore, there is no $w \in \mathcal{W}$ such that $w \leq^{+} w$. □

Now, I define *the set of refinement.*

Definition 3.11 *Suppose* $\leq^p_A, \leq^p_B \in \mathcal{P}$*. The* set of refinement of $\leq^p_A$ by $\leq^p_B$ *is* $RF(\leq^p_A, \leq^p_B) = PRF(\leq^p_A, \leq^p_B)^+$.

Note that there can be multiple elements in the set of refinement of $\leq^p_A$ by $\leq^p_B$. For example in the top of Figure 3.6, two relations satisfy the condition of Definition 3.10 and thus in the middle of Figure 3.6, there are two elements in the set of refinement of $\leq^p_A$ by $\leq^p_B$ by Definition 3.11. Therefore, I need a rationale to decide a unique result of fusion. If I assume an authority who choice an element within multiple candidates, I can formalize the selection function which accepts multiple preferential relations and returns one preferential relations. However, I propose that the result is the common relations in those multiple candidates, because agents A and B do not have the selection criteria for multiple candidates.

Definition 3.12 *Suppose* $\leq^p_A, \leq^p_B \in \mathcal{P}$*. The* refinement of $\leq^p_A$ by $\leq^p_B$ *is*

$$\leq_A \circledcirc^p \leq^p_B = \cap RF(\leq^p_A, \leq^p_B).$$

From the definition, the result of the example in Figure 3.6 is shown at the bottom of this figure. This operator is well defined.

Proposition 3.12 *If* $\leq^p_A, \leq^p_B \in \mathcal{P}$*, then* $\leq^p_A \circledcirc^p \leq^p_B \in \mathcal{P}$.

Proof. From Proposition 3.10 and Definition 3.11, for any $\leq^p_A, \leq^p_B \in \mathcal{P}$, there is an element in $RF(\leq^p_A, \leq^p_B)$. Therefore, it suffices to show that if $\leq \in RF(\leq^p_A, \leq^p_B)$, $\leq$ is a strict partial order over $\mathcal{W}$. Transitivity is straightforward. Irreflexivity is shown by contradiction. If I suppose there is $w \in \mathcal{W}$ such that $w \leq w$, then from Definition 3.11 there exists $\leq' \in PRF(\leq^p_A, \leq^p_B)$ such that $w \leq'^+ w$, and it contradicts Proposition 3.10. From Proposition 2.1, $\leq$ is a strict partial order. □

Obviously, $\leq^p_A \subseteq \leq^p_A \circledcirc^p \leq^p_B$.

Although this definition of refinement is reasonable, this definition has an important problem. It is about computational cost. When I calculate all elements of the set of primitive refinement of $\leq^p_A$ by $\leq^p_B$, if N is the number of elements in $\leq^p_B$, the complexity of enumeration $O(N!)$ times for generating $(w_{a_1}, w_{b_1}), (w_{a_2}, w_{b_2}), \cdots (w_{a_N}, w_{b_N})$. Therefore, I must consider more rapid refinment operator. I will define the simple refinement operator.

Definition 3.13 *Suppose* $\leq^p_A, \leq^p_B \in \mathcal{P}$*. The* simple refinement *of* $\leq^p_A$ *by* $\leq^p_B$*, written by* $\leq^p_A \bar{\circledcirc}^p \leq^p_B$ *is the set of the strict version of* $[\leq^p_A \widehat{\circledcirc}^p \leq^p_B]^+$.

This operator is well-defined as follows.

Proposition 3.13 *If* $\leq^p_A, \leq^p_B \in \mathcal{P}$*, then* $\leq^p_A \bar{\circledcirc} \leq^p_B \in \mathcal{P}$.

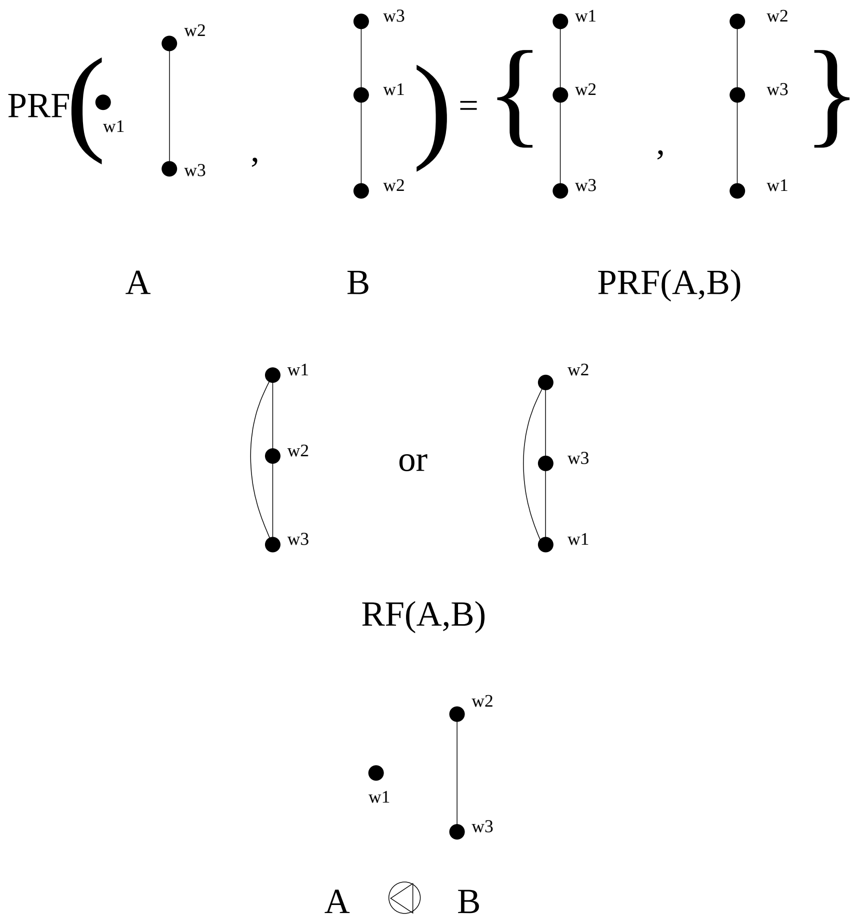


Figure 3.6: Example : there are two refinements of $\leq_A^p$ by $\leq_B^p$.

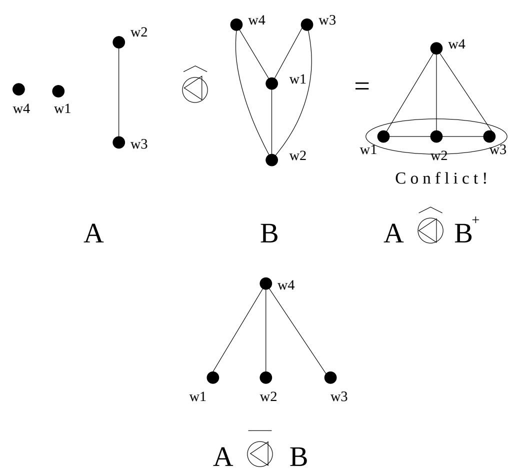


Figure 3.7: Example : the example of the simple refinement.

Proof. $\leq_A^p \bar{\circledcirc}^p \leq_B^p$ is the strict version of $[\leq_A^p \widehat{\circledcirc}^p \leq_B^p]^+$. Therefore, $\leq_A^p \bar{\circledcirc} \leq_B^p$ is irreflexive from Proposition 2.1. Moreover, $[\leq_A^p \widehat{\circledcirc}^p \leq_B^p]^+$ is the transitive closure of $\leq_A^p \widehat{\circledcirc}^p \leq_B^p$. Hence, $[\leq_A^p \widehat{\circledcirc}^p \leq_B^p]^+$ is transitive. Therefore, $\leq_A^p \bar{\circledcirc} \leq_B^p$ is transitive from Proposition 2.1. $\square$

However, this operator has a problem. See 3.7.In the top of Figure 3.7, there are conflictions in the transitive closure of the tentative refinement of agent A's preferential relation by agent B's preferential relation. Thus in the bottom of Figure 3.7, to solve this confliction, I take the strict version of the operation. However, this simple refinement of agent A's preferential relation by agent B's preferential relation does not preserve agent A's preferential relation. Therefore, $\leq_A^p \subseteq \leq_A^p \bar{\circledcirc}^p \leq_B^p$ may not be satisfied.

I put the merits and demerits of the above operators as follows.

- The merit of refinement is that this operator preserves information from sources which have high credibility. The demerit of it is that it has high computational cost.
- The merit of simple refinement is that it has low computational cost. The demerit of it is that it may not preserve information from sources which have high credibility.

3.4 Refinement of Generalized Preferential Relations

In the last section, I can not get an efficient and preservative refinement operation. I consider that it is because preferential relations are irreflexive, and do not allow conflict information. Therefore, if I find out confliction in the process of an operation, I must consider the way of eliminating it. In Definition 3.9, introducing a fixed point equation, I avoided confliction. In Definition 3.12, calculating the strict version, I eliminated confliction.

Therefore, if I allow agent's belief to be a conflict, I do not need the techniques for eliminating it. Generalized preferential relations do not require us irreflexibity and such an elimination. Now I define the refinement operator of generalized preferential relations.

Definition 3.14 *Suppose* $\leq_A^{gp}, \leq_B^{gp} \in \mathcal{GP}$. *The* tentative refinement *of* $\leq_A^{gp}$ *by* $\leq_B^{gp}$ *is* $\leq_A^{gp} \widehat{\circledcirc}^{gp} \leq_B^{gp} = \{(w_a, w_b) : w_a \leq_A^{gp} w_b \vee (w_a \sim_A^{gp} w_b \wedge w_a \leq_B^{gp} w_b)\}$.

However, this definition is not well-defined. For example, let $\mathcal{W} = \{w_a, w_b, w_c\}$, $\leq_A^{gp} = \{(w_a, w_b)\}$, and $\leq_B^{gp} = \{(w_b, w_c)\}$. Then $\leq_A^{gp} \widehat{\circledcirc}^{gp} \leq_B^{gp} = \{(w_a, w_b), (w_b, w_c)\}$. It is not transitive. Hence, I will take the transitive closure of the result of $\widehat{\circledcirc}^{gp}$.

Definition 3.15 *Suppose* $\leq_A^{gp}, \leq_B^{gp} \in \mathcal{GP}$. *The* refinement *of* $\leq_A^{gp}$ *by* $\leq_B^{gp}$ *is* $\leq_A^{gp} \circledcirc^{gp} \leq_B^{gp} = [\leq_A^{gp} \widehat{\circledcirc}^{gp} \leq_B^{gp}]^+$.

Thus, this operation is well-defined.

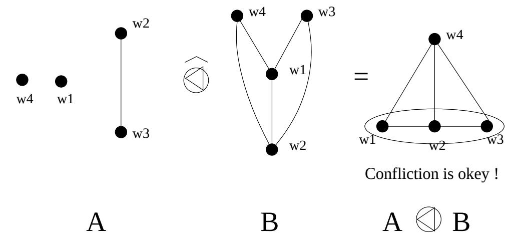


Figure 3.8: Example : the example of the refinement of generalized preferential relations.

Proposition 3.14 *If $\leq_A^{gp}, \leq_B^{gp} \in \mathcal{GP}$, then $\leq_A^{gp} \textcircled{$\lhd$}^{gp} \leq_B^{gp} \in \mathcal{GP}$.*

Proof. $\leq_A^{gp} \textcircled{$\lhd$}^{gp} \leq_B^{gp}$ is the transitive closure of $\leq_A^{gp} \widehat{\textcircled{$\lhd$}}^{gp} \leq_B^{gp}$. From Proposition 2.1, $\leq_A^{gp} \textcircled{$\lhd$}^{gp} \leq_B^{gp}$ is transitive. □

See Figure 3.8. This figure is similar with Figure 3.7. Because preferential relations are generalized preferential relations, beliefs of agent A and B must not be changed. However, I can abbreviate the process of the strict version, because generalized preferential relations are allowed to be a conflict. In addition, it preserve agent A's preferential relation. In fact, it is easy to show that $\leq_A^{gp} \subseteq \leq_A^{gp} \textcircled{$\lhd$}^{gp} \leq_B^{gp}$ for any $\leq_A^{gp}, \leq_B^{gp} \in \mathcal{GP}$. Therefore, whereas the definition of this operator is a curtailment of the definition of the simple refinement of preferential relations, this operator has an important property which the refinement of preferential relations also has.

3.5 Refinements of Belief States and Generalized Belief States

In the above discussion, I defined the refinement operators of preferential relation and generalized preferential relation. However, I did not construct the refinement operators of belief state and generalized belief state. Therefore, I will define them now. At first, I define the operator of belief states as Maynard-Reid II and Shoham did [29]. In the following definition, $<^b$ is the strict version of $\leq^b$

Definition 3.16 *[29] Suppose $\leq_A^b, \leq_B^b \in \mathcal{B}$. The refinement of $\leq_A^b$ by $\leq_B^b$ is $\leq_A^b \textcircled{$<$}^b \leq_B^b = \{(w_a, w_b) : w_a <_A^b w_b \vee (w_a \infty_A^b w_b \wedge w_a \leq_B^b w_b)\}$.*

For the following discussion, I expand this operation. Remember Proposition 3.6. The following definition of the refinement of the strict version of belief states is very similar with Proposition 3.6.

Definition 3.17 *[29] Suppose $<_A^b, <_B^b \in \mathcal{B}_<$, where $<_A^b, <_B^b$ are the strict version of $\leq_A^b, \leq_B^b \in \mathcal{B}$. The refinement of $<_A^b$ by $<_B^b$ is $<_A^b \textcircled{$<$}_<^b <_B = \{(w_a, w_b) : w_a <_A^b w_b \vee (w_a \sim_A^{sb} w_b \wedge w_a <_B^b w_b)\}$, where $w_a \sim^{sb} w_b$ is neither $w_a <^b w_b$ nor $w_b <^b w_a$.*

I can show the similarity of these two operators. For the verification, I use $Ref : \mathcal{B}_< \rightarrow \mathcal{B}$ which is a function such that $Ref(<^b) =< \cup \{(w_a, w_b) | w_a \sim^{sb} w_b\}$, where $w_a \sim^{sb} w_b$ iff $w_a \not<^b w_b \wedge w_b \not<^b w_a$.

Proposition 3.15 *Ref is a bijection from $\mathcal{B}_<$ to $\mathcal{B}$. $Ref(<^b) = \leq^b$.*

Proof. See the proof of Proposition 3.6. □

Ref shows that the iterated refinement of belief states is similar with the iterated refinement of the strict versions of belief states.

Proposition 3.16 *Suppose $<^b_1, <^b_2 ..., <^b_N \in \mathcal{B}_<$, where $<^b_1, <^b_2, ..., <^b_N$ are the strict version of $\leq^b_1, \leq^b_2, ..., \leq^b_N \in \mathcal{B}$. Therefore, $((\leq^b_1 \circledcirc^b \leq^b_2) \circledcirc^b \cdots \leq^b_N) = Ref((<^b_1 \circledcirc^b_< <^b_2) \circledcirc^b_< \cdots <^b_N)$.*

Proof. See Appendix A. □

Corollary 3.2 *Suppose $<^b_1, <^b_2 ..., <^b_N \in \mathcal{B}_<$, where $<^b_1, <^b_2, ..., <^b_N$ are the strict version of $\leq^b_1, \leq^b_2, ..., \leq^b_N \in \mathcal{B}$. Then $((<^b_1 \circledcirc^b_< <^b_2) \circledcirc^b_< \cdots <^b_N)$ is the strict version of $((\leq^b_1 \circledcirc^b \leq^b_2) \circledcirc^b \cdots \leq^b_N)$.*

Proof. *Ref* is is a bijection from $\mathcal{B}$ to $\mathcal{B}_<$ from Proposition 3.15. $((\leq^b_1 \circledcirc^b \leq^b_2) \circledcirc^b \cdots \leq^b_N) = Ref((<^b_1 \circledcirc^b_< <^b_2) \circledcirc^b_< \cdots <^b_N)$ from Proposition 3.16. Therefore, $((<^b_1 \circledcirc^b_< <^b_2) \circledcirc^b_< \cdots <^b_N)$ is the strict version of $((\leq^b_1 \circledcirc^b \leq^b_2) \circledcirc^b \cdots \leq^b_N)$. □

Therefore, these operators are interchangeable. I can also define the operator of generalized belief states in the similar way with the operator of the strict versions of belief states.

Definition 3.18 *Suppose $\leq^{gb}_A, \leq^{gb}_B \in \mathcal{GB}$. The refinement of $\leq^{gb}_A$ by $\leq^{gb}_B$ is $\leq^{gb}_A \circledcirc^{gb} \leq^{gb}_B = \{(w_a, w_b) : w_a \leq^{gb}_A w_b \vee (w_a \sim^{gb}_A w_b \wedge w_a \leq^{gb}_B w_b)\}$.*

They are well-defined operator.

Proposition 3.17

1. *If $\leq^b_A, \leq^b_B \in \mathcal{B}$, then $\leq^b_A \circledcirc^b \leq^b_B \in \mathcal{B}$.*
2. *If $<^b_A, <^b_B \in \mathcal{B}_<$, where $<^b_A, <^b_B$ are the strict version of $\leq^b_A, \leq^b_B \in \mathcal{B}$, then $<^b_A \circledcirc^b_< <^b_B \in \mathcal{B}_<$.*
3. *If $\leq^{gb}_A, \leq^{gb}_B \in \mathcal{GB}$, then $\leq^{gb}_A \circledcirc^{gb} \leq^{GB}_B \in \mathcal{GB}$.*

Proof. See Appendix A. □

At this point, I can define the following general refinement operator.

Definition 3.19 *Suppose $\leq_A, \leq_B \in \mathcal{R}$. The refinement of $\leq_A$ by $\leq_B$ is $\leq_A \circledcirc \leq_B = \{(w_a, w_b) : w_a \leq_A w_b \vee (w_a \sim_A w_b \wedge w_a \leq_B w_b)\}$.*

Note that this operator is also the tentative refinement of preferential relation, the refinement of the strict versions of belief states, and the refinement of generalized belief states, the tentative refinement of generalized preferential relations.

Table 3.1: The truth table.

	C	S	E
w_1	1	1	1
w_2	1	1	0
w_3	1	0	1

3.6 Why do I use preferential relations?

In the above discussion, I understood that the construction of the operator for preferential relations was more difficult than that for other representations. In fact, the definition of the aggregation of preferential relations will be complicated. In spite of the troubleness, I think that it is useful to define the operators for preferential relations, because the conditional beliefs generated by belief states satisfy the rule of RMO. For details, see [16, 39]. We already have explained that the conditionals generated by preferential relations may not satisfy RMO. I think that RMO is too strong to formalize the reasoning about incomplete information. The following example is reprinted from Antoniou [4].

- Typically, in case a child is in danger, I will try to save it even if this is risky.
- If my life could be endangered by helping, then typically I call the emergency.

In this example, the reasoner will want to save a child if it is in danger. However, he do not think that his life will not be endangered by helping even if a child is not safe. Unfortunately, if he think that RMO is a reasonable rule, he conclude that even if a child is in danger, and his life could be endangered by helping, he will still try to save it.

Therefore, RMO is unreasonable in this case. Using preferential relations, I can formalize this case. Let the proposition C mean "a child is in danger," S mean "I try to save a child," and E mean "my life can be endangered by helping." Also let $w_1, w_2, w_3 \in \mathcal{W}$. Table 3.1 is the interpretations of w_1, w_2, w_3.

Suppose that $\leq^p = \{(w_2, w_3)\}$. Thus, $C \mid\!\sim S$, $C \not\mid\!\sim \neg E$, but $C \wedge E \not\mid\!\sim S$. Such an incomplete information can not be formalized by belief states.

Chapter 4

Aggregations

Because refinement operators are not symmetric operators, I encounter the same problem as we already mentioned at the top of Chapter 2 when I iteratively apply the operators. Consider $\preceq_A^p$, $\preceq_B^p$, and $\preceq_C^p$ with increasing order of dominance ($\preceq_A^p$, dominated by $\preceq_B^p$, both by $\preceq_C^p$). Presumably, the above definition would give meaningful interpretation to $(\preceq_C^p \circledcirc \preceq_B^p) \circledcirc \preceq_A^p$, since all the information in $\preceq_C^p$ dominates all the information in $\preceq_A^p$ and $\preceq_B^p$. However in case $(\preceq_C^p \circledcirc \preceq_A^p) \circledcirc \preceq_B^p$, some of the information in $\preceq_C^p \circledcirc \preceq_A^p$ would dominate the information in $\preceq_B^p$ (because they originated from $\preceq_C^p$) and others are dominated by $\preceq_B^p$ (because they originated from $\preceq_A^p$).

In the similar way to [29], I introduce '*pedigree*.' The sources can be thought of as pedigrees with fixed relations. That is to say, each source has a relation from $\mathcal{R}$ as a belief. In this paper, I assume that the sources places a "credibility" preordering on the sources, but I do not assume that the preorder is total as the previous study [28].

Given such a credibility ordering, I can define the aggregation operator, by which an agent computes the induced belief from the informant sources. In this chapter, I will formalize the aggregation operators in the case of equally ranked sources, total strictly ranked sources, totally preordered sources, and partially preordered sources, respectively.
Whatever I use as the agents' beliefs, when I formalize the aggregation operators, I think that it is natural to define them as follows.

- If the credibility of an agent's belief is not comparable with, or as same as, that of an another agent's one, apply the principle of unanimity, respecting minority, or decision by majority.
- If an agent's belief is more credible than an another agent's belief, the former belief dominates the latter belief.

The second slogan is accomplished by using refinement operators. My problem is the first slogan. When I select (generalized) belief states, or (generalized) preferential relations, Which principle I can use? The main purpose of this chapter is to solve this problem, and to show that the aggregations constructed by these two slogans are well-defined.

4.1 Sources

As I discussed in the beginning of Chapter 1, if we can decide the credibility of sources, we tends to obey the opinion of experts. Therefore, before I formalize the aggregation operation, let me begin the formal development by defining sources:

Definition 4.1 *$\mathfrak{S}$ is a finite set of sources. Each source $s \in \mathfrak{S}$ is associated with a relation $\leq_s \in \mathcal{R}$.*

I denote the agnosticism and conflict relations of a source by $\sim_s$ and ∞_s, respectively. $<_s$ is the strict version of $\leq_s$. If neither $w_a <_s w_b$ nor $w_b <_s w_a$, then I denote it as $w_a \approx w_b$. If the relation is either belief state, generalized belief state, preferential relation, or generalized preferential relation, I denote it as $\leq_s^b$, $\leq_s^{gb}$, $\leq_s^p$, or $\leq_s^{gp}$, respectively.

As I have already mentioned, I assume that the credibility ranking over the sources is a partial preorder:

Definition 4.2 *$\mathcal{R}_{\mathcal{ANK}}$ is a finite set of* ranks *partially ordered by a relation $\geq$.*

Definition 4.3 *$rank : \mathfrak{S} \rightarrow \mathcal{R}_{\mathcal{ANK}}$ assigns to each source a rank.*

Definition 4.4 *$\sqsupseteq$ is the partial preorder over $\mathfrak{S}$ induced by the ordering over $\mathcal{R}_{\mathcal{ANK}}$. That is, $s \sqsupseteq s'$ iff $rank(s) \geq rank(s')$; we say s' is as credible as s. $\sqsupseteq_S$ is the restriction of $\sqsupseteq$ to $S \subseteq \mathfrak{S}$.*

I use $\sqsupset$ and $\equiv$ to denote the asymmetric and symmetric restrictions of $\sqsupseteq$, respectively.

Proposition 4.1 *Suppose that $\sqsupseteq$ is the partial preorder over $\mathfrak{S}$ induced by the ordering over $\mathcal{R}_{\mathcal{ANK}}$. Then $\sqsupseteq$ is a partial preorder over $\mathfrak{S}$.*

Proof. I will show transitivity. Suppose that $s \sqsupseteq s'$ and $s' \sqsupseteq s''$. Then $rank(s) \geq rank(s')$ and $rank(s') \geq rank(s'')$. Because $\geq$ is a partial order, $rank(s) \geq rank(s'')$. Then $s \sqsupseteq s''$. $\square$

I will define some concepts, i.e., *maximal prechain*, and *equivalence subset*. Let $S \subseteq \mathfrak{S}$. S_{pc} is a *prechain of S* iff $S_{pc} \subseteq S$, and $\sqsupset_{S_{pc}}$ is a total preorder. Given the notion of prechains, we can define the following concept.

Definition 4.5 *Let $S \subseteq \mathfrak{S}$. S_{mpc} is a* maximal prechain of *S iff*

1. *S_{mpc} is a prechain of S, and*
2. *for all $S_{pc} \subseteq S$ such that S_{pc} is a prechain of S, if $S_{mpc} \subseteq S_{pc}$, then $S_{pc} = S_{mpc}$.*

$MPC(S)$ is the set of all maximal prechain of S.

Definition 4.6 *Suppose that $S \subseteq \mathfrak{S}$. S_{eq} is an* equivalence subset of *S iff*

1. *S_{eq} is nonempty,*

2. $S_{eq} \subseteq S$, and

3. for all $s \in S_{eq}$, if $s \equiv s' \in S$, then $s' \in S_{eq}$.

$EQ(S)$ is the set of all the equivalence subsets of S.

When $S \subseteq \mathfrak{S}$, $S, S'' \in EQ(S)$, $s \in S'$, $s' \in S''$, and $s \sqsupset s'$, we denote it as $S' \sqsupset S''$.

In the case that $S \subseteq \mathfrak{S}$, $\sqsupseteq_S$ is a total preorder, and $EQ(S) = \{S_1, ..., S_N\}$ such that $S_i \sqsupset S_{i+1}$ for all $1 \leq^{\mathcal{N}} i \leq^{\mathcal{N}} N$[1], I denote $\mathcal{R}_{\mathcal{ANK}}{}^S = \{r_1, ..., r_N\}$ as a sequence of $r_i \in \mathcal{R}_{\mathcal{ANK}}$ such that $S_i = \{s \in S : rank(s) = r_i\}$ for all $1 \leq^{\mathcal{N}} i \leq^{\mathcal{N}} N$.

4.2 Aggregation of equally ranked sources

.

In this section, I will consider the aggregation operator with equally ranked sources. In such a case, I will take the principle of unanimity, respecting minority, or decision by majority, because I do not have any criteria in which I believe who. Suppose all the sources have the same rank. Therefore, $\sqsupseteq_S$ is fully connected. At first, I take the union of the relations:

Definition 4.7 *If $S \subseteq \mathfrak{S}$, then $Un(S)$ is the relation $\bigcup_{s \in S} \leq_s$.*

That is to say, this is the principle of respecting minority. Even if only one believes some preference, we should take it for the aggregation.

Assuming various restrictions to relations $\leq_s$, I can show various preservative powers of the aggregation, but they are not sufficient.

Proposition 4.2

1. *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^b$ is a belief state. If $S \subseteq \mathfrak{S}$, then $Un(S)$ is total and reflexive but not necessarily transitive.*

2. *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gb}$ is a generalized belief state. If $S \subseteq \mathfrak{S}$, then $Un(S)$ is modular but not necessarily transitive.*

3. *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. If $S \subseteq \mathfrak{S}$, then $Un(S)$ is irreflexive but not necessarily transitive.*

4. *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. If $S \subseteq \mathfrak{S}$, then $Un(S)$ is not necessarily transitive.*

[1] $\leq^{\mathcal{N}}$ means the ordering over natural numbers. In the following discussion, I will continue to use this notation.

Proof. For the case 2., see [28]. I will show the case 1. At first, I will show totality. For all $s \in S$, $w \leq^b_s w'$ or $w' \leq^b_s w$. Therefore, $(w, w') \in Un(S)$ or $(w', w) \in Un(S)$. I will show reflexivity. For all $s \in S$, $w \leq^b_s w$. Therefore, $(s, s) \in Un(S)$. I will show that $Un(S)$ is not necessarily transitive. Let $S = \{s_1, s_2\}$, $\leq^b_{s_1} = \{(w_a, w_a), (w_a, w_b), (w_b, w_b), (w_c, w_a), (w_c, w_b), (w_c, w_c)\}$ and $\leq^b_{s_2} = \{(w_a, w_a), (w_b, w_a), (w_b, w_b), (w_b, w_c), (w_c, w_a), (w_c, w_c)\}$. Then $(w_a, w_b), (w_b, w_c) \in Un(S)$ and $(w_a, w_c) \notin Un(S)$.

I will show the case 3. At first, I will show irreflexivity. For all $s \in S$, $w \not\leq^b_s w$. Therefore, $(s, s) \notin Un(S)$. I will show that $Un(S)$ is not necessarily transitive. Let $S = \{s_1, s_2\}$, $\leq^b_{s_1} = \{(a, b), (c, a), (c, b)\}$ and $\leq^b_{s_2} = \{(b, a), (b, c), (c, a)\}$. Then $(a, b), (b, c) \in Un(S)$ and $(a, c) \notin Un(S)$.

In the similar way to case 3, I can show the case 4, because a preferential relation is a generalized preferential relation. □

In the above discussion, it is important that the aggregation operator does not satisfy transitivity. Therefore, I will define the following operator.

Definition 4.8 *If $S \subseteq \mathfrak{S}$, then $AUn(S)$ is the relation $Un(S)^+$.*

Thus, I can show the following proposition.

Proposition 4.3

1. *Suppose that for all $s \in \mathfrak{S}$, $\leq^b_s$ is a belief state. If $S \subseteq \mathfrak{S}$, then $AUn(S) \in \mathcal{B}$.*
2. *Suppose that for all $s \in \mathfrak{S}$, $\leq^{gb}_s$ is a generalized belief state. If $S \subseteq \mathfrak{S}$, then $AUn(S) \in \mathcal{GB}$.*
3. *Suppose that for all $s \in \mathfrak{S}$, $\leq^{gp}_s$ is a generalized preferential relation. If $S \subseteq \mathfrak{S}$, then $AUn(S) \in \mathcal{GP}$.*

Proof. For the case 2, see [28]. I will show the case 1. $AUn(S)$ is the set of the transitive closure of $Un(S)$, and then transitive from Proposition 2.1. $Un(S) \subseteq AUn(S)$, and then $AUn(S)$ is reflexive and total.

I will show the case 3. $AUn(S)$ is the set of the transitive closure of $Un(S)$, and then transitive from Proposition 2.1. □

However, for the case of preferential relations, we have a problem.

Proposition 4.4 *Suppose that for all $s \in \mathfrak{S}$, $\leq^p_s$ is a preferential relation. If $S \subseteq \mathfrak{S}$, then $Un(S)$ is transitive but not necessarily irreflexive.*

Proof. $AUn(S)$ is the set of the transitive closure of $Un(S)$, and then transitive from Proposition 2.1. Let $S = \{s_1, s_2\}$, $\leq^b_{s_1} = \{(w_a, w_b), (w_c, w_a), (w_c, w_b)\}$ and $\leq^b_{s_2} = \{(w_b, w_a), (w_b, w_c), (w_c, w_a)\}$. Then $(w_a, w_b), (w_b, w_c), (w_c, w_a) \in Un(S)$ and $(w_a, w_a) \in AUn(S)$. Therefore, $AUn(S)$ is not irreflexive. □

Therefore, the principle of respecting minority is not applicable to preferential relations. Thus I will consider an another operator which eliminates confliction for preferential relations. I will consider that $AUn(S)$ is not applied to preferential relations, because of the strictness. When there is a strictness condition, adding elements and transitively closing is a bad strategy. Thus, when mixing strict beliefs, it is better to eliminate beliefs which is not agreed by someone than the previous strategy. That is to say, I will take the principle of unanimity.

Definition 4.9 *If $S \subseteq \mathfrak{S}$, then $AIn(S)$ is the relation $\bigcap_{s \in S} \leq_s$.*

According to the definition, the result of aggregation must be the belief with which everyone agree.

Proposition 4.5

1. *Suppose that for all $s \in \mathfrak{S}$, $\leq^{gp}_s$ is a generalized preferential relation. If $S \subseteq \mathfrak{S}$, then $AIn(S) \in \mathcal{GP}$.*
2. *Suppose that for all $s \in \mathfrak{S}$, $\leq^p_s$ is a preferential relation. If $S \subseteq \mathfrak{S}$, then $AIn(S) \in \mathcal{P}$.*
3. *Suppose that for all $s \in \mathfrak{S}$, $\leq^b_s$ is a belief state. If $S \subseteq \mathfrak{S}$, then $AIn(S)$ is transitive, but not total.*
4. *Suppose that for all $s \in \mathfrak{S}$, $\leq^{gb}_s$ is a generalized belief state. If $S \subseteq \mathfrak{S}$, then $AIn(S)$ is transitive, but not modular.*

Proof. See Appendix A. □

Therefore, whereas AUn is applicable with belief states, generalized belief states, and generalized preferential relations, AIn is applicable with preferential relations and generalized preferential relations. However, in section 4.4, I encounter the problem of applying AUn to generalized preferential relations.

About AUn, when I use belief states or generalized belief states, I verify that an element is created only if conflicts get created.

Proposition 4.6

1. *Suppose that for all $s \in \mathfrak{S}$, $\leq^b_s$ is a belief state. If $S \subseteq \mathfrak{S}$, $(x, y) \in AUn(S)$ for $x, y \in \mathcal{W}$, and $(x, y) \notin Un(S)$, then $(y, x) \in AUn(S)$.*
2. *Suppose that for all $s \in \mathfrak{S}$, $\leq^{gb}_s$ is a generalized belief state. If $S \subseteq \mathfrak{S}$, $(x, y) \in AUn(S)$ for $x, y \in \mathcal{W}$, and $(x, y) \notin Un(S)$, then $(y, x) \in AUn(S)$.*

Proof. For the case 2., see [28]. Let me consider the case 1. Because $(x, y) \notin Un(S)$ and if $s \in S$, then $\leq_s^b$ is total, for all $s \in S$, $y \leq_s^b x$, then $(y, x) \in Un(S)$, and $(y, x) \in AUn(S)$. $\square$

However, for the case of generalized preferential relations, such a property is not satisfied.

Proposition 4.7 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. It is not satisfied that if $S \subseteq \mathfrak{S}$, $(x, y) \in AUn(S)$ for $x, y \in \mathcal{W}$, and $(x, y) \notin Un(S)$, then $(y, x) \in AUn(S)$.*

Proof. Let $S = \{s_1, s_2\}$, $\leq_{s_1}^{gp} = \{(w_a, w_b)\}$ and $\leq_{s_1}^{gp} = \{(w_b, w_c)\}$. Then $(w_a, w_c) \notin Un(S)$ and $(w_a, w_c) \in AUn(S)$, but $(w_c, w_a) \notin AUn(S)$. $\square$

AUn process tends to be connected.

Example 4.1 *Suppose that $S = \{s_1, s_2\}$, and $\leq_{s_1}^b = \{(w_a, w_a), (w_a, w_b), (w_b, w_b), (w_c, w_a), (w_c, w_b), (w_c, w_c)\}$ and $\leq_{s_2}^b = \{(w_a, w_a), (w_b, w_a), (w_b, w_b), (w_b, w_c), (w_c, w_a), (w_c, w_c)\}$ are belief states, and hence, generalized belief states, and generalized preferential relations. Then $(w_a, w_b), (w_b, w_c) \in Un(S)$ and $(w_a, w_c) \notin Un(S)$. However, $AUn(S)$ is connected.*

In addition, AIn process tends to be empty.

Example 4.2 *Suppose that $S = \{s_1, s_2\}$, and $\leq_{s_1}^p = \{(w_a, w_b), (w_c, w_a), (w_c, w_b)\}$ and $\leq_{s_2}^p = \{(w_b, w_a), (w_b, w_c), (w_c, w_a)\}$ are preferential relations, and hence, generalized preferential relations. Then $AIn(S)$ is empty.*

For the following discussion, I define $SAUn$.

Definition 4.10 *If $S \subseteq \mathfrak{S}$, then $SAUn(S)$ is the strict version of $AUn(S)$.*

Moreover, we can assume the principle of decision by majority. If the number of agents who prefer w_a to w_b is more than the number of agents who do not prefer w_a to w_b, then all agents should prefer w_a to w_b.

Definition 4.11 *If $S \subseteq \mathfrak{S}$, then $Ma(S)$ is the relation $\{(w_a, w_b) |$*the number of $s \in S$ such that $w_a \leq_s w_b$ is more than half of the number of all elements in $S\}$.

Definition 4.12 *If $S \subseteq \mathfrak{S}$, then $AMa(S)$ is the relation $Ma(S)^+$.*

As $AUn(S)$, $AMa(S)$ is applicable to belief states, generalized belief states, and generalized preferential relation. However, $AMa(S)$ is not applicable to preferential relation.

Example 4.3 *Suppose that $S = \{s_1, s_2, s_3\}$, and $\leq_{s_1}^p = \{(w_a, w_b), (w_b, w_c)\}$, $\leq_{s_2}^p = \{(w_b, w_c), (w_c, w_a)\}$, and $\leq_{s_3}^p = \{(w_a, w_b), (w_c, w_a)\}$ are preferential relations. Then $AMa(S)$ is connected, and hence it is not preferential relation.*

Therefore, the principle of decision by majority is not applicable to preferential relation. $AMa(S)$ is more useful than $AUn(S)$, because $AMa(S)$ is more hard to be connected than $AUn(S)$.

Example 4.4 *Suppose that* $S = \{s_1, s_2, s_3\}$, *and* $\leq^p_{s_1} = \leq^p_{s_2} = \{(w_a, w_a), (w_a, w_b), (w_b, w_b)\}$, $\leq^p_{s_3} = \{(w_a, w_a), (w_b, w_a), (w_b, w_b)\}$ *are belief states. Then* $AMa(S)$ *is equal to* $\leq^p_{s_1}$*, but* $AUn(S)$ *is connected.*

Finally, we can complete the discussion of this section as follows.

Suppose that the sources are not comparable.

- If sources' beliefs can be partitioned by a total order, use the principle of respecting minority or decision by majority.
- Otherwise, use the principle of unanimity.

These policies are recycled in Section 4.4 and 4.5. In the following discussion, I will eliminate the case of decision by majority, because the formalization of such a case is similar with that of the case of respecting minority.

4.3 Aggregation of total strictly ranked sources

In the last section, if sources have the same credibility, then aggregation tends to be connected or empty. In this section, consider the case where the sources are strictly ranked, i.e., $\sqsupset_S$ is a total order. In this case, I will consider that a belief which a source with higher credibility has dominates another belief which a source with lower credibility. I define such an operator that the beliefs of lower ranked sources refine the beliefs of higher ranked sources. That is to say, the main purpose of this section is to show that the definitions of various aggregation operators in the case of strictly ranked sources mean the iterated application of the refinement operators. Because I already show that the refinement operators are well-defined, if the aggregation steps is equal to the iterated application of the refinements, then it is easy to show that the aggregation operators are well-defined. Now I construct the aggregation operator for totally ordered source.

Definition 4.13 *If* $S \subseteq \mathfrak{S}$, *then* $ARf(S)$ *is the relation*

$$\{(w_a, w_b) : \exists s \in S.w_a \leq_s w_b \wedge (\forall s' \sqsupset s \in S.w_a \sim_{s'} w_b)\}.$$

As I have already explained, I will show the equality between the definition and the refinement operator.

Lemma 4.1 *Let* $S \subseteq \mathfrak{S}$, $\sqsupset_S$ *be a total order, and* $S = \{s_1, ..., s_N\}$ *such that* $s_i \sqsupset s_{i+1}$ *for all* $1 \leq i < N$. *Then*

$$ARf(S) = \begin{cases} \leq_{s_1} & \text{if } N = 1 \\ ((\leq_{s_1} \oslash \leq_{s_2}) \oslash \ ... \ \leq_{s_N}) & \text{otherwise(i.e., } N >^{\mathcal{N}} 1) \end{cases}.$$

Proof. See Appendix A. $\square$

Thus, it is easy to show the following properties.

Proposition 4.8

1. *Suppose that for all $s \in \mathfrak{S}$, $<^b_s$ is the strict version of a belief state. If $S \subseteq \mathfrak{S}$, and $\sqsupset_S$ is a total order, then $ARf(S) \in \mathcal{B}_<$.*

2. *Suppose that for all $s \in \mathfrak{S}$, $\leq^b_s$ is a belief state. If $S \subseteq \mathfrak{S}$, and $\sqsupset_S$ is a total order, then $ARf(S) \in \mathcal{B}$.*

3. *Suppose that for all $s \in \mathfrak{S}$, $\leq^{gb}_s$ is a generalized belief state. If $S \subseteq \mathfrak{S}$, and $\sqsupset_S$ is a total order, then $ARf(S) \in \mathcal{GB}$.*

Proof. For the case 1., I can show it from Proposition 3.17 and Lemma 4.1. For the case 2, $ARf(S) = \leq^b_{s_1}$ is obvious. For the case 3, I can show it from Proposition 3.17 and Lemma 4.1. $\square$

For the case 2., such an aggregation is not appropriate for belief states, because $ARf(S)$ is equal to $\leq^b_{s_1}$. Therefore, I suppose another operator for belief states.

Definition 4.14 *Suppose that for all $s \in \mathfrak{S}$, $\leq^b_s$ is a belief state. If $S \subseteq \mathfrak{S}$, then $ARf^b_<(S)$ is the relation*

$$\{(w_a, w_b) : \exists s \in S.w_a <_s w_b \wedge (\forall s' \sqsupset s \in S.w_a \approx_{s'} w_b)\}.$$

Moreover, $ARf^b(S) = Ref(ARf^b_<(S))$.

I will show the equality between the definition and the refinement operator for belief states.

Lemma 4.2 *Suppose that for all $s \in \mathfrak{S}$, $\leq^b_s$ is a belief state. Let $S \subseteq \mathfrak{S}$, $\sqsupset_S$ be a total order, and $S = \{s_1, ..., s_N\}$ such that $s_i \sqsupset s_{i+1}$ for all $1 \leq i < N$. Then*

$$ARf^b(S) = \begin{cases} \leq^b_{s_1} & \text{if } N = 1 \\ ((\leq^b_{s_1} \otimes\, ^b\leq^b_{s_2}) \otimes\, ^b ... \leq^b_{s_N}) & \text{otherwise(i.e., } N >^{\mathcal{N}} 1) \end{cases}.$$

Proof. From Proposition 3.16, it suffices to show that

$$ARf^b_<(S) = \begin{cases} <^b_{s_1} & \text{if } N = 1 \\ ((<^b_{s_1} \otimes\, ^b<^b_{s_2}) \otimes\, ^b ... <^b_{s_N}) & \text{otherwise(i.e., } N >^{\mathcal{N}} 1) \end{cases}.$$

I can show it in the similar way to prove Lemma 4.1. $\square$

Therefore, I can show the following corollary.

Corollary 4.1 *Suppose that for all $s \in \mathfrak{S}$, $<^b_s$ is the strict version of $\leq^b_s$, where $\leq^b_s$ is a belief state. Let $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ be a total order. Then the strict version of $AGRRf^b(S)$, induced from belief states of sources, is equal to $AGRRf(S)$, induced from the strict versions of belief states of sources.*

Proof. It is easy to show from Lemma 4.1 and 4.2, and Proposition 3.15 and 3.16. □

Thus, it is easy to show the following property.

Proposition 4.9 *Suppose that for all $s \in \mathfrak{S}$, $\leq^b_s$ is a belief state. If $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ is a total order, then $ARf^b(S) \in \mathcal{B}$.*

Proof. I can show it from Proposition 3.17 and Lemma 4.2. □

In the above discussion, I did not mention the operators of preferential relations and generalized preferential relations. Now I will construct the aggregation operator of preferential relations, for totally ordered sources. However, because of the partiality of preferential relation of each source, the definition consists of the following multiple steps.

Definition 4.15 *Suppose that for all $s \in \mathfrak{S}$, $\leq^p_s$ is a preferential relation. Let $S \subseteq \mathfrak{S}$, $\sqsupseteq_S$ be a total order, and $S = \{s_1, ..., s_N\}$ such that $s_i \sqsupseteq s_{i+1}$ for all $1 \leq i \leq N$. The n-th refinement of preferential relations $O^{p,n}(S)$ is*

(I) if $n = 1$;
$O^{p,n}(S) = \leq^p_{s_1}$,

(II) else if $n > 1$; Suppose that for any $\leq \in \mathcal{R}$, $\leq \in SO^{p,n}(S)$ iff

$$\begin{aligned} \leq \;=\; & O^{p,n-1}(S) \cup \\ & \{(w_a, w_b) : (w_b, w_a) \notin O^{p,n-1}(S) \wedge \\ & \qquad w_a \leq_{s_n} w_b \wedge \\ & \qquad (w_b, w_a) \notin \leq^+\}. \end{aligned}$$

Then $O^{p,n}(S) = \cap([SO^{p,n}(S)]^+)$.

Definition 4.16 *Suppose that for all $s \in \mathfrak{S}$, $\leq^p_s$ is a preferential relation. Let $S \subseteq \mathfrak{S}$, $\sqsupseteq_S$ be a total order, and $S = \{s_1, ..., s_N\}$ such that $s_i \sqsupseteq s_{i+1}$ for all $1 \leq i \leq N$. Then*

$$ARf^p(S) = O^{p,N}(S).$$

Assuming such a complicated procedure, I can show the following lemma.

Lemma 4.3 *Suppose that for all $s \in \mathfrak{S}$, $\leq^p_s$ is a preferential relation. Let $S \subseteq \mathfrak{S}$, $\sqsupseteq_S$ be a total order, and $S = \{s_1, ..., s_N\}$ such that $s_i \sqsupseteq s_{i+1}$ for all $1 \leq^{\mathcal{N}} i <^{\mathcal{N}} N$. Then*

$$ARf^p(S) = \begin{cases} \leq^p_{s_1} & \text{if } N = 1 \\ ((\leq^p_{s_1} \circledcirc^p \leq^p_{s_2}) \circledcirc^p \ldots \ \leq^p_{s_N}) & \text{otherwise(i.e., } N >^{\mathcal{N}} 1) \end{cases}.$$

Proof. See Appendix A. □

Once the above lemma is shown, the following proposition is easy to prove.

Proposition 4.10 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. If $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ is a total order, then $ARf^p(S) \in \mathcal{P}$.*

Proof. I can show it from Proposition 3.17 and Lemma 4.3. □

For the simple refinement, I will define the following aggregation operator.

Definition 4.17 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. Let $S \subseteq \mathfrak{S}$, $\sqsupseteq_S$ be a total order, and $S = \{s_1, ..., s_N\}$ such that $s_i \sqsupseteq s_{i+1}$ for all $1 \leq i \leq N$. The n-th simple refinement of preferential relations $O_{sim}^{p;n}(S)$ is*

(I) *if $n = 1$;*
$O_{sim}^{p;n}(S) = \leq_{s_1}^p$,

(II) *else if $n > 1$; Suppose that*

$$TO_{sim}^{p;n}(S) = O_{sim}^{p;n-1}(S) \cup \{(w_a, w_b) : (w_b, w_a) \notin O_{sim}^{p;n-1}(S) \wedge w_a \leq_{s_n} w_b\}.$$

Then $O_{sim}^{p;n}(S)$ is the strict version of $[TO_{sim}^{p;n}(S)]^+$.

Definition 4.18 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. Let $S \subseteq \mathfrak{S}$, $\sqsupseteq_S$ be a total order, and $S = \{s_1, ..., s_N\}$ such that $s_i \sqsupseteq s_{i+1}$ for all $1 \leq i \leq N$. Then*

$$ARf_{sim}^p(S) = O_{sim}^{p,N}(S).$$

Also I can show the following lemma.

Lemma 4.4 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. Let $S \subseteq \mathfrak{S}$, $\sqsupseteq_S$ be a total order, and $S = \{s_1, ..., s_N\}$ such that $s_i \sqsupseteq s_{i+1}$ for all $1 \leq^{\mathcal{N}} i <^{\mathcal{N}} N$. Then*

$$ARf_{sim}^p(S) = \begin{cases} \leq_{s_1}^p & \text{if } N = 1 \\ ((\leq_{s_1}^p \bar{\circledcirc}^p \leq_{s_2}^p)\bar{\circledcirc}^p ... \leq_{s_N}^p) & \text{otherwise(i.e., } N >^{\mathcal{N}} 1) \end{cases}.$$

Proof. See Appendix A. □

Therefore, the following proposition is obvious.

Proposition 4.11 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. If $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ is a total order, then $ARf_{sim}^p(S) \in \mathcal{P}$.*

Proof. I can show it from Proposition 3.13 and Lemma 4.4. □

Now I will construct the aggregation operator of generalized preferential relations. for totally ordered source.

Definition 4.19 *Suppose that for all $s \in \mathfrak{S}$, $\leq^{gp}_s$ is a generalized preferential relation. Let $S \subseteq \mathfrak{S}$, $\sqsupseteq_S$ be a total order, and $S = \{s_1, ..., s_N\}$ such that $s_i \sqsupseteq s_{i+1}$ for all $1 \leq i \leq N$. The n-th refinement of generalized preferential relations $O^{gp,n}(S)$ is*

(I) if $n = 1$;
$O^{gp,n}(S) = \leq^{gp}_{s_1}$,

(II) else if $n > 1$; Suppose that

$$TO^{gp,n}(S) = O^{gp,n-1}(S) \cup \{(w_a, w_b) : (w_b, w_a) \notin O^{gp,n-1}(S) \wedge w_a \leq^{gp}_{s_n} w_b\}.$$

Then $O^{gp,n}(S) = [TO^{gp,n}(S)]^+$.

Definition 4.20 *Suppose that for all $s \in \mathfrak{S}$, $\leq^{gp}_s$ is a preferential relation. Let $S \subseteq \mathfrak{S}$, $\sqsupseteq_S$ be a total order, and $S = \{s_1, ..., s_N\}$ such that $s_i \sqsupseteq s_{i+1}$ for all $1 \leq i \leq N$. Then*

$$ARf^{gp}(S) = O^{gp,N}(S).$$

I can prove the following again.

Lemma 4.5 *Suppose that for all $s \in \mathfrak{S}$, $\leq^{gp}_s$ is a generalized preferential relation. Let $S \subseteq \mathfrak{S}$, $\sqsupseteq_S$ be a total order, and $S = \{s_1, ..., s_N\}$ such that $s_i \sqsupseteq s_{i+1}$ for all $1 \leq^{\mathcal{N}} i <^{\mathcal{N}} N$. Then*

$$ARf^{gp}(S) = \begin{cases} \leq^{gp}_{s_1} & \text{if } N = 1 \\ ((\leq^{gp}_{s_1} \circledcirc^{gp} \leq^{p}_{s_2}) \circledcirc^{gp} ... \ \leq^{p}_{s_N}) & \text{otherwise(i.e., } N >^{\mathcal{N}} 1) \end{cases}.$$

Proof. I can show it in the similar way to prove Lemma 4.4. □

Moreover, it is easy to show the following again.

Proposition 4.12 *Suppose that for all $s \in \mathfrak{S}$, $\leq^{gp}_s$ is a generalized preferential relation. If $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ is a total order, $ARf^{gp}(S) \in \mathcal{GP}$.*

Proof. I can show it from Proposition 3.14 and Lemma 4.5. □

After all, all the aggregation of beliefs for strictly ranked sources are formalized by the iterated refinement.

4.4 Aggregation of totally preordered sources

In the last section, if sources are strictlly ranked, then several aggregations are formalized from the refinement operators. In this section, I consider the case where the sources are totally preordered, i.e., $\sqsupseteq_S$ is a total preorder. At first, I construct the following operator, A^*: First combine equally ranked sources using the principle of respecting minority or unanimity, then aggregate the strictly ranked results using what is essentially ARf. That is to say, this strategy of the construction is the combination of the two slogans in the introduction of this chapter. Let $rank(S) = \{r \in \mathcal{R}_{\mathcal{ANK}} : rank(s) = r\})$.

Definition 4.21 *Let $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ be a total preorder. For any $r \in \mathcal{R}_{\mathcal{ANK}}$, let $\leq_r = AUn(\{s \in S : rank(s) = r\})$ and $\sim_r$, the corresponding agnosticism relation. $A(S)^*$ is the relation*

$$\{(w_a, w_b) : \exists r \in \mathcal{R}_{\mathcal{ANK}}.w_a \leq_r w_b \wedge (\forall r' > r \in rank(S).w_a \sim_{r'} w_b)\}.$$

I can show the relation between the definition and the refinement operator.

Lemma 4.6 *Let $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ be a total preorder. Also let $EQ(S) = \{S_1, ..., S_2\}$ such that $S_i \sqsupset S_{i+1}$ for all $1 \leq i < N$. Then*

$$A^*(S) = \begin{cases} AUn(S_1) & \text{if } N = 1 \\ (AUn(S_1) \oslash AUn(S_2)) \oslash \ldots AUn(S_N)) & \text{otherwise(i.e., } N >^{\mathcal{N}} 1) \end{cases}.$$

Proof. See Appendix A. □

That is to say, Definition 4.21 use the principle of respecting minority. Therefore, I can expect that this operator is available to generalized belief states and the strict version of belief states. Although this operation is also available to belief states, it is not intersting, because the result of this operator is $AUn(S_1)$, given belief states. Once the above lemma is proved, the following proposition is easy to show.

Proposition 4.13 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gb}$ is a generalized belief state. If $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ is a total preorder, then $A^*(S) \in \mathcal{GB}$.*

Proof. I can show it from Proposition 4.3,3.17 and Lemma 4.6. □

As the above discussion, it seems to be not hard to formalize the operator with the principle of decision by majority. Also I will define the aggregation operator of belief states for the total preordered sources.

Definition 4.22 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^b$ is a belief state. Let $S \subseteq \mathfrak{S}$. and $\sqsupseteq_S$ be a total preorder. For any $r \in \mathcal{R}_{\mathcal{ANK}}$, let $\leq_r = AUn(\{s \in : rank(s) = r\})$ and ∞_r, the corresponding conflict relation. $A^b_<(S)$ is the relation*

$$\{(w_a, w_b) : \exists r \in \mathcal{R}_{\mathcal{ANK}}.w_a <_r^b w_b \wedge (\forall r' > r \in rank(S).w_a \infty_{r'}^b w_b)\}.$$

Moreover, $A^b(S) = Ref(A^b_<(S))$.

I can also show the equation between the definition and the refinement operator of belief states.

Lemma 4.7 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^b$ is a belief state. Let $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ be a total preorder. Also let $EQ(S) = \{S_1, ..., S_2\}$ such that $S_i \sqsupset S_{i+1}$ for all $1 \leq i < N$. Then*

$$A^b(S) = \begin{cases} AUn(S_1) & \text{if } N = 1 \\ (AUn(S_1) \oslash {}^b AUn(S_2)) \oslash {}^b \ldots AUn(S_N)) & \text{otherwise(i.e., } N >^{\mathcal{N}} 1) \end{cases}.$$

Proof. From Proposition 3.16, it suffices to show that

$$A^b_<(S) = \begin{cases} SAUn(S_1) & \text{if } N = 1 \\ (SAUn(S_1) \circledless {}^b_< SAUn(S_2)) \circledless {}^b_< \ldots\ SAUn(S_N)) & \text{otherwise(i.e., } N >^{\mathcal{N}} 1) \end{cases}.$$

I can show it in the similar way to Lemma 4.6. □

That is to say, Definition 4.22 adopted the principle of respecting minority. Such an operation is well-defined.

Proposition 4.14 *Suppose that for all $s \in \mathfrak{S}$, $\leq^b_s$ is a belief state. If $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ is a total preorder, then $A^b(S) \in \mathcal{B}$.*

Proof. I can show it from Proposition 4.3, 3.17 and Lemma 4.7. □

In the above discussion, I did not mention the operator of preferential relations and generalized preferential relations. Now I will construct the aggregation operator of preferential relations for totally preordered source. The important point of the formalization for preferential relations is that the principle of unanimity must be used. However, because of the partiality of preferential relation of each source, the definition consists of the following multiple steps, as Definition 4.15 and 4.16.

Definition 4.23 *Suppose that for all $s \in \mathfrak{S}$, $\leq^p_s$ is a preferential relation. Let $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ be a total preorder. Also let $EQ(S) = \{S_1, ..., S_2\}$ such that $S_i \sqsupset S_{i+1}$ for all $1 \leq i < N$. The n-th refinement of preferential relations $O^{p,n}_{AIn}(S)$ is*

(I) if $n = 1$;
$O^{p,n}_{AIn}(S) = AIn(S_1)$,

(II) else if $n > 1$; Suppose that for any $\leq \in \mathcal{R}$, $\leq \in SO^{p,n}(S)$ iff

$$\begin{aligned} \leq \;=\; & O^{p,n-1}_{AIn}(S) \cup \\ & \{(w_a, w_b) : (w_b, w_a) \notin O^{p,n-1}_{AIn}(S) \wedge \\ & \qquad (w_a, w_b) \in AIn(S_n) \wedge \\ & \qquad (w_b, w_a) \notin \leq^+\}. \end{aligned}$$

Then $O^{p,n}_{AGRIn}(S) = \cap([SO^{p,n}_{AIn}(S)]^+)$.

Definition 4.24 *Suppose that for all $s \in \mathfrak{S}$, $\leq^p_s$ is a preferential relation. Let $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ be a total preorder. Then*

$$A^p(S) = O^{p,N}_{AIn}(S).$$

The following lemma will be shown as usual.

Lemma 4.8 *Suppose that for all $s \in \mathfrak{S}$, $\leq^p_s$ is a preferential relation. Let $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ be a total preorder. Also let $EQ(S) = \{S_1, ..., S_2\}$ such that $S_i \sqsupset S_{i+1}$ for all $1 \leq i < N$. Then*

$$A^p(S) = \begin{cases} AIn(S_1) & \text{if } N = 1 \\ ((AIn(S_1) \circledleft^p AIn(S_2)) \circledleft^p ... \ AIn(S_N)) & \text{otherwise(i.e., } N >^{\mathcal{N}} 1) \end{cases}.$$

Proof. I can show it in the similar way to Lemma 4.3. □

The following proposition is obvious.

Proposition 4.15 *Suppose that for all $s \in \mathfrak{S}$, $\leq^p_s$ is a preferential relation. If $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ is a total preorder, then $A^p(S) \in \mathcal{P}$.*

Proof. I can show it from Proposition 3.12 and Lemma 4.8. □

Now I want to show the example of the application of this operator, because my advantage is to use preferential relations, which breaks the rule of RMO. This is the arrangement of the example of space robot scenario in [28].

Example 4.5 *The robot sends to Earth a stream of telemetry data gathered by the spacecraft, as long as it receives positive feedback that the data is being received. At some point it loses contact with the automatic feedback system, so it sends a request for information to an agent on earth to the data retrieval system. In the former case, it would continue to send data, in the latter, desist. As it so happens, there has been no overload but the computer running the feedback system has hung. The agent consults the following three experts, aggregates their beliefs, and sends the results back to the robot:*

1. *s_p is the computer programmer that developed the feedback program. She is quite cocky and believes nothing could ever go wrong with her code, so there must have been an overload program. However, she concedes that if it were possible for her program to have crashed, it would have been highly unlikely for the feedback system to have crashed simultaneously.*

2. *s_m is the manager for the telemetry division. She was also told by the engineer who sold her the system that overloading could never happen. Not being too technical, she has no idea whether the feedback system crashed or not. However, she does not consider that even if the feedback system did not crash, the data retrieval system still was okay. Moreover, she feels that if the feedback system crashed, it was hard to say that the data retrieval system crashed simultaneously.*

3. *s_t is the technician working on the feedback system and, as a result, knows that it crashed. She does not know whether there was a data-overload in the retrieval system and, not being familiar with the system, is unable to speculate whether it could have overload had the feedback system not failed.*

This example is very important, because s_m does not satisfy RMO. Although Maynard-Reid II's representation did not deal with such incomplete information, I can represent the conditional beliefs in our space robot scenario with preferential relations.

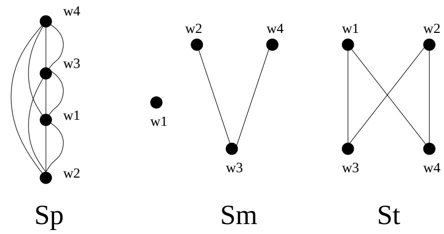


Figure 4.1: The preferential relations of s_p, s_m, s_t in Example 4.6.

Table 4.1: The truth table.

	F	D
w_1	1	1
w_2	1	0
w_3	0	1
w_4	0	0

Example 4.6 *Let F and D be propositional variables representing that the feedback and data retrieval system are okay, respectively. The preferential relations for the three sources are shown in Figure 4.2. The interpretation of $w_1, w_2, w_3, w_4 \in \mathcal{W}$ are shown in Table 4.1. They encode the following conditions in accordance with the descriptions of the sources' beliefs:*

- s_p : $Bel^p(true : F)$, $Bel^p(F : \neg D)$, $Bel^p(D : F)$, $Bel^p(\neg F : D)$.
- s_m : $Bel^p(true : D)$, $Agn^p(true : \neg F)$, $Agn^p(F : D)$, $Bel^p(\neg F : D)$.
- s_t : $Bel^p(true : \neg F)$, $Agn^p(true : D)$, $Agn^p(F : D)$.

He does not considered the aggregation of conditionals which does not satisfy RMO. However, we can operate such conditional beliefs, because we will use preferential relations:

Example 4.7 *In the space robot scenario of Example 4.10, suppose the technician is more credible than the programmer and the manager, but the latter two are considered equally credible. The totally preordered aggregated state, shown in Figure 4.2, gives the*

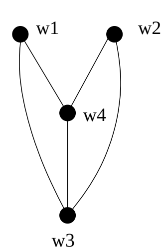


Figure 4.2: The result of the aggregation in Example 4.7.

robot correct information about the state of the system. The robot also learns for future reference that there is some disagreement over whether or not the feed back system would crash if the data retrieval system were not working.

However, although it is not important in this case, note that the principle of unanimity is only allowed to be used. Instead of getting the ability of avoiding RMO, I will not be able to use the principle of decision by majority, using preferential relations.

For the simple refinement, I will define the following aggregation operator, as Definition 4.17 and 4.18.

Definition 4.25 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. Let $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ be a total preorder. Also let $EQ(S) = \{S_1, ..., S_2\}$ such that $S_i \sqsupset S_{i+1}$ for all $1 \leq i < N$. The n-th simple refinement of preferential relations $O^{p,n}_{sim,AIn}(S)$ is*

(I) if $n = 1$;
$O^{p,n}_{sim,AIn}(S) = AIn(S_1)$,

(II) else if $n > 1$; Suppose that

$$TO^{p,n}_{sim,AIn}(S) = O^{p,n-1}_{sim,AIn}(S) \cup$$

$$\{(w_a, w_b) : (w_b, w_a) \notin O^{p,n-1}_{sim,AIn}(S) \wedge (w_a . w_b) \in AIn(S_n)\}.$$

Then $O^{p,n}_{sim,AIn}(S)$ is the strict version of $[TO^{p,n}_{sim,AIn}(S)]^+$.

Definition 4.26 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. Let $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ be a total preorder. Then*

$$A^p_{sim}(S) = O^{p,N}_{sim,AIn}(S).$$

I can show the following lemma as usual.

Lemma 4.9 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. Let $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ be a total preorder. Also let $EQ(S) = \{S_1, ..., S_2\}$ such that $S_i \sqsupset S_{i+1}$ for all $1 \leq i < N$. Then*

$$A^p_{sim}(S) = \begin{cases} AIn(S_1) & \text{if } N = 1 \\ ((AIn(S_1)\bar{\lhd}^p AIn(S_2)\bar{\lhd}^p ... \ AIn(S_N)) & \text{otherwise(i.e., } N >^{\mathcal{N}} 1) \end{cases}.$$

Proof. I can show it in the similar way to Lemma 4.4. □

Therefore, the following proposition is obvious as usual.

Proposition 4.16 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. If $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ is a total preorder, $A^p_{sim}(S) \in \mathcal{P}$.*

Proof. I can show it from Proposition 3.13 and Lemma 4.9. □

Now I will construct the aggregation operator of generalized preferential relations for pretotally ordered source, as Definition 4.19 and 4.20. First, I use the principle of respecting minority.

Definition 4.27 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. Let $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ be a total preorder. Also let $EQ(S) = \{S_1, ..., S_2\}$ such that $S_i \sqsupset S_{i+1}$ for all $1 \leq i < N$. The n-th tentative refinement of generalized preferential relations $O_{AUn}^{gp,n}(S)$ is*

(I) if $n = 1$;
$O_{AUn}^{gp,n}(S) = AUn(S_1)$,

(II) else if $n > 1$; Suppose that

$$TO_{AUn}^{gp,n}(S) = O_{AUn}^{gp,n-1}(S) \cup \{(w_a, w_b) : (w_b, w_a) \notin O_{AUn}^{gp,n-1}(S) \wedge (w_a, w_b) \in AUn(S_n)\}.$$

Then $O_{AUn}^{gp,n}(S) = [TO_{AUn}^{gp,n}(S)]^+$.

Definition 4.28 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. Let $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ be a total preorder. Then*

$$A^{*,gp}(S) = O_{AUn}^{gp,N}(S).$$

I may be repeating my self, but the following lemma is shown.

Lemma 4.10 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. Let $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ be a total preorder. Also let $EQ(S) = \{S_1, ..., S_2\}$ such that $S_i \sqsupset S_{i+1}$ for all $1 \leq i < N$. Then*

$$A^{*,gp}(S) = \begin{cases} AUn(S_1) & \text{if } N = 1 \\ ((AUn(S_1) \circledcirc^{gp} AUn(S_2)) \circledcirc^{gp} ... \; AUn(S_N)) & \text{otherwise(i.e., } N >^{\mathcal{N}} 1) \end{cases}.$$

Proof. I can show it in the similar way to prove Lemma 4.4. □

It is tedius to show the following proposition.

Proposition 4.17 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. If $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ is a total preorder, then $A^{*,gp}(S) \in \mathcal{GP}$.*

Proof. I can show it from Proposition 3.14 and Lemma 4.10. □

In the above definitions, we use $AUn(S_1)$ for generalized preferential relation. However, an undesirable effect is occurred by this prcedure. The problem is to introduce superfluous conflicts during the intermediate equally ranked aggregation step, as the following example shows:

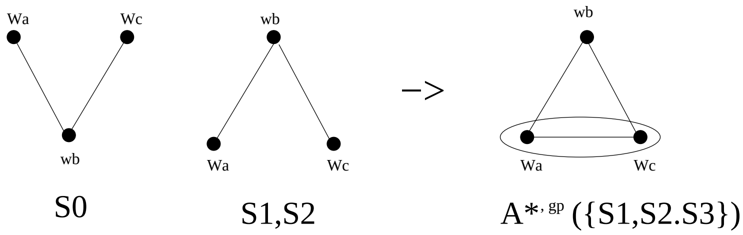


Figure 4.3: Sources and aggregate generalized preferential relations from Example 4.8 showing that $A^{*,gp}$ introduces superfluous conflicts if $s_2 \sqsupset s_1 \equiv s_0$.

Example 4.8 *Let $\mathcal{W} = \{w_a, w_b, w_c\}$. Suppose $S \subseteq \mathfrak{S}$ such that $S = \{s_0, s_1, s_2\}$ with generalized preferential relation shown in Figure 4.3. The result of applying $A^{*,gp}$ to S is also shown in the figure. All sources are agnostic over w_a and w_c, yet they are in conflict in the result. This unexpected development is due to the transitive closure in the lower rank involving opinions (w_b, w_c) and (w_b, w_a) which actually get overridden in the final result.*

That is to say, if we use the principle of respecting minority, we may encounter the unexpected conflict. Note that the relations in Figure 4.3 are also generalized belief states. Maynard-Reid II also indicated that this problem occurs in the aggregation of generalized belief state. Therefore, he defined another aggregation operator.

Definition 4.29 *The* rank-based aggregation *of a set of sources $S \subseteq \mathfrak{S}$, where $\sqsupseteq_S$ is total preorder, is $A(S) = ARf(S)^+$.*

Although Maynard-Reid II did not indicate it, I can show the following lemma.

Lemma 4.11 *Let $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ be a total preorder. Also let $EQ(S) = \{S_1, ..., S_2\}$ such that $S_i \sqsupset S_{i+1}$ for all $1 \leq i < N$. Then*

$$A(S) = \begin{cases} Un(S_1)^+ & \text{if } N = 1 \\ (Un(S_1) \lessdot\!\!\!\!\bigcirc\, Un(S_2)) \lessdot\!\!\!\!\bigcirc\, ...\, Un(S_N))^+ & \text{otherwise(i.e., } N >^{\mathcal{N}} 1) \end{cases}.$$

Proof. It suffices to show that

$$ARf(S) = \begin{cases} Un(S_1) & \text{if } N = 1 \\ (Un(S_1) \lessdot\!\!\!\!\bigcirc\, Un(S_2)) \lessdot\!\!\!\!\bigcirc\, ...\, Un(S_N)) & \text{otherwise(i.e., } N >^{\mathcal{N}} 1) \end{cases}.$$

See the proof of Lemma 4.6. □

Thus he avoided the problem with using the less restricted principle of respecting minority.

Proposition 4.18 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gb}$ is a generalized belief state. If $S \subseteq \mathfrak{S}$, and where $\sqsupseteq_S$ is total preorder, then $A(S) \in \mathcal{GB}$.*

Proof. see [28]. □

However, I did not use $ARf(S)$ for the aggregation of generalized preferential relations, because it does not correspond the refinement operator of generalized preferential relations. Therefore, we define the another aggregation operation of generalized preferential relations for totally preordered source. That is to say, instead of $AUn(S)$, we use the principle of unanimity.

Definition 4.30 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. Let $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ be a total preorder. Also let $EQ(S) = \{S_1, ..., S_2\}$ such that $S_i \sqsupset S_{i+1}$ for all $1 \leq i < N$. The n-th refinement of generalized preferential relations $O_{AIn}^{gp,n}(S)$ is*

(I) *if $n = 1$;*
$O_{AIn}^{gp,n}(S) = AIn(S_1)$,

(II) *else if $n > 1$; Suppose that*

$$TO_{AIn}^{gp,n}(S) = O_{AIn}^{gp,n-1}(S) \cup \{(w_a, w_b) : (w_b, w_a) \notin O_{AIn}^{gp,n-1}(S) \wedge (w_a, w_b) \in AIn(S_n)\}.$$

Then $O_{AIn}^{gp,n}(S) = [TO_{AIn}^{gp,n}(S)]^+$.

Definition 4.31 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. Let $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ be a total preorder. Then*

$$A^{gp}(S) = O_{AIn}^{gp,N}(S).$$

It is needless to say that I can show the following lemma.

Lemma 4.12 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. Let $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ be a total preorder. Also let $EQ(S) = \{S_1, ..., S_2\}$ such that $S_i \sqsupset S_{i+1}$ for all $1 \leq i < N$. Then*

$$A^{gp}(S) = \begin{cases} AIn(S_1) & \text{if } N = 1 \\ ((AIn(S_1) \circledcirc^{gp} AIn(S_2)) \circledcirc^{gp} ... \ AIn(S_N)) & \text{otherwise(i.e., } N >^{\mathcal{N}} 1) \end{cases}.$$

Proof. We can show it in the similar way to prove Lemma 4.4. □

In the similar way, it is easy to show the following proposition.

Proposition 4.19 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. If $S \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ is a total preorder, then $A^{gp}(S) \in \mathcal{GP}$.*

Proof. We can show it from Proposition 3.14 and Lemma 4.12. □

I observe that A^{gp}, when applied to the set of sources in Example 4.8, does indeed bypass the problem described above of extraneous opinion introduction:

Example 4.9 *Assume $\mathcal{W}$, S, and $\sqsubseteq$ are as in Example 4.8.* $A^{gp}(S) = \{(w_a, w_b), (w_c, w_b)\}$.

I also show that A behaves well in the special cases I have already considered.

Proposition 4.20 *Suppose $S \subseteq \mathfrak{S}$.*

1. *[28] If $\sqsupseteq_S$ is fully connected, $A(S) = AUn(S)$.*

2. *[28] If $\sqsupseteq_S$ is a total order, $A(S) = ARf(S)$.*

3. *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^b$ is a belief state. If $\sqsupseteq_S$ is fully connected, $A^b(S) = AUn(S)$.*

4. *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^b$ is a belief state. If $\sqsupseteq_S$ is a total order, $A^b(S) = ARf^b(S)$.*

5. *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. If $\sqsupseteq_S$ is fully connected, $A^p(S) = AIn(S)$.*

6. *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. If $\sqsupseteq_S$ is a total order, $A^p(S) = ARf^p(S)$.*

7. *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. If $\sqsupseteq_S$ is fully connected, $A^p_{sim}(S) = AIn(S)$.*

8. *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. If $\sqsupseteq_S$ is a total order, $A^p_{sim}(S) = ARf^p_{sim}(S)$.*

9. *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. If $\sqsupseteq_S$ is fully connected, $A^{gp}(S) = AIn(S)$.*

10. *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. If $\sqsupseteq_S$ is a total order, $A^{gp}(S) = ARf^p(S)$.*

Proof. For the case 1. and 2., see [28]. For the case 3., 5., 7., and 9., I can show it from the fact that for any $S \subseteq \mathfrak{S}$, $S = \bigcup\{S_1\}$ such that for any $s, s' \in S_1$, $rank(s) = rank(s')$. For the case 4., 6., 8., and 10., I can show it from the fact that for any $S \subseteq \mathfrak{S}$, $S = \bigcup\{S_1, ..., S_N\}$ such that for any $1 \leq^{\mathcal{N}} i \leq^{\mathcal{N}} N$, S_i has only one element s_i and If $i >^{\mathcal{N}} j$, $rank(s_i) > rank(s_j)$. $\square$

4.5 General aggregation

In the last section, if sources are totally preordered, then several aggregations are formalized from amalgamating equally ranked source version and strictly ranked version. In this section, I consider the case where the sources are partially preordered, i.e., $\sqsupseteq_S$ is a partial preorder. At first, I construct the following operator, AGe: First aggregate the totally preordered results using what is essentially A for each chain, then combine all these results with the principle of unanimity (or respecting minority). This procedure is the combination of the two slogans again. Now I can define the following general aggregation operator:

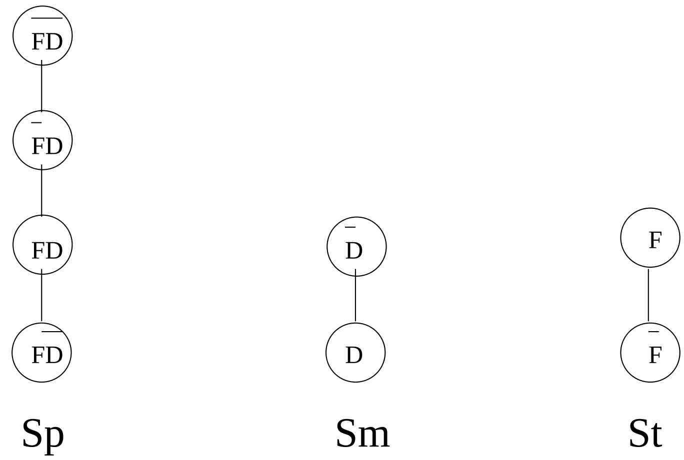


Figure 4.4: The generalized belief states of s_p, s_m, and s_t in Example 4.9.

Definition 4.32 *The* general aggregation *of a set of sources $S \subseteq \mathfrak{S}$ is*

$$AGe(S) = [\bigcup_{S_{mpc} \in MPC(S)} A(S_{mpc})]^{+}.$$

Proposition 4.21 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gb}$ is a generalized belief state. If $S \subseteq \mathfrak{S}$, then $AGe(S) \in \mathcal{GB}$.*

Proof. I can show it from Proposition 4.8 and the proof of 4.3. □

For the application of this operator, I arrange the example of space robot scenario in [28].

Example 4.10 *This scenario is almost as same as Example 4.5. However, the manager's opinion is different with the previous one:*

- *s_m is the manager for the telemetry division. She was also told by the engineer who sold her the system that overloading could never happen. Not being too technical, she has no idea what would happen if there was an overload.*

I can represent the sources' beliefs in our space robot scenario using generalized belief states.

Example 4.11 *Let F and D be propositional variables representing that the feedback and data retrieval system are okay, respectively. The generalized belief states for the three sources are shown in Figure 4.4[2]. They encode the following conditions in accordance with the descriptions of the sources' beliefs:*

[2]Each circle represents all the worlds in $\mathcal{W}$ which satisfy the sentence inside.

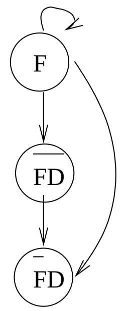


Figure 4.5: The generalized belief states after aggregation in Example 4.11.

- s_p : $Bel^{gb}(true : F)$, $Bel^{gb}(F : \neg D)$, $Bel^{gb}(D : F)$, $Bel^{gb}(\neg F : D)$.
- s_m : $Bel^{gb}(true : D)$, $Agn^{gb}(true : \neg F)$, $Agn^{gb}(\neg F : D)$.
- s_t : $Bel^{gb}(true : \neg F)$, $Agn^{gb}(true : D)$, $Agn^{gb}(F : D)$.

Maynard-Reid II [28] considered the aggregation of strictly ranked case and totally preordered case. However, the following case was not considered, because it is partially ordered:

Example 4.12 *In the space robot scenario of 4.10, suppose the technician is more credible than the programmer, but the manager is not comparable with the two. The generally aggregated state, shown in Figure 4.5, gives the robot correct information about the state of the system. The robot also learns for future reference that there is some disagreement over whether or not there would have been a data overload if the feedback system were working.*

In this case, the principle of decision by majority may be available.

I also observe that AGe behaves well in the special case I have already considered, reducing to A when the sources are totally preordered:

Proposition 4.22 *Suppose $S \subseteq \mathfrak{S}$. If $\sqsupseteq_S$ is totally preordered, $AGe(S) = A(S)$.*

Proof. If $\sqsupseteq_S$ is totally preordered, $MPC(S) = \{S\}$ is obvious. Therefore, $AGe(S) = A(S)$. □

That is to say, I also observe that AGe behaves well in the special case I have already considered, reducing to AUn when the sources have an equal rank, and to ARf when the sources are totally ranked:

Proposition 4.23 *Suppose $S \subseteq \mathfrak{S}$.*

1. *If $\sqsupseteq_S$ is fully connected, $AGe(S) = AUn(S)$.*

2. *If $\sqsupseteq_S$ is a total order, $AGe(S) = ARf(S)$.*

Proof. They are derived from Proposition 4.20 and 4.22. □

I can define the following general aggregation operator of belief states:

Definition 4.33 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^b$ is a belief state. Let $S \subseteq \mathfrak{S}$. $AGe^p(S)$ is the relation such that*

$$AGe^b(S) = [\bigcup_{S_{mpc} \in MPC(S)} A^b(S_{mpc})]^+.$$

In the definition, I use the principle of respecting minority, but that of decision by majority also may be available.

Proposition 4.24 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^b$ is a belief state. If $S \subseteq \mathfrak{S}$, then $AGe^b(S) \in \mathcal{B}$.*

Proof. For every case, I can show it from Proposition 4.14 and the proof of 4.3. □

I also observe that AGe behaves well in the special case I have already considered, reducing to A^b when the sources are totally preordered:

Proposition 4.25 *Suppose $S \subseteq \mathfrak{S}$. If $\sqsupseteq_S$ is totally preordered, $AGe^b(S) = A^b(S)$.*

Proof. If $\sqsupseteq_S$ is totally preordered, $MPC(S) = \{S\}$ is obvious. Therefore, $AGe^b(S) = A^b(S)$. □

That is to say, I also observe that AGe^b behaves well in the special case I have already considered, reducing to AUn when the sources have an equal rank, and to ARf^b when the sources are totally ranked:

Proposition 4.26 *Suppose $S \subseteq \mathfrak{S}$.*

1. *If $\sqsupseteq_S$ is fully connected, $AGe^b(S) = AUn^b(S)$.*
2. *If $\sqsupseteq_S$ is a total order, $AGe^b(S) = ARf^b(S)$.*

Proof. They are derived from Proposition 4.20 and 4.25. □

I can define the following general aggregation operator of preferential relations:

Definition 4.34 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. Let $S \subseteq \mathfrak{S}$. $AGe^p(S)$ is the relation such that*

$$AGe^p(S) = \bigcap_{S_{mpc} \in MPC(S)} A^p(S_{mpc}).$$

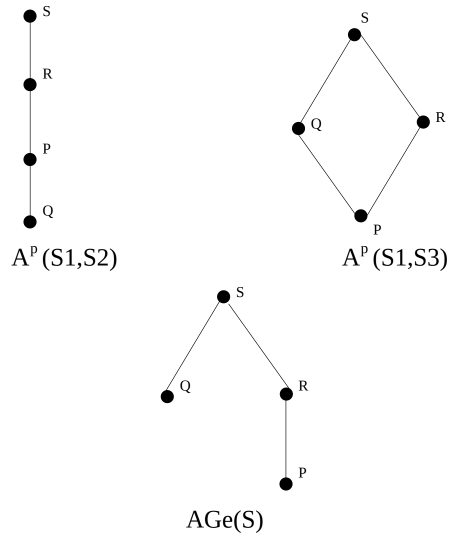


Figure 4.6: The preferential relations after aggregation in Example 4.12.

In the formalization, the principle of unanimity is only allowed to be used, because the formalizations with the principle of respecting minority or decision by majority is not well-defined from the second section of this chapter.

Proposition 4.27 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. If $S \subseteq \mathfrak{S}$, then $AGe^p(S) \in \mathcal{P}$.*

Proof. For every case, I can show it from Proposition 4.15 and the proof of Proposition 4.5. □

I revisit Example 1.1 and 1.2.

Example 4.13 *In the detective story scenario of Example 1.1, $\sqsupseteq = \{(s_1, s_1), (s_1, s_2), (s_1, s_3), (s_2, s_2), (s_3, s_3)\}$. See Example 3.1, for all preferential relations which sources have. In the top of Figure 4.6, $A^p(S_{mpc})$ is calculated for each $S_{mpc} \in MPC(S)$. In the bottom of this figure, the result of aggregation $AGe^p(S)$ is given. In this figure, the investigation headquarter should examine P and Q at first.*

Example 4.14 *In the TV show scenario of Example 1.2, $\sqsupseteq = \{(s_1, s_1), (s_1, s_2), (s_2, s_2), (s_3, s_3), (s_3, s_4), (s_4, s_4)\}$. See Example 3.2, for all preferential relations which sources have. In the top of Figure 4.7, $A^p(S_{mpc})$ is calculated for each $S_{mpc} \in MPC(S)$. In the bottom of this figure, the result of aggregation $AGe^p(S)$ is given. In this figure, P should be the muster.*

I also observe that AGe^p behaves well in the special case I have already considered, reducing to A^p when the sources are totally preordered:

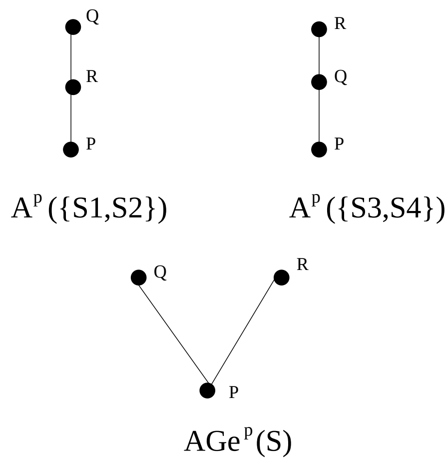


Figure 4.7: The preferential relations after aggregation in Example 4.13.

Proposition 4.28 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. Let $S \subseteq \mathfrak{S}$. If $\sqsupseteq$ is totally preordered, $AGe^p(S) = A^p(S)$.*

Proof. If $\sqsupseteq$ is totally preordered, $MPC(S) = \{S\}$ is obvious. Therefore, $AGe^p(S) = A^p(S)$. □

That is to say, I also observe that AGe^p behaves well in the special case I have already considered, reducing to AIn when the sources have an equal rank, and to ARf^p when the sources are totally ranked:

Proposition 4.29 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. Let $S \subseteq \mathfrak{S}$.*

1. *If $\sqsupseteq_S$ is fully connected, $AGe^p(S) = AIn(S)$.*
2. *If $\sqsupseteq_S$ is a total order, $AGe^p(S) = ARf^p(S)$.*

Proof. They are derived from Proposition 4.20 and 4.28. □

I can define the following general simple aggregation operator of preferential relations:

Definition 4.35 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. Let $S \subseteq \mathfrak{S}$. $AGe^p_{sim}(S)$ is the relation such that*

$$AGe^p_{sim}(S) = \bigcap_{S_{mpc} \in MPC(S)} A^p_{sim}(S_{mpc}).$$

Proposition 4.30 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. If $S \subseteq \mathfrak{S}$, then $AGe^p(S)_{sim} \in \mathcal{P}$.*

Proof. For every case, I can show it from Proposition 4.16 and the proof of Proposition 4.5. □

I also observe that AGe^p_{sim} behaves well in the special case I have already considered, reducing to A^p_{sim} when the sources are totally preordered:

Proposition 4.31 *Suppose that for all $s \in \mathfrak{S}$, $\leq^p_s$ is a preferential relation. Let $S \subseteq \mathfrak{S}$. If $\sqsupseteq$ is totally preordered, $AGe^p_{sim}(S) = A^p_{sim}(S)$.*

Proof. If $\sqsupseteq$ is totally preordered, $MPC(S) = \{S\}$ is obvious. Therefore, $AGe^p(S) = A^p(S)$. □

That is to say, I also observe that AGe^p_{sim} behaves well in the special case I have already considered, reducing to AIn when the sources have an equal rank, and to ARf^p_{sim} when the sources are totally ranked:

Proposition 4.32 *Suppose that for all $s \in \mathfrak{S}$, $\leq^p_s$ is a preferential relation. Let $S \subseteq \mathfrak{S}$.*

1. *If $\sqsupseteq_S$ is fully connected, $AGe^p_{sim}(S) = AIn(S)$.*
2. *If $\sqsupseteq_S$ is a total order, $AGe^p_{sim}(S) = ARf^p_{sim}(S)$.*

Proof. They are derived from Proposition 4.20 and 4.31. □

I can define the following general aggregation operator of generalized preferential relations:

Definition 4.36 *Suppose that for all $s \in \mathfrak{S}$, $\leq^{gp}_s$ is a generalized preferential relation. Let $S \subseteq \mathfrak{S}$. $AGe^{gp}(S)$ is the relation such that*

$$AGe^{gp}(S) = \bigcap_{S_{mpc} \in MPC(S)} A^{gp}(S_{mpc}).$$

It is obvious that the principle of unanimity is only allowed to be assumed.

Proposition 4.33 *Suppose that for all $s \in \mathfrak{S}$, $\leq^{gp}_s$ is a generalized preferential relation. If $S \subseteq \mathfrak{S}$, then $AGe^{gp}(S) \in \mathcal{GP}$.*

Proof. I can show it from Proposition 4.17 and the proof of Proposition 4.17. □

I assume a more powerful detective story for this operator.

Example 4.15 *P and Q are suspects of a murder case, and they are brother. s_1 and s_2 are acquaintances. s_3 is a mother of the two. s_4 is an anonymous telephone.*

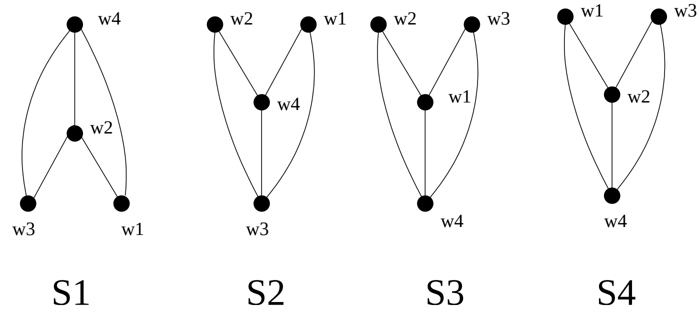


Figure 4.8: The generalized belief states of s_1, s_2, s_3, and s_4 in Example 4.15.

s_1 *"I think that P is the murderer, because P bought a weapon. if he is not the murderer, Q may be it. Q hates the victim very much."*

s_2 *"It seems to be that P is not a murderer, because I saw that P went to the opposite side of the crime scene. However, Q may be it, because Q was a bad boy."*

s_3 *"P and Q are very good boys, and they are not murderers. But one of them cannot be a murder, because they are friendly."*

s_4 *"Q cannot be a murder, because he robed a jewelly shop of a valuable article at the time. P may be accomplice. He always keep a his company.*

Who are the murderer?

Example 4.16 *Let P and Q be propositional variables representing thatP and data Q are murderers, respectively. The generalized preferential relations for the four sources are shown in Figure 4.8. In this figure,*

- *For w_1, P and Q are true.*
- *For w_2, P is true, and Q is false.*
- *For w_3, P is false, and Q is true.*
- *For w_4, P and Q are false.*

They encode the following conditions in accordance with the descriptions of the sources' beliefs:

- $s_1 : Bel^{gp}(true : P)$, $Agn^{gp}(true : \neg P)$, $Agn^{gp}(true : P)$.
- $s_2 : Bel^{gp}(true : \neg P)$, $Bel^{gp}(true : Q)$, $Bel^{gp}(\neg Q : \neg P)$, $Agn^{gp}(P : Q)$.
- $s_3 : Bel^{gp}(true : \neg P)$, $Bel^{gp}(F : \neg Q)$, $Bel^{gp}(P : Q)$, $Bel^{gp}(\neg Q : P)$.
- $s_4 : Bel^{gp}(true : \neg P)$, $Bel^{gp}(F : \neg Q)$, $Bel^{gp}(P : \neg Q)$, $Agn^{gp}(Q : P)$.

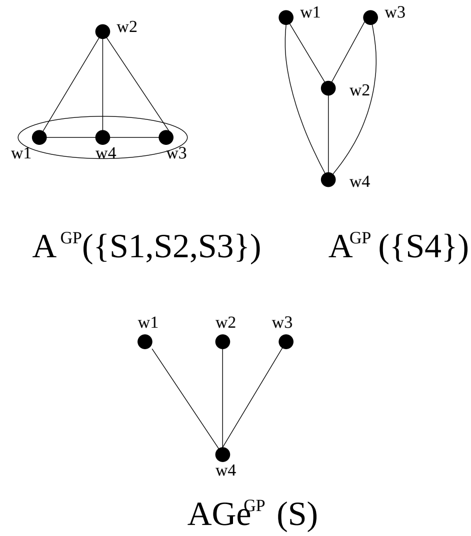


Figure 4.9: The generalized preferential relations after aggregation in Example 4.16.

I can show the step of aggregation of generalized preferential relations.

Example 4.17 *In the detective story scenario of 4.14,* $\sqsupseteq = \{(s_1, s_1), (s_1, s_2), (s_1, s_3), (s_2, s_1), (s_2, s_2), (s_2, s_3), (s_3, s_3), (s_4, s_4)\}$. *See Example 4.15, for all generalized preferential relations which sources have. In the top of Figure 4.9,* $A^p(S_{mpc})$ *is calculated for each* $S_{mpc} \in MPC(S)$. *In the bottom of this figure, the result of aggregation* $AGe^p(S)$ *is given. In this figure, neither* P *nor* Q *is the murderer.*

I also observe that AGe^{gp} behaves well in the special case I have already considered, reducing to AGe^{gp} when the sources are totally preordered:

Proposition 4.34 *Suppose that for all* $s \in \mathfrak{S}$, $\leq_s^{gp}$ *is a generalized preferential relation. Let* $S \subseteq \mathfrak{S}$. *If* $\sqsupseteq$ *is totally preordered,* $AGe^{gp}(S) = A^{gp}(S)$.

Proof. If $\sqsupseteq$ is totally preordered, $MPC(S) = \{S\}$ is obvious. Therefore, $AGe^{gp}(S) = A^{gp}(S)$. □

That is to say, I also observe that AGe^{gp} behaves well in the special case I have already considered, reducing to AIn when the sources have an equal rank, and to ARf^{gp} when the sources are totally ranked:

Proposition 4.35 *Suppose that for all* $s \in \mathfrak{S}$, $\leq_s^{gp}$ *is a preferential relation. Let* $S \subseteq \mathfrak{S}$.

1. *If* $\sqsupseteq_S$ *is fully connected,* $AGe^{gp}(S) = AIn(S)$.
2. *If* $\sqsupseteq_S$ *is a total order,* $AGe^{gp}(S) = ARf^{gp}(S)$.

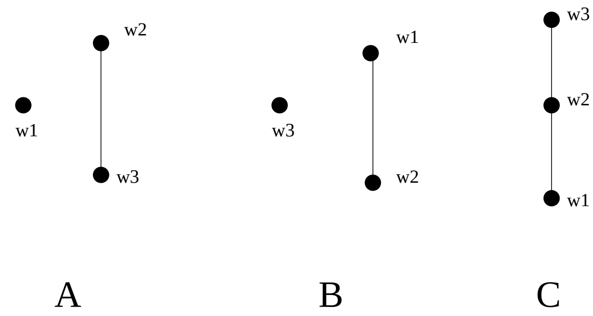


Figure 4.10: Example : $AGe^p(S)$ is empty.

Proof. They are derived from Proposition 4.20 and 4.34. $\Box$

Note that the aggregation of preferential relations and generalized preferential relations may be empty. For example in Figure 4.10, let $s_1 \sqsupseteq s_2$, but $s_1 \not\sqsupseteq s_3$, $s_3 \not\sqsupseteq s_1$, $s_2 \not\sqsupseteq s_3$, and $s_3 \not\sqsupseteq s_2$. Then the aggregation is empty.

I need to show what kind of preferential relations induce the nonempty ordering. I define the maximum-ordered set of sources.

Definition 4.37 *Let $S \subseteq \mathfrak{S}$. $\sqsupseteq_S$ is maximum-ordered iff S has a maximum source s_m, that is, for all $s \in S$, if $s \sqsupseteq_S s_m$, then $s = s_m$.*

Proposition 4.36 *Let $S \subseteq \mathfrak{S}$.*

1. *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. If $\sqsupseteq_S$ is a maximum-ordered, $\leq_{s_m}^p \subseteq AGe^p(S)$.*

2. *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. If $\sqsupseteq_S$ is a maximum-ordered, $\leq_{s_m}^{gp} \subseteq AGe^{gp}(S)$.*

proof. I show only the case of preferential relation. For all $S_{mpc} = MPC(S)$, $AIn(S_1) = \leq_{s_m}^p$, and then $\leq_{s_m}^p \subseteq A^p(S_{mpc})$ from Lemma 4.8. It follows that $\leq_{s_m}^p \subseteq AGe^p(S)$. $\Box$

Therefore, an aggregation of preferential relations and generalized preferential relations is nonempty, when its maximal source's ordering is nonempty.

Chapter 5

Fusion

In the previous chapter, I considered the case where a single agent must construct or update her belief state once informed by a set of sources. However, as I already see Example 1.1, aggregation step may be multiple agents, for example, agent A aggregates information of s_1 and s_2, and agent B aggregates information of s_1 and s_3. Therefore, I must consider the multi-agent case of aggregation. Multi-agent fusion is the process of aggregating the belief states of a set of agents, each with its respective set of informant sources.

Example 5.1 *In Example 4.10, I consider the single agent's aggregation. However, in this example, I consider the multi-agent version of Example 4.10. Suppose that the experts in the space robot scenario work at different centers or for different companies. The robot will need to request information from different agents, one to aggregate the sources from each center. Agent A1 consults the company of a source s_p, and Agent A2 consults the company of sources s_m and s_t. How does the space robot fuse all agent's aggregating information?*

In such a scenario, the robot needs a mechanism for combining the beliefs of multiple agents potentially arriving at different times. Moreover, the belief state output by the mechanism should be invariant with respect to the order of arrivals of agents. In this chapter, a simple fusion operator will be given by the set-theoretical mechanism. Then we will describe the procedure of computing the resulting belief.

5.1 Formalization

In the following discussion, we assume that each agent combines the beliefs of sources with the aggregation operator from the previous chapter to compute its own beliefs.

Definition 5.1 *An agent A is* informed by *a set of sources $S \subseteq \mathfrak{S}$.*

1. *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gb}$ is a generalized belief state. Agent A's* induced generalized belief state *is the generalized belief state formed by aggregating the generalized belief states of its sources, i.e., $AGe(S)$.*

2. *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^b$ is a belief state. Agent A's* induced belief state *is the belief state formed by aggregating the belief states of its sources, i.e., $AGe^b(S)$.*

3. *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. Agent A's* induced preferential relation *is the preferential relation formed by aggregating the preferential relation of its sources, i.e., $AGe^p(S)$.*

4. *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. Agent A's* induced generalized preferential relation *is the generalized preferential relation formed by aggregating the generalized preferential relation of its sources, i.e., $AGe^{gp}(S)$.*

I will use $A_\emptyset$ and $A_\mathfrak{S}$ to denote special agents informed by $\emptyset$ and $\mathfrak{S}$, respectively. Each sources can be thought of as a primitive agent with fixed belief state.

Now suppose that a new agent aggregates information of, not sources, but a set of other agents with pre-aggregated beliefs. In addition, I assume that all the agents have the same ranking over sources, i.e., $\sqsupseteq$. I will define the fusion of the set of agents to be an agent informed by the combination of informant sources:

Definition 5.2 *Let $\mathcal{A} = \{A_1, ..., A_n\}$ be a set of agents such that each agent A_i is informed by $S_i \subseteq \mathfrak{S}$. The* fusion *of $\mathcal{A}$, written $\oplus(\mathcal{A})$, is an agent informed by $S = \bigcup_{i=1}^n S_i$.*

My formalization have the same merit as Maynard-Reid II's formalization. That is to say, iterated fusion is formally well-defined: the output of fusion is an agent, a legitimate input to another fusion operation. Also this operator does not depend on the order of inputs. This point justified the discussion of Maynard-Reid II for Example 5.1 in [28]. In addition, $\oplus$ is idempotent, commutative, and associative. Therefore, a set of agents $\mathcal{A}$ with informant sources from $\mathfrak{S}$ and closed under $\oplus$ forms a *semi-lattice* (see [6]). In this formalization, agents higher in the lattice contain better information than lower ones. $\oplus$ accepts a n-tuple of agents and returns the least agent that contains at least as much information as all of them. This semi-lattice has a "unit" element, $A_\emptyset$m and an "annihilator" element, $A_\mathfrak{S}$.

I have some advantage that Maynard-Reid II does not have. That is, my formalization allows the sources to be partially preordered, because I expand $AGR(S)$ into $AGRGe(S)$ in the previous chapter, and this expansion does not affect the definition of fusion operator. However, this relaxation of the constraint will complicate the following discussion.

5.2 Computing Fusion

In this section, I will discuss the possibility of computing the beliefs induced by the agents' fusion solely from their initial beliefs, that is, without having to reference to the belief states of their sources. This is highly desirable because of the expense of storing all source beliefs; we would like to represent each agent's knowledge as compact as possible.

In fact, I can do this if all sources have equal rank or incomparable. Then I will introduce the following concept:

Definition 5.3 *Let $S \subseteq \mathfrak{S}$. $\sqsupseteq_S$ is* isolated *iff for all $s, s' \in S$, $s \equiv s'$, or $s \not\sqsupseteq s'$ and $s' \not\sqsupseteq s$.*

Proposition 5.1 *Let $\mathcal{A} = \{A_1, ..., A_n\}$ be a set of agents such that each agent A_i is informed by $S_i \subseteq \mathfrak{S}$, and $\sqsupseteq_S$ be isolated.*

1. *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^b$ (or $\leq_s^{gb}$) is a belief state (or a generalized belief state). let $\preceq_{\mathcal{A}_\rangle}^b$ (or $\preceq_{\mathcal{A}_\rangle}^{gb}$ be agent A_i's induced belief state (or induced generalized belief state). If $A = \oplus(\mathcal{A})$, then $(\bigcup_{A_i \in \mathcal{A}} \preceq_{\mathcal{A}_\rangle}^b)^+$ (or $(\bigcup_{A_i \in \mathcal{A}} \preceq_{\mathcal{A}_\rangle}^{gb})^+$) is A's induced belief state (or induced generalized belief state).*

2. *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ (or $\leq_s^{gp}$) is a preferential relation (or a generalized preferential relation). let $\preceq_{\mathcal{A}_\rangle}^p$ (or $\preceq_{\mathcal{A}_\rangle}^{gp}$ be agent A_i's induced preferential relation (or induced generalized preferential relation). If $A = \oplus(\mathcal{A})$, then $\bigcap_{A_i \in \mathcal{A}} \preceq_{\mathcal{A}_\rangle}^b$ (or $\bigcap_{A_i \in \mathcal{A}} \preceq_{\mathcal{A}_\rangle}^{gb}$) is A's induced belief state (or induced generalized belief state).*

Proof. I will show the case 1 with belief states. For all $S_{mpc} \in MPC(\bigcup_{i=0}^n S_i)$, $A^b(S_{mpc}) = [\bigcup_{s \in S_{mpc}} \leq_s^b]^+$. Therefore, $AGe^b(S) = [\bigcup_{S_{mpc} \in MPC(\bigcup_{i=0}^n S_i)} [\bigcup_{s \in S_{mpc}} \leq_s^b]^+]^+ = (\bigcup_{s \in \bigcup_{i=0}^n S_i} \leq_s^b)^+$. By the way, $(\bigcup_{A_i \in \mathcal{A}} \preceq_{\mathcal{A}_\rangle}^b)^+ = [\bigcup_{i=0}^n AGe^b(S_i)]^+ = (\bigcup_{s \in \bigcup_{i=0}^n S_i} \leq_s^b)^+$.

We will show the case 2 with preferential relations. For all $S_{mpc} \in MPC(\bigcup_{i=0}^n S_i)$, $A^p(S_{mpc}) = \bigcap_{s \in S_{mpc}} \leq_s^p$. Therefore, $AGe^p(S) = \bigcap_{S_{mpc} \in MPC(\bigcup_{i=0}^n S_i)} \bigcap_{s \in S_{mpc}} \leq_s^p = \bigcup_{s \in \bigcup_{i=0}^n S_i} \leq_s^p$. By the way, $\bigcap_{A_i \in \mathcal{A}} \preceq_{\mathcal{A}_\rangle}^p = \bigcap_{i=0}^n AGe^p(S_i) = \bigcap_{s \in \bigcup_{i=0}^n S_i} \leq_s^p$. □

Unfortunately, the equal rank case is special. If I have sources of different ranks, I generally cannot compute the induced beliefs after fusion using only the agent beliefs before fusion, as the following simple example demonstrates:

Example 5.2 *Let $\mathcal{W} = \{w_a, w_b\}$. Suppose two agents A_1 and A_2 are informed by sources s_1 with preferential relation $\leq_{s_1}^p = \{(w_a, w_b)\}$ and s_2 with preferential relation $\leq_{s_2}^p = \{(w_b, w_a)\}$, respectively. A_1's preferential relation is the same as s_1's and A_2's is the same as s_2's. If $s_1 \sqsupset s_2$, then the preferential relation induced by $\oplus(A_1, A_2)$ is $\leq_{s_1}^p$, whereas if $s_2 \sqsupset s_1$, then it is $\leq_{s_2}^p$. Thus, just knowing the preferential relations of the fused agents is not sufficient for computing the induced preferential relation. I need more information about the original sources.*

Thus, I need to maintain some information for computing the induced belief without mentioning the beliefs of sources. How do I better than storing all the original sources?

I can consider the procedure which computes a credibility for each agent based on the credibility of her sources, then simply apply AGe to the agents' induced beliefs. However, this does not work in general if each agent can have sources both more and less credible than those of another agent as the following example demonstrates:

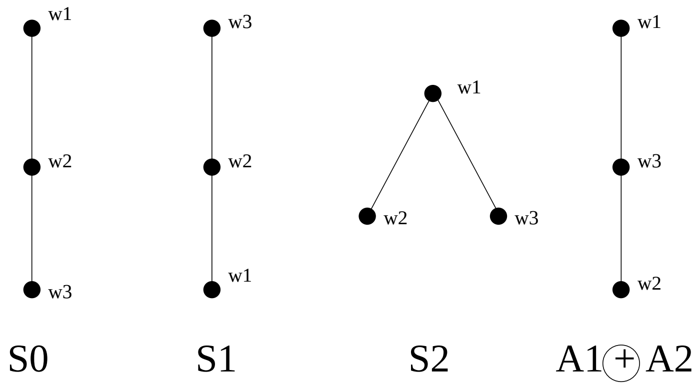


Figure 5.1: Example : $AGRGe^p(S)$ is empty.

Example 5.3 *Let $\mathcal{W} = \{w_1, w_2, w_3\}$. Suppose that agent A_1 is informed by source s_1 and s_2, where the sources' preferential relations are those in Figure 5.1. Let $s_2 \sqsupset s_1 \sqsupset s_0$. Then A_1's induce belief state is $\leq^p_{s_1}$ and A_2's is $\leq^p_{s_2}$. The preferential relation induced by $\oplus(A_1, A_2)$ is shown in the figure. However, if I rank A_1 over A_2 and apply AGe to their induced preferential relations, the result of the operator is $\leq^p_{s_1}$; if I rank A_2 over A_1, the result is $\leq^p_{s_0}$; and, if I rank them equally, I get the empty relation. All of these are obviously incorrect.*

Therefore, I need more information about the source. Now I define *pedigreed relations* which have more information than simple relation. This representation enable me to compute the induced beliefs without reference to sources.

Definition 5.4 *Suppose that for all $s \in \mathfrak{S}$, $\leq^b_s$ is a belief state. Let A be an agent informed by a set of sources $S \subseteq \mathfrak{S}$. A's* pedigreed belief state *is a pair $(\preceq^b, l^b)$ where $\preceq^b = ARf^b_{<}(S)$ and $l : \preceq^b \rightarrow \mathcal{R_{ANK}}$ such that $l((x,y)) = max\{rank(s) : x <^b_s y, s \in S\}$, where $max : 2^{\mathcal{R_{ANK}}} \rightarrow 2^{\mathcal{R_{ANK}}}$ is a function which select maximal elements with respect to $>$. We use $\preceq^{b,A}_r$ to denote the* restriction of A's pedigreed belief state to r, *that is, $\preceq^{b,A}_r = \{(x,y) \in \preceq^b : r \in l((x,y)) \wedge (r' \in l((y,x)) \rightarrow r' \not> r)\}$, and $\sim^{b,A}_r$ to denote its agnosticism relation.*

Definition 5.5 *Suppose that for all $s \in \mathfrak{S}$, $\leq^{gb}_s$ is a generalized belief state. Let A be an agent informed by a set of sources $S \subseteq \mathfrak{S}$. A's* pedigreed generalized belief state *is a pair $(\preceq^{gb}, l^{gb})$ where $\preceq^{gb} = ARf(S)$ and $l : \preceq^{gb} \rightarrow \mathcal{R_{ANK}}$ such that $l((x,y)) = max\{rank(s) : x \leq^{gb}_s y, s \in S\}$, where $max : 2^{\mathcal{R_{ANK}}} \rightarrow max : 2^{\mathcal{R_{ANK}}}$ is a function which select maximal elements with respect to $>$. We use $\preceq^{gb,A}_r$ to denote the* restriction of A's pedigreed generalized belief state to r, *that is, $\preceq^{gb,A}_r = \{(x,y) \in \preceq^{gb} : r \in l((x,y)) \wedge (r' \in l((y,x)) \rightarrow r' \not> r)\}$, and $\sim^{gb,A}_r$ to denote its agnosticism relation.*

Note that it seems to be similar with Maynard-Reid II, but I apply ARf to the partially preordered sources. In addition, when I defines ARf, I apply it only to the totally preordered source, but now ARf is applied to partially preordered sources.

The aggregation operators of preferential relations and generalized preferential relations were complicated. Hence, the definition of pedigreed ones are also complicated.

Then, I can define the pedigreed preferential relation and pedigreed generalized preferential relation, as follows:

Definition 5.6 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. Let A be an agent informed by a set of sources $S \subseteq \mathfrak{S}$. A's* pedigreed preferential relation *is a pair $(\preceq^p, l^p)$ where $\preceq^p = \cup_{S_{eq} \in EQ(S)} AIn(S_{eq})$ and $l : \preceq^p \rightarrow \mathcal{R}_{\mathcal{ANK}}$ such that $l((x,y)) = \{rank(s) : x \leq_s^p y \wedge (\forall S_{eq} \in EQ(S).s \in S_{eq} \Rightarrow (x,y) \in AIn(S_{eq}), s \in S\}$. We use $\preceq_r^{p,A}$ to denote the* restriction of A's pedigreed preferential relation to r, *that is, $\preceq_r^{p,A} = \{(x,y) \in \preceq^p : r \in l((x,y))\}$, and $\sim_r^{p,A}$ to denote its agnosticism relation.*

Definition 5.7 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. Let A be an agent informed by a set of sources $S \subseteq \mathfrak{S}$. A's* pedigreed generalized preferential relation *is a pair $(\preceq^{gp}, l^{gp})$ where $\preceq^{gp} = \cup_{S_{eq} \in EQ(S)} AIn(S_{eq})$ and $l : \preceq^{gp} \rightarrow \mathcal{R}_{\mathcal{ANK}}$ such that $l((x,y)) = \{rank(s) : x \leq_s^{gp} y \wedge (\forall S_{eq} \in EQ(S).s \in S_{eq} \Rightarrow (x,y) \in AIn(S_{eq}), s \in S\}$. We use $\preceq_r^{gp,A}$ to denote the* restriction of A's generalized pedigreed preferential relation to r, *that is, $\preceq_r^{gp,A} = \{(x,y) \in \preceq^{gp} : r \in l((x,y))\}$, and $\sim_r^{gp,A}$ to denote its agnosticism relation.*

I verify that a pair's label is equal to the rank of the source used to determine the pair's membership.

Proposition 5.2 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^b$ is a belief state. Let A be an agent informed by a set of sources $S \subseteq \mathfrak{S}$ and with pedigreed belief state $(\preceq^b, l^b)$. Then*

$$x \preceq_r^{b,A} y \qquad \textit{iff}$$

$$\exists s \in S.x <_s^b y \wedge r = rank(s) \wedge (\forall s' \sqsupset s \in S.x \approx_{s'}^b y).$$

Proof. See Appendix A. □

Proposition 5.3 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gb}$ is a generalized belief state. Let A be an agent informed by a set of sources $S \subseteq \mathfrak{S}$ and with pedigreed belief state $(\preceq^{gb}, l^{gb})$. Then*

$$x \preceq_r^{gb,A} y \qquad \textit{iff}$$

$$\exists s \in S.x \leq^{gb} y \wedge r = rank(s) \wedge (\forall s' \sqsupset s \in S.x \sim_{s'}^{gb} y).$$

Proof. I can prove it as the proof of Proposition 5.2. □

Proposition 5.4 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. Let A be an agent informed by a set of sources $S \subseteq \mathfrak{S}$ and with pedigreed preferential relation $(\preceq^p, l^p)$. Then*

$$x \preceq_r^{p,A} y \qquad \textit{iff}$$

$$\exists s \in S.r = rank(s) \wedge (\forall S_{eq} \in EQ(S).s \in S_{eq} \Rightarrow (x,y) \in AIn(S_{eq})).$$

Proof. See Appendix A. $\square$

Proposition 5.5 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a generalized preferential relation. Let A be an agent informed by a set of sources $S \subseteq \mathfrak{S}$ and with pedigreed generalized preferential relation $(\preceq^{gp}, l^{gp})$. Then*

$$x \preceq_r^{gp,A} y \qquad \textit{iff}$$

$$\exists s \in S.r = rank(s) \wedge (\forall S_{eq} \in EQ(S).s \in S_{eq} \Rightarrow (x, y) \in AIn(S_{eq})).$$

Proof. I can prove it as the proof of Proposition 5.4. $\square$

I can compute the agent's induced belief states given its pedigreed belief state $(\preceq^b, l^b)$, as follows.

Proposition 5.6 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^b$ is a belief state. Let A be an agent informed by a set of sources $S \subseteq \mathfrak{S}$ and with pedigreed belief state $(\preceq^b, l^b)$. Then A's induced belief state is*

$$[\bigcup_{S_{mpc} \in MPC(S)} Ref(\bigcup_{r \in \mathcal{RANK}^{S_{mpc}}} \preceq_r^{b,A} y)]^+$$

.

Proof.

$$\begin{aligned} [\textstyle\bigcup_{S_{mpc} \in MPC(S)} Ref(\bigcup_{r \in \mathcal{RANK}^{S_{mpc}}} \preceq_r^{b,A} y)]^+ &= [\textstyle\bigcup_{S_{mpc} \in MPC(S)} Ref(ARf_<^b(S_{mpc}))]^+ \\ &= [\textstyle\bigcup_{S_{mpc} \in MPC(S)} ARf^b(S_{mpc})]^+ \\ &= AGe^b(S) \end{aligned}$$

$\square$

To compute the agent's induced generalized belief states given its pedigreed generalized belief state $(\preceq^{gb}, l^{gb})$, it suffices to take the transitive closure of $\preceq^{gb}$.

Proposition 5.7 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gb}$ is a generalized belief state. Let A be an agent informed by a set of sources $S \subseteq \mathfrak{S}$ and with pedigreed generalized belief state $(\preceq^{gb}, l^{gb})$. Then A's induced generalized belief state is $\preceq^{gb+}$.*

Proof.

$$\begin{aligned} \preceq^{gb+} &= [ARf(S)]^+ \\ &= [\textstyle\bigcup_{S_{mpc} \in MPC(S)} ARf(S_{mpc}]^+ \\ &= [\textstyle\bigcup_{S_{mpc} \in MPC(S)} [ARf(S_{mpc}]^+]^+ \\ &= AGe(S) \end{aligned}$$

$\square$

To compute the agent's induced preferential relations and generalized preferential relations given its pedigreed preferential relation $(\preceq^p, l^p)$ and generalized preferential relation $\preceq^{gp}, l^{gp}$, respectively, I need more complicated process than the previous two.

Definition 5.8 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. Let A be a set of agent, informed by some subset $S \subseteq \mathfrak{S}$, let $\sqsupseteq_S$ be a partial preorder, and let $(\preceq^p, l^p)$ be the pedigreed preferential relation of the agent A. Given $S_{mpc} \in MPC(S)$, let ${\mathcal{R}_{\mathcal{ANK}}}^{S_{mpc}} = \{r_1, ..., r_N\}$. The n th ranked refinement of preferential relations $O^{p,n}_{{\mathcal{R}_{\mathcal{ANK}}}^{S_{mpc}}}((\preceq^p, l^p))$ is*

(I) if $n = 1$;
$O^{p,n}_{{\mathcal{R}_{\mathcal{ANK}}}^{S_{mpc}}}((\preceq^p, l^p)) = \preceq^{p,A}_{r_1}$,

(II) else if $n >^{\mathcal{N}} 1$; Suppose that for any $\leq \in \mathcal{R}$, $\leq \in SO^n_{\mathcal{A}}(S)$ iff

$$\begin{aligned} \leq \; = \; & O^{p,n-1}_{{\mathcal{R}_{\mathcal{ANK}}}^{\mathcal{S}_{\Updownarrow\surd\rfloor}}}((\preceq^p, l^p)) \cup \\ & \{(w_a, w_b) : (w_b, w_a) \notin O^{p,n-1}_{{\mathcal{R}_{\mathcal{ANK}}}^{S_{mpc}}}((\preceq^p, l^p)) \wedge \\ & \qquad w_a \preceq^{p,A}_{r_n} w_b \wedge \\ & \qquad (w_b, w_a) \notin \leq^+\}. \end{aligned}$$

Then $O^{p,n}_{{\mathcal{R}_{\mathcal{ANK}}}^{S_{mpc}}}((\preceq^p, l^p)) = \cap([SO^n_{\mathcal{A}}({\mathcal{R}_{\mathcal{ANK}}}^{S_{mpc}})]^+)$.

Definition 5.9 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. Let A be an agent, informed by some subset $S \subseteq \mathfrak{S}$, let $\sqsupseteq_S$ be a partial preorder. Given $S_{mpc} \in MPC(S)$, let ${\mathcal{R}_{\mathcal{ANK}}}^{S_{mpc}} = \{r_1, ..., r_N\}$. Then*

$$A^p_{{\mathcal{R}_{\mathcal{ANK}}}^{S_{mpc}}}((\preceq^p, l^p)) = O^{p,N}_{{\mathcal{R}_{\mathcal{ANK}}}^{S_{mpc}}}((\preceq^p, l^p)).$$

Proposition 5.8 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^p$ is a preferential relation. Let A be an agent informed by a set of sources $S \subseteq \mathfrak{S}$ and with pedigreed preferential relation $(\preceq^p, l^p)$. Then A's induced preferential relation is $\bigcap_{S_{mpc} \in MPC(S)} A^p_{{\mathcal{R}_{\mathcal{ANK}}}^{S_{mpc}}}((\preceq^p, l^p))$,*

Proof. See Appendix A. □

I also can show that the induced generalized preferential relation is computed with the pedigreed generalized preferential relation.

Definition 5.10 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. Let A be a set of agent, informed by some subset $S \subseteq \mathfrak{S}$, let $\sqsupseteq_S$ be a partial preorder, and let $(\preceq^{gp}, l^{gp})$ be the generalized pedigreed preferential relation of the agent A. Given $S_{mpc} \in MPC(S)$, let ${\mathcal{R}_{\mathcal{ANK}}}^{S_{mpc}} = \{r_1, ..., r_N\}$. The n th refinement $O^{gp,n}_{{\mathcal{R}_{\mathcal{ANK}}}^{S_{mpc}}}((\preceq^{gp}, l^{gp}))$ is*

(I) if $n = 1$;
$O^{p,n}_{{\mathcal{R}_{\mathcal{ANK}}}^{S_{mpc}}}((\preceq^{gp}, l^{gp})) = \preceq^{gp,A}_{r_1}$,

(II) else if $n >^{\mathcal{N}} 1$; Suppose that for any

$$\begin{aligned} TO^{p,n}_{{\mathcal{R}_{\mathcal{ANK}}}^{S_{mpc}}}((\preceq^{gp}, l^{gp})) \; = \; & O^{gp,n-1}_{\mathcal{R}_{\mathcal{ANK}mpc}}((\preceq^{gp}, l^{gp})) \cup \\ & \{(w_a, w_b) : (w_b, w_a) \notin O^{gp,n-1}_{{\mathcal{R}_{\mathcal{ANK}}}^{S_{mpc}}}((\preceq^{gp}, l^{gp})) \wedge \\ & \qquad w_a \preceq^{gp,A}_{r_n} w_b\}. \end{aligned}$$

Then $O^{p,n}_{{\mathcal{R}_{\mathcal{ANK}}}^{S_{mpc}}}((\preceq^p, l^p)) = [TO^{p,n}_{{\mathcal{R}_{\mathcal{ANK}}}^{S_{mpc}}}((\preceq^p, l^p))]^+$.

Definition 5.11 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. Let A be an agent, informed by some subset $S \subseteq \mathfrak{S}$, let $\sqsupseteq_S$ be a partial preorder. Given $S_{mpc} \in MPC(S)$, let $\mathcal{RANK}^{S_{mpc}} = \{r_1, ..., r_N\}$. Then*

$$A^{gp}_{\mathcal{RANK}^{S_{mpc}}}((\preceq^{gp}, l^{gp})) = O^{gp,N}_{\mathcal{RANK}_{mpc}}((\preceq^{gp}, l^{gp})).$$

Proposition 5.9 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. Let A be an agent informed by a set of sources $S \subseteq \mathfrak{S}$ and with pedigreed generalized preferential relation $(\preceq^{gp}, l^{gp})$. Then A's induced generalized preferential relation is $\bigcap_{S_{mpc} \in MPC(S)} A^p_{\mathcal{RANK}^{S_{mpc}}}((\preceq^p, l^p))$,*

Proof. I can prove it as the proof of Proposition 5.8. □

Now, given only the pedigreed beliefs of a set of agents, I can compute the new pedigreed beliefs. I use the strategy which I already developed.

Definition 5.12 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^b$ is a belief state. Let $\mathcal{A} = \{A_1, ..., An\}$ be a set of agent, each informed by some subset $S \subseteq \mathfrak{S}$, let $\sqsupseteq_S$ be a partial preorder, and let $\mathcal{P}^b_{\mathcal{A}}$ be the set of pedigreed belief state of the agents in $\mathcal{A}$. The* pedigreed fusion *of $\mathcal{P}^b_{\mathcal{A}}$, written $\oplus^b_p(\mathcal{P}^b_{\mathcal{A}})$, is $(\preceq^b, l^b)$ where*

1. *$\preceq^b$ is the relation*

$$\{(x,y) : \exists A_i \in \mathcal{A}, r \in \mathcal{RANK}.x \preceq_r^{b,A_i} y \wedge (\forall A_j \in \mathcal{A}, r' > r \in \mathcal{RANK}.x \sim_{r'}^{b,A_j} y)\}$$

   over $\mathcal{W}$ and

2. *$l^b : \preceq^b \rightarrow \mathcal{RANK}$ such that $l^b((x,y)) = max\{r : x \preceq_r^{b,A_i} y, A_i \in \mathcal{A}\}$.*

Proposition 5.10 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^b$ is a belief state. Let $\mathcal{A} = \{A_1, ..., An\}$ be a set of agent such that each agent A_i is informed by some subset $S_i \subseteq \mathfrak{S}$, let $\sqsupseteq_{\bigcup S_i}$ be a partial preorder, and let $\mathcal{P}^b_{calA}$ be the set of pedigreed belief state of the agents in $\mathcal{A}$. Then $\oplus^b_p(\mathcal{P}^b_{\mathcal{A}})$ is the pedigreed belief state of $\oplus(\mathcal{A})$.*

Proof. See Appendix A. □

Definition 5.13 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gb}$ is a generalized belief state. Let $\mathcal{A} = \{A_1, ..., An\}$ be a set of agent, each informed by some subset $S \subseteq \mathfrak{S}$, let $\sqsupseteq_S$ be a partial preorder, and let $\mathcal{P}^{gb}_{\mathcal{A}}$ be the set of pedigreed generalized belief state of the agents in $\mathcal{A}$. The* pedigreed fusion *of $\mathcal{P}^{gb}_{\mathcal{A}}$, written $\oplus^{gb}_p(\mathcal{P}^{gb}_{\mathcal{A}})$, is $(\preceq^{gb}, l^{gb})$ where*

1. *$\preceq^{gb}$ is the relation*

$$\{(x,y) : \exists A_i \in \mathcal{A}, r \in \mathcal{RANK}.x \preceq_r^{b,A_i} y \wedge (\forall A_j \in \mathcal{A}, r' > r \in \mathcal{RANK}.x \sim_{r'}^{b,A_j} y)\}$$

   over $\mathcal{W}$ and

2. *$l^{gb} : \preceq^{gb} \rightarrow \mathcal{RANK}$ such that $l^{gb}((x,y)) = max\{r : x \preceq_r^{gb,A_i} y, A_i \in \mathcal{A}\}$.*

Proposition 5.11 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gb}$ is a generalized belief state. Let $\mathcal{A} = \{A_1, ..., An\}$ be a set of agent such that each agent A_i is informed by some subset $S_i \subseteq \mathfrak{S}$, let $\sqsupseteq_{\bigcup S_i}$ be a partial preorder, and let $\mathcal{P}_{\mathcal{A}}^{gb}$ be the set of pedigreed generalized belief state of the agents in $\mathcal{A}$. Then $\oplus_p^{gb}(\mathcal{P}_{\mathcal{A}}^{gb})$ is the pedigreed generalized belief state of $\oplus(\mathcal{A})$.*

Proof. I can prove it as the proof of Proposition 5.10. □

Before I define the pedigreed fusion of preferential relations and generalized preferential relations, I will prepare some concepts.

Definition 5.14 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{p}$ is a preferential relation. Let $\mathcal{A} = \{A_1, ..., An\}$ be a set of agent, each informed by some subset $S \subseteq \mathfrak{S}$, let $\sqsupseteq_S$ be a partial preorder, and let $\mathcal{P}_{\mathcal{A}}^{p}$ be the set of pedigreed preferential relation of the agents in $\mathcal{A}$. Then $\preceq_r^{p,\mathcal{A}} = \bigcap_{A_i \in \mathcal{A}} \preceq_r^{p,A_i}$.*

Then I define the pedigreed fusion of preferential relations.

Definition 5.15 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{p}$ is a preferential relation. Let $\mathcal{A} = \{A_1, ..., An\}$ be a set of agent, each informed by some subset $S \subseteq \mathfrak{S}$, let $\sqsupseteq_S$ be a partial preorder, and let $\mathcal{P}_{\mathcal{A}}^{p}$ be the set of pedigreed preferential relation of the agents in $\mathcal{A}$. The* pedigreed fusion *of $\mathcal{P}_{\mathcal{A}}^{p}$, written $\oplus_p^{p}(\mathcal{P}_{\mathcal{A}}^{p})$, is $(\preceq^p, l^p)$ where*

1. $\preceq^p = \{(x, y) | \exists r \in \mathcal{R}_{\mathcal{ANK}}.x \preceq_r^{p,\mathcal{A}} y\}$ *and*
2. $l^p : \preceq^p \to \mathcal{R}_{\mathcal{ANK}}$ *such that* $l^p((x, y)) = \{r : x \preceq_r^{p,\mathcal{A}} y\}$.

Proposition 5.12 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{p}$ is a preferential relation. Let $\mathcal{A} = \{A_1, ..., An\}$ be a set of agent such that each agent A_i is informed by some subset $S_i \subseteq \mathfrak{S}$, let $\sqsupseteq_{\bigcup S_i}$ be a partial preorder, and let $\mathcal{P}_{\mathcal{A}}^{p}$ be the set of pedigreed preferential relations of the agents in $\mathcal{A}$. Then $\oplus_p^{p}(\mathcal{P}_{\mathcal{A}}^{p})$ is the pedigreed preferential relation of $\oplus(\mathcal{A})$.*

Proof. See Appendix A. □

Also I can define the fusion of pedigreed generalized preferential relations.

Definition 5.16 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. Let $\mathcal{A} = \{A_1, ..., An\}$ be a set of agent, each informed by some subset $S \subseteq \mathfrak{S}$, let $\sqsupseteq_S$ be a partial preorder, and let $\mathcal{P}_{cal A}^{gp}$ be the set of pedigreed generalized preferential relation of the agents in $\mathcal{A}$. Then $\preceq_r^{gp,\mathcal{A}} = \bigcap_{A_i \in \mathcal{A}} \preceq_r^{gp,A_i}$.*

Definition 5.17 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. Let $\mathcal{A} = \{A_1, ..., An\}$ be a set of agent, each informed by some subset $S \subseteq \mathfrak{S}$, let $\sqsupseteq_S$ be a partial preorder, and let $\mathcal{P}_{cal A}^{gp}$ be the set of generalized pedigreed preferential relation of the agents in $\mathcal{A}$. The* pedigreed fusion *of $\mathcal{P}_{\mathcal{A}}^{gp}$, written $\oplus_p^{gp}(\mathcal{P}_{\mathcal{A}}^{gp})$, is $(\preceq^{gp}, l^{gp})$ where*

1. $\preceq^{gp} = \{(x, y) | \exists r \in \mathcal{R}_{\mathcal{ANK}}.x \preceq_r^{gp,\mathcal{A}} y\}$ *and*

2. $l^{gp} : \preceq^{gp} \rightarrow \mathcal{R}_{\mathcal{ANK}}$ such that $l^{gp}((x, y)) = \{r : x \preceq_r^{gp,\mathcal{A}} y\}$.

Proposition 5.13 *Suppose that for all $s \in \mathfrak{S}$, $\leq_s^{gp}$ is a generalized preferential relation. Let $\mathcal{A} = \{A_1, ..., An\}$ be a set of agent such that each agent A_i is informed by some subset $S_i \subseteq \mathfrak{S}$, let $\sqsupseteq_{\bigcup S_i}$ be a partial preorder, and let $\mathcal{P}_{\mathcal{A}}^{gp}$ be the set of pedigreed generalized preferential relations of the agents in $\mathcal{A}$. Then $\oplus_p^{gp}(\mathcal{P}_{\mathcal{A}}^{p})$ is the pedigreed generalized preferential relation of $\oplus(\mathcal{A})$.*

Proof. I can prove it as the proof of Proposition 5.12. $\square$

Chapter 6

The various relation of aggregations

In this chapter, I show the connection between the aggregation operator, especially, the relation about the operator for preferential relations and the operator for belief states.

6.1 the relation about operation of preferential relations and that of belief states

Let $<^b \in \mathcal{B}_<$ is the strict version of $\leq^b \in \mathcal{B}$. By Proposition 3.8, note that $\mathcal{B}_< \subseteq \mathcal{P}$. Can I show any relation between the refinement for preferential relation and the refinement for belief states? This answer is yes. I can use the refinement operator for the strict versions of belief states as the refinement operator for belief states as follows:

Proposition 6.1 *Suppose $\leq^b_A$ and $\leq^b_B \in \mathcal{B}$. Then the strict version of $\leq^b_A \olessthan^{\,b} \leq^b_B$ is equal to $<^b_A \circledtriangleleft^{\,p} <^b_B$.*

The following corollary is immediate from the above proposition.

Corollary 6.1 *Suppose $\leq^b_A$ and $\leq^b_B \in \mathcal{B}$. Then $<^b_A \olessthan^{\,b}_{\,<} <^b_B$ is equal to $<^b_A \circledtriangleleft^{\,p} <^b_B$.*

This proposition is generalized by the iterative refinement.

Corollary 6.2 *Suppose $\leq^b_1, \leq^b_2, ..., \leq^b_N \in \mathcal{B}$. Then the strict version of $((\leq^b_1 \olessthan^{\,b} \leq^b_2) \olessthan^{\,b} \cdots \leq^b_N)$ is equal to $((<^b_1 \circledtriangleleft^{\,p} <^b_2) \circledtriangleleft^{\,p} \cdots <^b_N)$.*

I can translate not only refinement; the aggregation for belief states can also be translated to the aggregation for preferential relations.

Proposition 6.2 *Suppose that for all $s \in \mathfrak{S}$, $\leq^b_s$ is a belief state, and $<^b_s$ is the strict version of $\leq^b_s$. If $S \subseteq \mathfrak{S}$ and $\sqsupseteq_S$ is total order, then the strict version of $ARf^b(S)$, applied to the belief states of sources, is equal to $ARf^p(S)$, applied to the strict version of belief states of sources.*

Proof. From Proposition 3.16, and Lemma 4.2 and 4.3,

$$
\begin{array}{ll}
\text{the strict version of } AGRRf^b(S) & \Leftrightarrow \\
\text{the strict version of } ((\leq^b_{s_1} \circledless{}^b \leq^b_{s_2}) \circledless{}^b \cdots \leq^b_{s_N}) & \Leftrightarrow \\
((<^b_{s_1} \circledtriangleleft{}^p <^b_{s_2}) \circledtriangleleft{}^p \cdots <^b_{s_N}) & \Leftrightarrow \\
AGRRf^p(S). &
\end{array}
$$

□

Proposition 6.3 *Suppose that for all $s \in \mathfrak{S}$, $\leq^b_s$ is a belief state, and $<^b_s$ is the strict version of $\leq^b_s$. If $S \subseteq \mathfrak{S}$ and $\sqsupseteq_S$ is total order, then the strict version of $AGe^b(S)$, applied to the belief states of sources, is equal to $AGe^p(S)$, applied to the strict version of belief states of sources.*

Proof. From Proposition 4.26, 4.29 and 6.2. □

6.2 the relation about operation of generalized preferential relations and that of belief states

In this section, I will show some relation between the refinement for generalized preferential relation and the refinement for belief states.

Proposition 6.4 *Suppose $\leq^b_A$ and $\leq^b_B \in \mathcal{B}$. Then the strict version of $\leq^b_A \circledless{}^b \leq^b_B$ is equal to $<^b_A \circledtriangleleft{}^{gp} <^b_B$.*

Proof. See Appendix A. □

The following corollary is immediate from the above proposition.

Corollary 6.3 *Suppose $\leq^b_A$ and $\leq^b_B \in \mathcal{B}$. Then $<^b_A \circledless{}^b_< <^b_B$ is equal to $<^b_A \circledtriangleleft{}^{gp} <^b_B$.*

This proposition is generalized by the iterative refinement.

Corollary 6.4 *Suppose $\leq^b_1, \leq^b_2, ..., \leq^b_N \in \mathcal{B}$. Then the strict version of $((\leq^b_1 \circledless{}^b \leq^b_2) \circledless{}^b \cdots \leq^b_N)$ is equal to $((<^b_1 \circledtriangleleft{}^{gp} <^b_2) \circledtriangleleft{}^{gp} \cdots <^b_N)$.*

I can translate not only the refinement; the aggregation for belief states can also be translated to the aggregation for preferential relations.

Proposition 6.5 *Suppose that for all $s \in \mathfrak{S}$, $\leq^b_s$ is a belief state, and $<^b_s$ is the strict version of $\leq^b_s$. If $S \subseteq \mathfrak{S}$ and $\sqsupseteq_S$ is total order, then the strict version of $ARf^b(S)$, applied to the belief states of sources, is equal to $ARf^{gp}(S)$, applied to the strict version of belief states of sources.*

Proof. From Proposition 3.16, and Lemma 4.2 and 4.5,

$$\begin{array}{ll}
\text{the strict version of } ARf^b(S) & \Leftrightarrow \\
\text{the strict version of } ((\leq^b_{s_1} \circledless {}^b \leq^b_{s_2}) \circledless {}^b \cdots \leq^b_{s_N}) & \Leftrightarrow \\
((<^b_{s_1} \circledless {}^{gp} <^b_{s_2}) \circledless {}^{gp} \cdots <^b_{s_N}) & \Leftrightarrow \\
ARf^{gp}(S). &
\end{array}$$

□

Proposition 6.6 *Suppose that for all $s \in \mathfrak{S}$, $\leq^b_s$ is a belief state, and $<^b_s$ is the strict version of $\leq^b_s$. If $S \subseteq \mathfrak{S}$ and $\sqsupseteq_S$ is total order, then the strict version of $AGe^b(S)$, applied to the belief states of sources, is equal to $AGe^{gp}(S)$, applied to the strict version of belief states of sources.*

Proof. From Proposition 4.29, 4.35, and 6.5. □

Chapter 7

Conclusion and Discussion

This paper's contribution is the following. Maynard-Reid II et al. [29] proposed the idea of pedigreed sources to solve the inconsistency of knowledge amalgamation when the refinement operator is iteratively applied. However in the theory, all the sources must be totally ordered and this fact restricts the area of application. Maynard-Reid II [28] researched the aggregation operator for the totally preordered sources. In [44], we realized the partiality of order of sources, and showed the procedure of fusion of preferential relations. In this paper, I showed that several operation can be formalized with belief state, generalized belief state, preferential relation, and generalized preferential relation.

In my method of fusion of them, there were still several issues. First, the definition of the aggregation operator of (generalized) belief state and (generalized) preferential relation is very different. Although some relation about the operation of belief state and that of (generalized) preferential relation was proved in 6, I need to generalize this relation in order to cover the operation for partially preordered sources. Secondly, I defined the refinement of preferential relations by the intersection of all the possible refinements. Of course, this is not an only method to uniquely decide the refinement, and I can consider other ways to select one refinement among other refinements.

In future, I am to study all these branches and to evaluate the adequateness, considering the applicability of practical problems.

Appendix A

Proofs

A.1 Proofs of Chapter 2

Proof of Proposition 2.1. For 1., 2., and 7., see [28]. 5. is obvious. I will show 3. Let $\leq$ be cyclic, and $\leq^+$ be the transitive closure of $\leq$. Then there is some $w_0, ..., w_n \in \Omega$ such that $w_0 \leq \cdots \leq w_n$ and $w_n \leq w_0$. Therefore, $w_0 \leq^+ w_0$, and $\{w_0\}$ satisfies the condition of partially connectedness.

I will show 4. At first, I will show if-part. Let $\leq$ be irreflexive and transitive. Suppose that $\leq' = \leq \cup \{(w, w) | w \in \mathcal{W}\}$. I will show that $\leq'$ is partial order. Reflexivity is obvious from the definition. I will show anti-symmetricity. Suppose that $w_a \leq' w_b$ and $w_b \leq' w_a$. Therefore, $w_a \leq w_b$ or $w_a = w_b$, and $w_b \leq w_a$ or $w_b = w_a$ from the definition. Thus, it suffices to consider the case $w_a \leq w_b$ and $w_b \leq w_a$. However, from transivity, $w_a \leq w_a$, and it contradicts irreflexivity of $\leq$. It follows that for every case, $w_a = w_b$. I will show transitivity. Suppose that $w_a \leq' w_b$ and $w_b \leq' w_c$. Then $w_a \leq w_b$ or $w_a = w_b$, and $w_b \leq w_c$ or $w_b = w_c$ from the definition.

- Suppose $w_a \leq w_b$ and $w_b \leq w_c$. From transitivy of $\leq$, $w_a \leq w_c$. Therefore, $w_a \leq' w_c$.
- Suppose $w_a \leq w_b$ and $w_b = w_c$. It is obvious that $w_a \leq w_c$. Therefore, $w_a \leq' w_c$.
- Suppose $w_a = w_b$ and $w_b \leq w_c$. It is obvious that $w_a \leq w_c$. Therefore, $w_a \leq' w_c$.
- Suppose $w_a = w_b$ and $w_b = w_c$. It is obvious that $w_a = w_c$. Therefore, $w_a \leq' w_c$.

For every case, $w_a \leq' w_c$. It follows that $\leq'$ is a partial order. I will show that $\leq$ is the strict version of $\leq'$. Suppose that $w_a \leq w_b$. $w_a \leq' w_b$ is obvious from the definition. If $w_b \leq' w_a$, then $w_b \leq w_a$ or $w_b = w_a$. For the first case, from transitivity, $w_a \leq w_a$, and it contradicts irreflexivity of $\leq$. For the second case, $w_a \leq w_a$, and it contradicts irreflexivity of $\leq$. It follows that $w_b \not\leq' w_a$. Suppose that $w_a \leq' w_b \wedge w_b \not\leq' w_a$. Then $w_a \leq w_b$ or $w_a = w_b$, and $w_b \not\leq w_a$ and $w_b \neq w_a$ from the definition. It follows that $w_a \leq w_b$.

I will show the only-if part. Suppose that $\leq$ is a strict partial order. Let be $\leq'$ such that $\leq$ is a strict version of $\leq'$. I will show transitivity. Suppose that $w_a \leq w_b$ and $w_b \leq w_c$. Then $w_a \leq' w_b$ and $w_b \not\leq' w_a$, and $w_b \leq' w_c$ and $w_c \not\leq' w_b$. From transitivity,

$w_a \leq' w_c$. Suppose that $w_c \leq' w_a$. Then $w_b \leq' w_a$ from transitivity, and it contradicts $w_b \not\leq' w_a$. Therefore, $w_c \not\leq' w_a$. It follows that $w_a \leq w_c$. I will show irreflexivity. For any $w \in \mathcal{W}$, $w \leq' w$ is true iff $w \not\leq' w$ is false. It follows that $w \leq w$.

I will show 6. Suppose that $\leq$ is a strict total order. Let $\leq'$ be the relation such that $\leq$ is the strict version of $\leq'$. $\leq'$ is a total order, and it is also a partial order. Therefore, $\leq$ is a strict partial order. I will show modularlity. Suppose that $w_a \leq w_b$. Then $w_a \leq' w_b$ and $w_b \not\leq' w_a$. From totality, $w_a \leq' w_c$ or $w_c \leq' w_a$, and $w_b \leq' w_c$ or $w_c \leq' w_b$.

- Suppose that $w_a \leq' w_c$ and $w_b \leq' w_c$. Then $w_c \not\leq' w_a$ from transitivity. Therefore, $w_a \leq w_c$.
- Suppose that $w_c \leq' w_a$ and $w_b \leq' w_c$. Then $w_b \leq' w_a$ from transitivity. It contradicts the supposition $w_b \not\leq' w_a$.
- Suppose that $w_a \leq' w_c$ and $w_c \leq' w_b$. If $w_c \leq' w_a$, then $w_b \not\leq' w_c$ from transitivity. Therefore, $w_a \leq w_c \vee w_c \leq w_b$.
- Suppose that $w_c \leq' w_a$ and $w_c \leq' w_b$. Then $w_b \not\leq' w_c$ from transitivity. Therefore, $w_c \leq w_b$.

For every case, $w_a \leq w_c \vee w_c \leq w_b$.

I will show 8. Suppose that $\leq$ is the strict version of $\leq'$. Let $w \leq w$ for some $w \in \Omega$. Then $w \leq' w$ and $w \not\leq' w$. Contradiction. Therefore, $\leq$ is irreflexive.

I will show 9. Suppose that $\leq$ is the strict version of $\leq'$, and $\leq'$ is transitive. Let $\leq$ be not acyclic. Then for some $w_0, ..., w_n \in \Omega$, $w_0 \leq \cdots \leq w_n$ and $w_n \leq w_0$. Then $w_0 \leq' \cdots \leq' w_n$ and $w_n \not\leq' \cdots \not\leq' w_0$, and $w_n \leq' w_0$ and $w_0 \not\leq' w_n$. However, from transitivity, $w_0 \leq' w_n$. Contradiction. Therefore, $\leq$ is acyclic.

I will show transitivity of $\leq$. Suppose that $w_a \leq w_b$ and $w_b \leq w_c$. Then $w_a \leq' w_b$ and $w_b \not\leq w_a$. Moreover, $w_b \leq' w_c$ and $w_c \not\leq w_b$. Because $\leq'$ is transitive, $w_a \leq' w_c$. If $w_c \leq' w_a$, then $w_c \leq' w_b$ from transitivity, and it contradicts $w_c \not\leq w_b$. Therefore,$w_c \not\leq w_a$. It follws that $w_a \leq w_c$. $\square$

A.2 Proofs of Chapter 3

Proof of Proposition 3.5. We refer to the conditions in the proposition as condition 1. and 2., respectively. We prove **Only-If-Part** and **If-Part** of the proposition separately.

Only-If-Part. Suppose that $\leq^b \in \mathcal{B}$, that is $\leq^b$ is a total preorder over $\mathcal{W}$. We use a series of definitions and lemmas to show that for some partition of $\mathcal{W}$, conditions 1. and 2. is satisfied. We first show that ∞^b is an equivalent relation by which we will partition $\mathcal{W}$.

Lemma A.1 *∞ is an equivalence relation over $\mathcal{W}$.*

Proof. We will show that ∞^b is reflexive. $\leq^b$ is reflexive. Thus, $w \leq^b w$. Therefore, $w \infty^b w$.

We will show that ∞^b is symmetric. Suppose that $w \infty^b w'$. Then $w \leq^b w'$ and $w' \leq^b w$. Therefore, $w' \infty^b w$.

We will show that ∞^b is transitive. Suppose that $w \infty^b w'$ and $w' \infty^b w''$. From $w \infty^b w'$, $w \leq^b w'$ and $w' \leq^b w$. From $w' \infty^b w''$, $w' \leq^b w''$ and $w'' \leq^b w'$. From transitivity of $\leq^b$, $w \leq^b w''$ and $w'' \leq^b w$. Therefore, $w \infty^b w''$. $\Box$.

∞^b partitions $\mathcal{W}$ into its equvalence classes. We use $[w]$ to denote the equivalence class containing w, that is, the set $\{w' \in \mathcal{W} : w \equiv w'\}$. We now define a total order over these equvalence classes:

Definition A.1 *For all $w, w' \in \mathcal{W}$, $[w] \leq [w']$ iff $w \leq^b w'$.*

Lemma A.2 *$\leq$ is well-defined, that is, if $w_a \infty^b w_b$ and $w'_a \infty^b w'_b$, then $w_a \leq^b w'_a$ iff $w_b \leq^b w'_b$, for all $w_a, w_b, w'_a, w'_b \in calW$.*

Proof. Let $w_a \infty^b w_b$ and $w'_a \infty^b w'_b$. Thus, $w_a \leq^b w_b$ and $w_b \leq^b w_a$. In addition, $w'_a \leq^b w'_b$ and $w'_b \leq^b w'_a$. Suppose that $w_a \leq^b w'_a$. From transitivity of $\leq^b$, $w_b \leq^b w'_a$. Therefore, $w_b \leq^b w'_b$ from transitivity. Also suppose that $w_b \leq^b w'_b$. From transitivity of $\leq^b$, $w_b \leq^b w'_a$. Therefore, $w_a \leq^b w'_a$ from transitivity. $\Box$

Lemma A.3 *$\leq$ is a total order over the equivalence classes of $\mathcal{W}$ defined by ∞^b.*

Proof. Suppose that $w, w', w'' \in \mathcal{W}$. We first show that $\leq$ is total. By totality of $\leq$, $w \leq^b w'$ or $w' \leq^b w$. Hence, $[w] \leq [w']$ or $[w'] \leq [w]$.

We will show that $\leq$ is anti-symmetric. Suppose that $[w] \leq [w']$ and $[w'] \leq [w]$. Then $w \leq^b w'$ and $w' \leq^b w$. Therefore, $w \infty^b w'$, and then $[w] = [w']$.

We will show that $\leq$ is transitive. Suppose that $[w] \leq [w']$ and $[w'] \leq [w'']$. Then $w \leq^b w'$ and $w' \leq^b w''$. From transitivity of $\leq^b$, $w \leq^b w''$. Therefore, $[w] \leq [w'']$. $\Box$

Let $\mathbf{W}$ be the set of equivalence classes. We show that $\mathbf{W}$ and $\leq$ satisfy conditions 1. and 2. For condition 1., suppose that $W \in \mathbf{W}$ and $w, w' \in W$. It suffices to show that $w \leq^b w'$. Since w, w' are in the same equivalence class, $x \infty^b y$. Therefore, $w \leq^b w'$. Condition 2. is obvious from Definition A.1.

If-Part. Suppose that $\mathbf{W}$ is a partition of $\mathcal{W}$, $\leq$ is a total order over $\mathbf{W}$, and $\leq^b$ is a relation over $\mathcal{W}$, toghether satisfying conditions 1 and 2. We want to show that $\leq^b$ is total, reflexive, and transitive. At first, we show that $\leq^b$ is total. Suppose that $w \in W$, $w' \in W'$ and $W, W' \in \mathbf{W}$. From totality of $\leq$, $W \leq W'$ or $W' \leq W$. From condition 2., $w \leq^b w'$ or $w' \leq^b w$.

We will show $\leq^b$ is reflexive. Suppose that $w \in W$ and $W \in\mathbf{W}$. From condition 1., W is fully connected. Therefore, $w \leq^b w$.

We will show $\leq^b$ is transitive. Suppose that $w \in W$, $w' \in W'$, $w'' \in W''$, $w \leq^b w'$, and $w' \leq^b w''$. From condition 2., $W \leq W'$ and $W' \leq W''$. From transitivity of $\leq$, $W \leq W''$. Therefore, $w \leq^b w''$ from condition 2. □

Proof of Proposition 3.6. Suppose that $trans : \mathcal{B} \to \mathcal{B}_<$ is such that for all $\leq^b \in \mathcal{B}$, $trans(\leq^b)$ is the strict version of $\leq^b$. Because $trans(\leq^b)$ is a strict total preoreder over $\mathcal{W}$, $trans(\leq^b) \in \mathcal{B}_<$. For all $\leq^b \in \mathcal{B}$, $trans(\leq^b) \in \mathcal{B}_<$, and then $trans$ is injection.

We will show that $trans$ is surjection. Suppose that $\leq^{sb} \in \mathcal{B}_<$. Let $\leq^b = \leq^{sb} \cup \{(w_a, w_b) | w_a \sim^{sb} w_b\}$. We will show $\leq^b \in \mathcal{B}$. From irreflexivity, for any $w \in \mathcal{W}$, $w \sim^{sb} w$. Hence, $w \leq^b w$. It follows that $\leq^b$ is reflexive. Suppose that $w_a \not\leq^b w_b \wedge w_b \not\leq^b w_a$. Then $w_a \not\leq^{sb} w_b \wedge w_b \not\leq^{sb} w_a$, and it follows that $w_a \sim^{sb} w_b$. That is to say, $w_a \leq^b w_b$, and it contradicts the supposition. It follows that $\leq^b$ is total. Suppose that $w_a \leq^b w_b$ and $w_b \leq^b w_c$. If $w_a \leq^{sb} w_b$ and $w_b \leq^{sb} w_c$, then $w_a \leq^{sb} w_c$, and $w_a \leq^b w_c$. If $w_a \sim^{sb} w_b$ and $w_b \leq^{sb} w_c$, we can assume that there is a total preorder $\leq$ such that $w_a \leq w_b$, $w_b \leq w_a$, $w_b \leq w_c$ and $w_c \not\leq w_b$, and the strict version of $\leq$ is $\leq^{sb}$, because $\leq^{sb}$ is a strict total preorder. From $w_a \leq w_b$ and $w_b \leq w_c$, we can show $w_a \leq w_c$. From $w_a \leq w_b$ and $w_c \not\leq w_b$, we can show $w_c \not\leq w_a$. Therefore, we can show $w_a \leq^{sb} w_c$. If $w_a \leq^{sb} w_b$ and $w_b \sim^{sb} w_c$, we can show $w_a \leq^{sb} w_c$ as well. If $w_a \sim^{sb} w_b$ and $w_b \sim^{sb} w_c$, we can show $w_a \sim^{sb} w_c$ as well. Therefore, $w_a \leq^b w_c$, and it follows that $\leq^b$ is transitive. Therefore, $\leq^b \in \mathcal{B}$.

I will show that $trans(\leq^b) = \leq^{sb}$. '$\supseteq$' is obvious from the definition. I will show '$\subseteq$.' Suppose that $w_a trans(\leq^b) w_b$ and $w_a \not\leq^{sb} w_b$. From $w_a trans(\leq^b) w_b$, $w_a \leq^b w_b$ and $w_b \not\leq^b w_a$. From $w_a \leq^b w_b$, $w_a \leq^{sb} w_b$ or $w_a \sim^{sb} w_b$. From $w_a \not\leq^{sb} w_b$, $w_a \sim^{sb} w_b$. However, from $w_b \not\leq^b w_a$, $w_b \not\leq^{sb} w_a$ and $w_b \not\sim^{sb} w_a$, and then $w_a \not\sim^{sb} w_b$. Contradiction. I proved that $trans(\leq^b) = \leq^{sb}$, and I also proved that $trans$ is a surjection. □

Proof of Proposition 3.7. For the case 1., suppose that $\leq^{gb}$ is a generalized belief state. At first, I show that ∞^{gb} is symmetric. Let $w_a \infty^{gb} w_b$. Then $w_b \leq^{gb} w_a$ and $w_a \leq^{gb} w_b$. Therefore, $w_b \infty^{gb} w_a$.

I will show that ∞^{gb} is transitive. Let $w_a \infty^{gb} w_b$ and $w_b \infty^{gb} w_c$. Then $w_a \leq^{gb} w_b$, $w_b \leq^{gb} w_a$, $w_b \leq^{gb} w_c$, and $w_c \leq^{gb} w_b$. From transitivity of $\leq^{gb}$, $w_a \leq^{gb} w_c$, and $w_c \leq^{gb} w_a$. It follows that $w_a \infty^{gb} w_c$.

In the similar way, I can show that $\sim^{gp}$ is a symmetric and transitive relation.

For the case 2., suppose that $\leq^{gb}$ is a generalized belief state. At first, I show that $\sim^{gb}$ is symmetric. Let $w_a \sim^{gb} w_b$. Then $w_b \not\leq^{gb} w_a$ and $w_a \not\leq^{gb} w_b$. Therefore, $w_b \sim^{gb} w_a$.

I will show that $\sim^{gb}$ is transitive. Let $w_a \sim^{gb} w_b$ and $w_b \sim^{gb} w_c$. Then $w_a \not\leq^{gb} w_b$, $w_b \not\leq^{gb} w_a$, $w_b \not\leq^{gb} w_c$, and $w_c \not\leq^{gb} w_b$. From modularity of $\leq^{gb}$, $w_a \not\leq^{gb} w_c$, and $w_c \not\leq^{gb} w_a$. It follows that $w_a \sim^{gb} w_c$. □

Proof of Proposition 3.9. 1. is obvious from Proposition 2.1 that $\mathcal{P}$ is the set of irreflexive and transitive relation over W. 2. is obvious from the fact that $\mathcal{GB}$ is the set of modular versions of GP. I will show $\mathcal{P} \cap \mathcal{GB} = \mathcal{B}_<$. At first, we will show $\mathcal{P} \cap \mathcal{GB} \subseteq \mathcal{B}_<$. Suppose that $\leq \in \mathcal{P} \cap \mathcal{GB}$. From $\leq \in \mathcal{P}$, it is irreflexive. Let $\leq' = \leq \cup \{(w_a, w_b) | w_a \sim w_b\}$. I must show $\leq' \in B$.

I will show that $\leq'$ is reflexive. From irreflexivity, $w \not\leq w$ is obvious. Therefore, $w \sim w$, and then $w \leq' w$.

I will show that $\leq'$ is total. Suppose that for some $(w_a, w_b) \in \mathcal{W} \times \mathcal{W}$, $w_a \not\leq' w_b \wedge w_b \not\leq' w_a$. From the supposition, $w_a \not\leq w_b \wedge w_b \not\leq w_a$ and $w_a \not\sim w_b \wedge w_b \not\sim w_a$. it contradicts the definition of $\sim$. I will show that $\leq'$ is transitive. Suppose that $w_a \leq' w_b$ and $w_b \leq' w_c$. If $w_a \leq w_b$ and $w_b \leq w_c$, then $w_a \leq w_c$, and $w_a \leq' w_c$. If $w_a \sim w_b$ and $w_b \leq w_c$, then $w_b \leq w_a$ or $w_a \leq w_c$ from modularity, and $w_a \leq w_c$ from $w_a \sim w_b$. If $w_a \leq w_b$ and $w_b \sim w_c$, then $w_a \leq w_c$ as well. If $w_a \sim w_b$ and $w_b \sim w_c$, then from Proposition 3.6, $\sim$ is transitive, and $w_a \sim w_c$. Therefore, $w_a \leq' w_c$.

I will show that $\leq$ is a strict version of $\leq'$. Suppse that $w_a \leq w_b$. From the definition of $\leq'$, $w_a \leq' w_b$. Because $\leq$ is the strict version of a transitive relation, it is acyclic from Proposition 2.1. Thus, $w_b \not\leq w_a$. From the definition of $\leq'$, $w_b \not\leq' w_a$. Suppose that $w_a \leq' w_b \wedge w_b \not\leq' w_a$. From the definition of $\leq'$ and $w_a \leq' w_b$, $w_a \leq w_b$ or $w_a \sim w_b$. Suppose that $w_a \sim w_b$. From Proposition 3.6, $\sim$ is symmetric, and then $w_b \sim w_a$. Therefore, from the definition of $\leq'$, $w_b \leq' w_a$. It contradicts the supposition $w_b \not\leq' w_a$. Then $w_a \leq w_b$. It follows that $\leq$ is a strict version of $\leq'$, and $\leq \in B_<$.

Secondly, I will show $\mathcal{P} \cap \mathcal{GB} \supseteq \mathcal{B}_<$. Suppose that $\leq \in \mathcal{B}_<$. It is obvious that $\leq$ is acyclic. From $\leq \in \mathcal{B}_<$, there is some $\leq' \in \mathcal{B}$ such that $\leq$ is the strict version of $\leq'$. At first, I will show $\mathcal{GB} \supseteq \mathcal{B}_<$. I will show the transitivity. Suppose that $w_a \leq w_b$ and $w_b \leq w_c$. Then $w_a \leq' w_b$, $w_b \not\leq' w_a$, $w_b \leq' w_c$, and $w_c \not\leq' w_b$. From transitivity, $w_a \leq' w_c$. Suppose that $w_c \leq' w_a$. Thus, from transitivty, $w_c \leq' w_b$, and it contradicts $w_c \not\leq' w_b$. Therefore, $w_c \not\leq' w_a$. It follows that $w_a \leq w_c$. We will show the modularity. Suppose that $w_a \leq w_b$. Thus, $w_a \leq' w_b$ and $w_b \not\leq' w_a$. $\leq'$ is total, therefore $w_a \leq' w_c$ or $w_c \leq' w_a$, and $w_b \leq' w_c$ or $w_c \leq' w_b$.

1. Suppose that $w_a \leq' w_c$ and $w_b \leq' w_c$. From transitivity, if $w_c \leq' w_a$, then $w_b \leq' w_a$ and it contradicts the supposition. It follows that $w_c \not\leq' w_a$. And then $w_a \leq w_c$.

2. Suppose that $w_c \leq' w_a$ and $w_b \leq' w_c$. From transitivity, $w_b \leq' w_a$. It contradicts the supposition.

3. Suppose that $w_a \leq' w_c$ and $w_c \leq' w_b$. From transitivity, if $w_c \leq' w_a$, then $w_b \not\leq' w_c$. Therefore, $w_a \leq w_c \vee w_c \leq w_b$.

4. Suppose that $w_c \leq' w_a$ and $w_c \leq' w_b$. From transitivity, if $w_b \leq' w_c$, then $w_b \leq' w_a$, and it contradicts $w_b \not\leq' w_a$. Therefore, $w_b \not\leq' w_c$, and then $w_c \leq w_b$.

For every case, $w_a \leq w_c \vee w_c \leq w_b$. Therefore, $\leq$ is modular.

I will show $P \supseteq B_<$. Transitivity is already shown. Therefore, it suffices to show irreflexivity from Proposition 2.1. Because $\leq$ is the strict of $\leq'$, $w \not\leq w$. $\square$

Proof of 3.10. Let N be the number of all elements in $\leq^p_B$. Suppose that $(w_{a_1}, w_{b_1}), (w_{a_2}, w_{b_2}), \cdots (w_{a_N}, w_{b_N}$ is an arbitrary sequence of elements in $\leq^p_B$. I define the upward permutation $\leq^p_A = \leq_0 \subseteq \leq_1 \subseteq \leq_2 \subseteq \cdots \subseteq \leq^p_N$ as follows.

$$\leq_{i+1} = \begin{cases} \leq_i & \text{if} w_{a_{i+1}} \not\sim^p_A w_{b_{i+1}} \text{or} \\ & w_{b_{i+1}} \leq^+_i w_{a_{i+1}}; \\ \leq_i \cup (w_{a_{i+1}}, w_{b_{i+1}}) & \text{Otherwise.} \end{cases}$$

By the definition, it is obvious that for any $\leq_i$ $(0 \leq^{\mathcal{N}} i \leq^{\mathcal{N}} N)$[1], $(w_{b_i}, w_{a_i}) \notin \leq^+_i$. Now, it suffices to show that $\leq = \leq_N \in PRF(\leq^p_A, \leq^p_B)$. That is to say, I will show that for any $(w'_a, w'_b) \in \mathcal{W} \times \mathcal{W}$, $(w'_a, w'_b) \in \leq \Leftrightarrow (w'_a, w'_b) \in \leq^p_A \cup \{(w_a, w_b) : w_a \sim^p_A w_b \wedge w_a \leq^p_B w_b \wedge (w_b, w_a) \notin \leq^+\}$.

($\Rightarrow$) Let $(w'_a, w'_b) \in \leq$. Then there is some minimum $i (0 \leq^{\mathcal{N}} i \leq^{\mathcal{N}} N)$ such that $(w'_a, w'_b) = (w_{a_i}, w_{b_i}) \in \leq_i$. If $i = 0$, then $(w'_a, w'_b) \in \leq^p_A$, and it is okey. If $i >^{\mathcal{N}} 0$, then it is obvious from the definition that $w'_a \sim^p_A w'_b$ and $w_a \leq^p_B w'_b$. Assume that $(w'_b, w'_a) \in \leq^+$. Because $\leq^+$ is constructed by finite steps, there is some minimum k $(i <^{\mathcal{N}} k \leq^{\mathcal{N}} N)$ such that $(w'_b, w'_a) \in \leq^+_k$. That is to say, I can show $(w'_b, w'_a) \in \leq^+$ with all elements in $\leq_k$. It follows that I can show the following sequence from $\leq_k$.

$$w'_b \cdots w_{a_k}, w_{b_k} \cdots w'_a$$

By $(w'_a, w'_b) \in \leq_i \subseteq \leq_k$, I can construct the following sequence from $\leq_k$.

$$w_{b_k} \cdots w'_a, w'_b \cdots w_{a_k}$$

It contradicts $(w_{b_k}, w_{a_k}) \notin \leq^+_k$. Therefore, $(w'_b, w'_a) \notin \leq^+$.

($\Leftarrow$) If $(w'_a, w'_b) \in \leq^p_A$, then $\leq^p_A = \leq^p_0$, and it is okey. Suppose that $(w'_a, w'_b) \in \{(w_a, w_b) : w_a \sim^p_A w_b \wedge w_a \leq^p_B w_b \wedge (w_b, w_a) \notin \leq^+\}$. By $w'_a \leq^p_B w'_b$, there is some $i \geq^{\mathcal{N}} 0$ such that $(w'_a, w'_b) = (w_{a_{i+1}}, w_{b_{i+1}}) \in \leq^p_B$. By $w'_a \sim^p_A w'_b$, $w_{a_{i+1}} \sim^p_A w_{b_{i+1}}$. If $w_{b_{i+1}} \leq^+_i w_{a_{i+1}}$, then $(w_{a_{i+1}}, w_{b_{i+1}}) \in \leq^+$, and it contradict the supposition. Then $(w_{a_{i+1}}, w_{b_{i+1}}) \in \leq_{i+1} \subseteq \leq$. $\square$

Proof of Proposition 3.16. I will show that for $1 \leq^{\mathcal{N}} n \leq^{\mathcal{N}} N$, $((\leq^b_1 \otimes^b \leq^b_2) \otimes^b \cdots \leq^b_n) = Ref((<^b_1 \otimes^b_< <^b_2) \otimes^b_< \cdots <^b_n)$. For the case $n = 1$, $\leq^b_1 = Ref(<^b_1)$ is obvious from Proposition 3.15. For the case $N >^{\mathcal{N}} 1$, suppose that $((\leq^b_1 \otimes^b \leq^b_2) \otimes^b \cdots \leq^b_{n-1}) = Ref((<^b_1 \otimes^b_< <^b_2) \otimes^b_< \cdots <^b_{n-1})$. Let $\leq^b = ((\leq^b_1 \otimes^b \leq^b_2) \otimes^b \cdots \leq^b_{n-1})$ and $<^b = ((<^b_1 \otimes^b_< <^b_2) \otimes^b_< \cdots <^b_{n-1})$. I will show that $\leq^b \otimes^b \leq^b_n = Ref(<^b \otimes^b_< <^b_n)$.

('$\subseteq$') Suppose that $(w, w') \in \leq^b \otimes^b \leq^b_n = \{(w_a, w_b) : w_a <^b w_b \vee (w_a \infty^b w_b \wedge w_a \leq^b_n w_b)\}$. If $w <^b w'$, then $(w, w') \in <^b \otimes^b_< <^b_n$, and then $(w, w') \in Ref(<^b \otimes^b_< <^b_n)$. Also suppose

[1] Note that $\leq^{\mathcal{N}}$ means the ordering over natural numbers.

that $w\infty^b w'$ and $w \leq^b_n w'$. From $w\infty^b w'$, neither $w <^b w'$ nor $w' <^b w$. Therefore, if $w' \not\leq^b_n w$, then $w <^b_n w'$, and hence $(w,w') \in <^b \otimes {}^b_< <^b_n$. It follows that $(w,w') \in Ref(<^b \otimes {}^b_< <^b_n)$. If $w' \leq^b_n w$, $(w,w') \notin <^b \otimes {}^b_< <^b_n$. Suppose that $(w',w) \in <^b \otimes {}^b_< <^b_n$. Then $w' <^b w$, or $w' \sim^{sb} w$ and $w' <^b_n w$. In either case, it contradicts the supposition $w\infty^b w'$ and $w \leq^b_n w'$. Therefore, $(w',w) \notin <^b \otimes {}^b_< <^b_n$. Then $(w,w'),(w',w) \notin <^b \otimes {}^b_< <^b_n$. It follows that $(w,w') \in Ref(<^b \otimes {}^b_< <^b_n)$.

('$\supseteq$') Suppose that $(w,w') \in Ref(<^b \otimes {}^b_< <_n)$. Also suppose that $(w,w') \in <^b \otimes {}^b_< <^b_n$. If $w <^b w'$, then $(w,w') \in \leq^b \otimes {}^b \leq^b_n$. Suppose that $w \sim^{sb} w'$ and $w <^b_n w'$. From $w \sim^{sb} w'$, neither $w <^b w'$ nor $w' <^b w$. Hence $w \not\leq^b w'$ or $w' \leq^b w$, and $w' \not\leq^b w$ or $w \leq^b w'$. Because $\leq^b$ is a total preorder, $w' \leq^b w$ and $w \leq^b w'$. It follows that $w\infty^b w'$. From $w <^b_n w'$, $w \leq^b_n w'$. Therefore, $(w,w') \in \leq^b \otimes {}^b \leq^b_n$. Suppose that $(w,w'),(w'w) \notin <^b \otimes {}^b_< <_n$. Then $w \not<^b w'$, and $w \not\sim^{sb} w'$ or $w \not<^b_n w'$ from $(w,w') \notin <^b \otimes {}^b_< <_n$. In addition, $w' \not<^b w$, and $w' \not\sim^{sb} w$ or $w' \not<^b_n w$ from $(w',w) \notin <^b \otimes {}^b_< <_n$. If $w \not\sim^{sb} w'$ (or $w' \not\sim^{sb} w$), then either $w <^b w'$ or $w <^b w'$, and it contradicts $w \not<^b w'$ and $w' \not<^b w$. Therefore, I consider the case $w \not<^b w'$, $w' \not<^b w$, $w \not<^b_n w'$. and $w' \not<^b_n w$. From $w \not<^b w'$ and $w' \not<^b w$, $w\infty^b w'$. From $w \not<^b_n w'$ and $w' \not<^b_n w$, $w\infty^b_n w'$, and $w \leq^b_n w'$. Therefore, $(w,w') \in \leq^b \otimes {}^b \leq^b_n$. □

Proof of Proposition 3.17. For the case 1., see [29]. Let me consider the case 2. From the case 1.,$\leq^b_A \otimes {}^b \leq^b_B \in \mathcal{B}$. From Proposition 3.15, $\leq^b_A \otimes {}^b \leq^b_B = Ref(<^b_A \otimes {}^b_< <^b_B)$. From Proposition 3.17, Ref is a bijection from $\mathcal{B}_<$ to $\mathcal{B}$. Therefore, $<^b_A \otimes {}^b_< <^b_B \in \mathcal{B}_<$.

Let me consider the case 3. Suppose that $\leq^{gb}_A, \leq^{gb}_B \in \mathcal{GB}$. I will show the modularity. Suppose that $(w,w') \in \leq^{gb}_A \otimes {}^{gb} \leq^{gb}_B$ and $(w,w'') \notin \leq^{gb}_A \otimes {}^{gb} \leq^{gb}_B$. Then $w \leq^{gb}_A w'$, or $w \sim^{gb}_A w'$ and $w \leq^{gb}_B w'$. In addition, $w \not\leq^{gb}_A w''$, and $w \not\sim^{gb}_A w''$ or $w \not\leq^{gb}_B w''$.

1. The case of $w \leq^{gb}_A w'$, $w \not\leq^{gb}_A w''$, and $w \not\sim^{gb}_A w''$. From $w \not\leq^{gb}_A w''$ and $w \not\sim^{gb}_A w''$, $w'' \leq^{gb}_A w$. From transitivity, $w'' \leq^{gb}_A w'$. It follows that $(w'',w') \in \leq^{gb}_A \otimes {}^{gb} \leq^{gb}_B$.

2. The case of $w \sim^{gb}_A w'$, $w \leq^{gb}_B w'$, $w \not\leq^{gb}_A w''$, and $w \not\sim^{gb}_A w''$. From $w \sim^{gb}_A w'$, neither $w \leq^{gb}_A w'$ nor $w' \leq^{gb}_A w$. From $w \not\leq^{gb}_A w''$ and $w \not\sim^{gb}_A w''$, $w'' \leq^{gb}_A w$. From $w'' \leq^{gb}_A w$, $w' \not\leq^{gb}_A w$ and modularity, $w'' \leq^{gb}_A w'$. It follows that $(w'',w') \in \leq^{gb}_A \otimes {}^{gb} \leq^{gb}_B$.

3. The case of $w \leq^{gb}_A w'$, $w \not\leq^{gb}_A w''$, $w \not\leq^{gb}_B w''$. From $w \not\leq^{gb}_A w''$, $w \leq^{gb}_A w'$ and modularity, $w'' \leq^{gb}_A w'$. It follows that $(w'',w') \in \leq^{gb}_A \otimes {}^{gb} \leq^{gb}_B$.

4. The case of $w \sim^{gb}_A w'$, $w \leq^{gb}_B w'$, $w \not\leq^{gb}_A w''$, and $w \not\leq^{gb}_B w''$. Suppose that $w'' \leq^{gb}_A w$. Then $w'' \leq^{gb}_A w'$ from $w' \not\leq^{gb}_A w$ and modularity. It follows that $(w'',w') \in \leq^{gb}_A \otimes {}^{gb} \leq^{gb}_B$. Suppose that $w'' \not\leq^{gb}_A w$. Then $w \sim^{gb}_A w''$. From Proposition 3.7,$w' \sim^{gb}_A w''$. From $w \leq^{gb}_B w'$, $w \not\leq^{gb}_B w''$, and modularity, $w'' \leq^{gb}_B w'$. It follows that $(w'',w') \in \leq^{gb}_A \otimes {}^{gb} \leq^{gb}_B$.

For every case, $(w'',w') \in \leq^{gb}_A \otimes {}^{gb} \leq^{gb}_B$.

I will show the transitivity. Suppose that $(w,w') \in \leq^{gb}_A \otimes {}^{gb} \leq^{gb}_B$ and $(w',w'') \in \leq^{gb}_A \otimes {}^{gb} \leq^{gb}_B$. From modurality, $(w'',w') \in \leq^{gb}_A \otimes {}^{gb} \leq^{gb}_B$ or $(w,w'') \in \leq^{gb}_A \otimes {}^{gb} \leq^{gb}_B$, and $(w,w'') \in \leq^{gb}_A \otimes {}^{gb} \leq^{gb}_B$ or $(w',w) \in \leq^{gb}_A \otimes {}^{gb} \leq^{gb}_B$. Suppose that $(w'',w'),(w',w) \in \leq^{gb}_A \otimes {}^{gb} \leq^{gb}_B$. Then $(w',w''),(w'',w'),(w,w'),(w',w) \in \leq^{gb}_A \otimes {}^{gb} \leq^{gb}_B$. From the definition of $\leq^{gb}_A \otimes {}^{gb} \leq^{GB}_B$, $(w',w''),(w'',w'),(w,w'),(w',w) \in \leq^{gb}_A$ or $(w',w''),(w'',w'),(w,w'),(w',w) \notin \leq^{gb}_A$ and (w',w''),

$(w'', w'), (w, w'), (w', w) \in \leq_B^{gb}$. For the first case, $(w, w'') \in \leq_A^{gb}$ from transitivity. For the second case, $(w, w'') \notin \leq_A^{gb}$ from modularity, and $(w, w'') \in \leq_B^{gb}$ from transitivity. Therefore, $(w, w'') \in \leq_A^{gb} \oslash {}^{gb}\leq_B^{GB}$. $\Box$

A.3 Proofs of Chapter 4

Proof of Proposition 4.5. For the case 1. and 2., it suffices to show the following properties.

- For any transitive relation $\leq_A, \leq_B \in \mathcal{R}$, $\leq_A \cap \leq_B$ is tranisitive.
- For any irreflexive relation $\leq_A, \leq_B \in \mathcal{R}$, $\leq_A \cap \leq_B$ is irreflexive.

For the case of transitivity, Let $\leq_A, \leq_B \in \mathcal{R}$ be transitive. Suppose that $(w_a, w_b) \in \leq_A \cap \leq_B$ and $(w_b, w_c) \in \leq_A \cap \leq_B$. Then $w_a \leq_A w_b$ and $w_a \leq_B w_b$, and $w_b \leq_A w_c$ and $w_b \leq_B w_c$. From tranisitivity, $w_a \leq_A w_c$ and $w_a \leq_B w_c$. Therefore, $(w_a, w_c) \in \leq_A \cap \leq_B$.

For the case of irreflexivity, Let $\leq_A, \leq_B \in \mathcal{R}$ be irreflexive. Then neither $w \leq_A w$ nor $w \leq_B w$. Then $(w, w) \notin \leq_A \cap \leq_B$.

For the case 3. and 4., Let $S = \{s_1, s_2\}$, $\leq_{s_1} = \{(a,a), (a,b), (b,b)\}$ and $\leq_{s_2} = \{(a,a), (b,a), (b,b)\}$. Then $AIn(S) = \{(a,a), (b,b)\}$. However, $(a,b), (b,a) \notin AIn(S)$, and it is neither total nor modular. $\Box$

Proof of Lemma 4.1. I will show that for $1 \leq i \leq N$,

$$\begin{array}{l}((\leq_{s_1} \oslash \leq_{s_2}) \oslash \ ... \ \leq_{s_n}) = \\ \{(w_a, w_b) : \exists s \in S \backslash \{s_{i+1}, ..., s_N\}.w_a \leq_s w_b \wedge (\forall s' \sqsupset s \in S \backslash \{s_{i+1}, ..., s_N\}.w_a \sim_{s'} w_b)\}\end{array}.$$

The base case $i = 1$ obviously holds. For the case $i >^{\mathcal{N}} 1$, let

$$\begin{array}{l}((\leq_{s_1} \oslash \leq_{s_2}) \oslash \ ... \ \leq_{s_{i-1}}) = \\ \{(w_a, w_b) : \exists s \in S \backslash \{s_i, ..., s_N\}.w_a \leq_s w_b \wedge (\forall s' \sqsupset s \in S \backslash \{s_i, ..., s_N\}.w_a \sim_{s'} w_b)\}\end{array}.$$

Then it suffices to show

$$\begin{array}{l}((\leq_{s_1} \oslash \leq_{s_2}) \oslash \ ... \ \leq_{s_i}) = \\ \{(w_a, w_b) : \exists s \in S \backslash \{s_{i+1}, ..., s_N\}.w_a \leq_s w_b \wedge (\forall s' \sqsupset s \in S \backslash \{s_{i+1}, ..., s_N\}.w_a \sim_{s'} w_b)\}\end{array}.$$

('$\subseteq$') Let $(w_a, w_b) \in ((\leq_{s_1} \oslash \leq_{s_2}) \oslash \ ... \ \leq_{s_i})$. Then $(w_a, w_b) \in ((\leq_{s_1} \oslash \leq_{s_2}) \oslash \ ... \ \leq_{s_{i-1}})$ and $(w_b, w_a) \notin ((\leq_{s_1} \oslash \leq_{s_2}) \oslash \ ... \ \leq_{s_{i-1}})$, or $(w_a, w_b) \notin ((\leq_{s_1} \oslash \leq_{s_2}) \oslash \ ... \ \leq_{s_{i-1}})$, $(w_b, w_a) \notin ((\leq_{s_1} \oslash \leq_{s_2}) \oslash \ ... \ \leq_{s_{i-1}})$, and $w_a \leq_{s_i} w_b$. For the first case, For some $j <^{\mathcal{N}} i$, $w_a \leq_{s_j} w_b$ and for all $s' \sqsupset s_j \in S \backslash \{s_{i+1}, ..., s_N\}.w_a \sim_{s'} w_b)$ from the assumption. For the second case, It suffices to show that for all $s' \sqsupset s_i \in S \backslash \{s_{i+1}, ..., s_N\}.w_a \sim_{s'} w_b)$. Suppose that for some least $s_j \sqsupset s_i \in S \backslash \{s_{i+1}, ..., s_N\}.w_a \leq_{s_j} w_b \vee w_b \leq_{s_j} w_a)$. If $w_a \leq_{s_j} w_b$, then for all $s' \sqsupset s_j \in S \backslash \{s_{j+1}, ..., s_N\}.w_a \sim_{s'} w_b$, because s_j is a least element such that

$w_a \leq_{s_j} w_b \vee w_b \leq_{s_j} w_a$. Then $(w_a, w_b) \in ((\leq_{s_1} \varolessthan \leq_{s_2}) \varolessthan \ldots \leq_{s_{i-1}})$, and it contradicts the supposition $(w_a, w_b) \notin ((\leq_{s_1} \varolessthan \leq_{s_2}) \varolessthan \ldots \leq_{s_{i-1}})$. In the similar way, I can show that $w_b \leq_{s_j} w_a$ is not possible. Therefore, for all $s' \sqsupset s_i \in S \backslash \{s_{i+1}, ..., s_N\}.w_a \sim_{s'} w_b)$.

('$\supseteq$') Let $(w_a, w_b) \in \{(w_a, w_b) : \exists s \in S \backslash \{s_{i+1}, ..., s_N\}.w_a \leq_s w_b \wedge (\forall s' \sqsupset s \in S \backslash \{s_{i+1}, ..., s_N\}.w_a \sim_{s'} w_b)\}$. If for some $k <^{\mathcal{N}} i$, $w_a \leq_{s_k} w_b \wedge (\forall s' \sqsupset s \in S \backslash \{s_{i+1}, ..., s_N\}.w_a \sim_{s'} w_b)$, then $(w_a, w_b) \in ((\leq_{s_1} \varolessthan \leq_{s_2}) \varolessthan \ldots \leq_{s_{i-1}})$ from the assumption. $(w_a, w_b) \notin ((\leq_{s_1} \varolessthan \leq_{s_2}) \varolessthan \ldots \leq_{s_{i-1}})$ is derived from the definition. Hence $(w_a, w_b) \in ((\leq_{s_1} \varolessthan \leq_{s_2}) \varolessthan \ldots \leq_{s_i})$. Suppose that $w_a \leq_{s_i} w_b \wedge (\forall s' \sqsupset s \in S \backslash \{s_{i+1}, ..., s_N\}.w_a \sim_{s'} w_b)$. Then $(w_a, w_b) \notin ((\leq_{s_1} \varolessthan \leq_{s_2}) \varolessthan \ldots \leq_{s_{i-1}})$ from the definition. Therefore, $(w_a, w_b) \in ((\leq_{s_1} \varolessthan \leq_{s_2}) \varolessthan \ldots \leq_{s_i})$. $\square$

Proof of Lemma 4.3. I will show that for $1 \leq^{\mathcal{N}} i \leq^{\mathcal{N}} N$, $O^{p,i}(S) = ((\leq^p_{s_1} \varolessthan^p \leq^p_{s_2}) \varolessthan^p \ldots \leq^p_{s_i})$. If $i = 1$, then $O^{p,1}(S) = \leq^p_{s_1}$ is obvious. If $i >^{\mathcal{N}} 1$, suppose $O^{p,i-1}(S) = ((\leq^p_{s_1} \varolessthan^p \leq^p_{s_2}) \varolessthan^p \ldots \leq^p_{s_i-1})$. If you want to show $O^{p,i}(S) = ((\leq^p_{s_1} \varolessthan^p \leq^p_{s_2}) \varolessthan^p \ldots \leq^p_{s_i})$, then it suffices to show that $SO^{p,i}(S) = PRF(((\leq^p_{s_1} \varolessthan^p \leq^p_{s_2}) \varolessthan^p \ldots \leq^p_{s_i-1}), \leq^p_{s_i})$. That is, it suffices to show that for any $\leq \in \mathcal{R}$, $\leq \in SO^{p,i}(S)$ iff $\leq^p \in PRF(((\leq^p_{s_1} \varolessthan^p \leq^p_{s_2}) \varolessthan^p \ldots \leq^p_{s_i-1}), \leq^p_{s_i})$. Then,

$$
\begin{array}{ll}
\leq \in SO^{p,i}(S) & \Leftrightarrow \\
\leq = O^{p,i-1}(S) \cup & \\
\quad \{(w_a, w_b) : (w_b, w_a) \notin O^{p,i-1}(S) \wedge & \\
\qquad\qquad w_a \leq^p_{s_i} w_b \in \Phi_S \wedge (w_b, w_a) \notin \leq^+\} & \Leftrightarrow \\
\leq = ((\leq^p_{s_1} \varolessthan^p \leq^p_{s_2}) \varolessthan^p \ldots \leq^p_{s_n-1}) \cup & \\
\quad \{(w_a, w_b) : (w_b, w_a) \notin ((\leq^p_{s_1} \varolessthan^p \leq^p_{s_2}) \varolessthan^p \ldots \leq^p_{s_i-1}) \wedge & \\
\qquad\qquad w_a \leq^p_{s_i} w_b \wedge (w_b, w_a) \notin \leq^+\} & \Leftrightarrow \\
r \in PRF(((\leq^p_{s_1} \varolessthan^p \leq^p_{s_2}) \varolessthan^p \ldots \leq^p_{s_n-1}), \leq^p_{s_n}) &
\end{array}
$$

$\square$

Proof of Lemma 4.4. I will show that for $1 \leq^{\mathcal{N}} i \leq^{\mathcal{N}} N$, $O^{p,i}_{sim}(S) = ((\leq^p_{s_1} \bar{\varolessthan}^p \leq^p_{s_2}) \bar{\varolessthan}^p \ldots \leq^p_{s_i})$. If $i = 1$, then $O^{p,1}_{sim}(S) = \leq^p_{s_1}$ is obvious. If $i >^{\mathcal{N}} 1$, suppose $O^{p,i-1}_{sim}(S) = ((\leq^p_{s_1} \bar{\varolessthan}^p \leq^p_{s_2}) \bar{\varolessthan}^p \ldots \leq^p_{s_i-1})$. If I want to show $O^{p,i}_{sim}(S) = ((\leq^p_{s_1} \bar{\varolessthan}^p \leq^p_{s_2}) \bar{\varolessthan}^p \ldots \leq^p_{s_i})$, then it suffices to show that $TO^{p,i}_{sim}(S) = ((\leq^p_{s_1} \bar{\varolessthan}^p \leq^p_{s_2}) \bar{\varolessthan}^p \ldots \leq^p_{s_i-1}) \widehat{\varolessthan}^p \leq^p_{s_i}$. Then,

$$
\begin{array}{rcl}
TO^{p,i}_{sim}(S) & = & O^{p,i-1}_{sim}(S) \cup \{(w_a, w_b) : (w_b, w_a) \notin O^{p,i-1}_{sim}(S) \wedge w_a \leq^p_{s_i} w_b\} \\
& = & ((\leq^p_{s_1} \bar{\varolessthan}^p \leq^p_{s_2}) \bar{\varolessthan}^p \ldots \leq^p_{s_n-1}) \cup \\
& & \{(w_a, w_b) : (w_b, w_a) \notin ((\leq^p_{s_1} \bar{\varolessthan}^p \leq^p_{s_2}) \bar{\varolessthan}^p \ldots \leq^p_{s_i-1}) \wedge w_a \leq^p_{s_i} w_b\} \\
& = & ((\leq^p_{s_1} \varolessthan^p \leq^p_{s_2}) \varolessthan^p \ldots \leq^p_{s_n-1}) \widehat{\varolessthan}^p \leq^p_{s_n}
\end{array}
$$

$\square$

Proof of Lemma 4.6. Let $\mathcal{R}_{\mathcal{ANK}}{}^S = \{r_1, ..., r_N\}$. I will show that for $1 \leq i \leq N$,

$$
\begin{array}{l}
((AUn(S_1) \varolessthan AUn(S_2)) \varolessthan \ldots AUn(S_i)) = \\
\{(w_a, w_b) : \exists r \in \mathcal{R}_{\mathcal{ANK}} \backslash \{r_{i+1}, ..., r_N\}.w_a \leq_r w_b \wedge (\forall r' > r \in \mathcal{R}_{\mathcal{ANK}} \backslash \{r_{i+1}, ..., r_N\}.w_a \sim_{r'} w_b)\}
\end{array} .
$$

. The base case $i = 1$ obviously holds. For the case $i >^{\mathcal{N}} 1$, let

$$((AUn(S_1) \otimes AUn(S_2)) \otimes \ldots AUn(S_{i-1})) = \\ \{(w_a, w_b) : \exists r \in \mathcal{R}_{\mathcal{ANK}} \backslash \{r_i, \ldots, r_N\}.w_a \leq_r w_b \wedge (\forall r' > r \in \mathcal{R}_{\mathcal{ANK}} \backslash \{r_i, \ldots, r_N\}.w_a \sim_{r'} w_b)\}.$$

Then it suffices to show $((AUn(S_1) \otimes AUn(S_2)) \otimes \ldots AUn(S_i)) =$

$\{(w_a, w_b) : \exists r \in \mathcal{R}_{\mathcal{ANK}} \backslash \{r_{i+1}, \ldots, r_N\}.w_a \leq_r w_b \wedge (\forall r' > r \in \mathcal{R}_{\mathcal{ANK}} \backslash \{r_{i+1}, \ldots, r_N\}.w_a \sim_{r'} w_b)\}$.

('$\subseteq$') Let $(w_a, w_b) \in ((AUn(S_1) \otimes AUn(S_2)) \otimes \ldots AUn(S_i))$. Then $(w_a, w_b) \in ((AUn(S_1) \otimes AUn(S_2)) \otimes \ldots AUn(S_{i-1}))$, or $(w_a, w_b) \notin ((AUn(S_1) \otimes A\ Un(S_2)) \otimes \ldots AUn(S_{i-1}))$, $(w_b, w_a) \notin ((AUn(S_1) \otimes AUn(S_2)) \otimes \ldots AUn(S_{i-1}))$, and $w_a \leq_{r_i} w_b$. For the first case, For some $j <^{\mathcal{N}} i$, $w_a \leq_{r_j} w_b$ and for all $r' > r_j \in \mathcal{R}_{\mathcal{ANK}} \backslash \{r_{i+1}, \ldots, r_N\}.w_a \sim_{r'} w_b)$ from the assumption. For the second case, It suffices to show that for all $r' > r_i \in \mathcal{R}_{\mathcal{ANK}} \backslash \{r_{i+1}, \ldots, r_N\}.w_a \sim_{r'} w_b$. Suppose that for some least $r_j > r_i \in \mathcal{R}_{\mathcal{ANK}} \backslash \{r_{i+1}, \ldots, r_N\}.w_a \leq_{r_j} w_b \vee w_b \leq_{r_j} w_a)$. If $w_a \leq_{r_j} w_b$, then for all $r' > r_j \in \mathcal{R}_{\mathcal{ANK}} \backslash \{r_{j+1}, \ldots, r_N\}.w_a \sim_{r'} w_b$, because r_j is a least element such that $w_a \leq_{r_j} w_b \vee w_b \leq_{r_j} w_a$. Then $(w_a, w_b) \in ((AUn(S_1) \otimes AUn(S_2)) \otimes \ldots AUn(S_{i-1}))$, and it contradicts the supposition $(w_a, w_b) \notin ((AUn(S_1) \otimes AUn(S_2)) \otimes \ldots AUn(S_{i-1}))$. In the similar way, I can show that $w_b \leq_{r_j} w_a$ is not possible. Therefore, for all $r' > r_i \in \mathcal{R}_{\mathcal{ANK}} \backslash \{r_{i+1}, \ldots, r_N\}.w_a \sim_{r'} w_b)$.

('$\supseteq$') Let $(w_a, w_b) \in \{(w_a, w_b) : \exists r \in R_{ANK} \backslash \{r_{i+1}, \ldots, r_N\}.w_a \leq_r w_b \wedge (\forall r' > r \in \mathcal{R}_{\mathcal{ANK}} \backslash \{r_{i+1}, \ldots, r_N\}.w_a \sim_{r'} w_b)\}$. If for some $k <^{\mathcal{N}} i$, $w_a \leq_{r_k} w_b \wedge (\forall r' \sqsupset r \in \mathcal{R}_{\mathcal{ANK}} \backslash \{r_{i+1}, \ldots, r_N\}.w_a \sim_{r'} w_b)$, then $(w_a, w_b) \in ((AUn(S_1) \otimes AUn(S_2)) \otimes \ldots A\ Un(S_{i-1}))$ from the assupmption. Hence $(w_a, w_b) \in ((AUn(S_1) \otimes AUn(S_2)) \otimes \ldots AUn(S_i))$. Suppose that $w_a <_{r_i} w_b \wedge (\forall r' \sqsupset s \in \mathcal{R}_{\mathcal{ANK}} \backslash \{r_{i+1}, \ldots, r_N\}.w_a \sim_{r'} w_b)$. Then $(w_a, w_b) \in ((AUn(S_1) \otimes AUn(S_2)) \otimes \ldots AUn(S_i))$ is obvious. $\square$

A.4 Proofs of Chapter 5

Proof of Proposition 5.2. Now suppose that for all $s \in S$, if $r = rank(s)$, then $x \not<^b_s y$, or, for some $s' \sqsupset s \in S$, $x \not\approx^b_{s'} y$. Suppose that $x \preceq^b y$. Then there is some $S_{mpc} \in MPC(S)$ such that $(x, y) \in ARf^b_<(S_{mpc})$, because if there is no $S_{mpc} \in MPC(S)$ such that $(x, y) \in ARf^b_<(S_{mpc})$, then $(x, y) \notin ARf^b_<(S)$, and it contradicts $x \preceq^b y$. Suppose that $r = rank(s)$. From the supposition, if $x <^b_s y$, then for some $s' \sqsupset s \in S$, $x <^b_{s'} y$ or $y <^b_{s'} x$. For the case $x <^b_{s'} y$, we can say $rank(s') > rank(s)$, and then $r \notin l^b((x, y))$. For the case $y <^b_{s'} x$, there is some $r' \in l((y, x))$ such that $r' \geq rank(s')$, and then, even if $r \in l^b((x, y))$, $r' > r$. Therefore, $x \not\preceq^{b,A}_r y$.

Now suppose there exists $s \in S$ such that $x <^b_s y$, $r = rank(s)$, and, for every $s' \sqsupset s \in S$, $x \approx^b_{s'} y$. Then $x \preceq^b y$. Moreover, since for every $s' \in S$, $x <^b_{s'} y$ or $x <^b_{s'} y$ implies $s \sqsupseteq s'$ which implies $rank(s) \geq rank(s')$,

$$l((x,y)) = max\{rank(s') : x <^b_{s'} y, s' \in S\} \supseteq \{rank(s)\} = \{r\}.$$

and if $r' \in l((y,x))$, $r' \not> r$. Therefore, $x \preceq^{b,A}_r y$. $\Box$

Proof of Proposition 5.4. I will show only-if-part. Suppose that $x \preceq^{p,A}_r y$. Then $x \preceq^p y$ and $r \in l((x,y))$. By Definition 4.6, there is some $s \in S$ such that $r = rank(s) \land x \leq^s y \land (\forall S_{eq} \in EQ(S).s \in S_{eq} \Rightarrow (x,y) \in AIn(S_{eq})$.

We will show if-part. Suppose that there exists $s \in S$ such that $r = rank(s)$ and for all $S_{eq} \in EQ(S)$, if $s \in S_{eq}$, then $(x,y) \in AIn(S_{eq})$. $\sqsupseteq$ is a partial preorder, then $s \equiv s$. Therefore, there is a $S_{eq} \in EQ(S)$ such that $s \in S_{eq}$. Hence, $(x,y) \in AIn(S_{eq})$. It follows that $x \preceq y$, and $x \leq^p_s y$. Therefore, $r \in l((x,y))$. $\Box$

Proof of Proposition 5.8. I will show that for all $S_{mpc} \in MPC(S)$, $A^p(S_{mpc}) = A^p_{\mathcal{RANK}^{S_{mpc}}}((\preceq^p, l^p))$. Let $\mathcal{RANK}^{S_{mpc}} = \{r_1, ..., r_N\}$ such that $r_i > r_{i+1}$ for all $1 \leq^{\mathcal{N}} i <^{\mathcal{N}} N$. Also let $S_i = \{s \in S_{mpc} | r_i = rank(s)\}$. Then it suffices to show that for all $1 \leq^{\mathcal{N}} i <^{\mathcal{N}} N$, $O^{p,i}_{AIn}(S_{mpc}) = O^{p,i}_{\mathcal{RANK}^{S_{mpc}}}((\preceq^p, l^p))$. If $i = 1$, Then $O^{p,i}_{AIn}(S_{mpc}) = AIn(S_1) = \preceq^{p,A}_{r_1} = O^{p,i}_{\mathcal{RANK}^{S_{mpc}}}((\preceq^p, l^p))$. If $n >^{\mathcal{N}} 1$, suppose $O^{p,i-1}_{AIn}(S_{mpc}) = O^{p,i-1}_{\mathcal{RANK}^{S_{mpc}}}((\preceq^p, l^p))$. we must show for any $\leq \in \mathcal{R}$,

$$\begin{aligned} \leq \ = \ & O^{p,i-1}_{AIn}(S) \cup \\ & \{(w_a, w_b) : (w_b, w_a) \notin O^{p,i-1}_{AIn}(S) \land \\ & \qquad (w_a, w_b) \in AIn(S_i) \land \\ & \qquad (w_b, w_a) \notin \leq^+\}. \end{aligned}$$

iff

$$\begin{aligned} \leq \ = \ & O^{p,i-1}_{\mathcal{RANK}^{S_{mpc}}}((\preceq^p, l^p)) \cup \\ & \{(w_a, w_b) : (w_b, w_a) \notin O^{p,i-1}_{\mathcal{RANK}^{S_{mpc}}}((\preceq^p, l^p)) \land \\ & \qquad w_a \preceq^{b,A}_{r_i} w_b \land \\ & \qquad (w_b, w_a) \notin \leq^+\}. \end{aligned}$$

It suffices to show that $\preceq^{b,A}_{r_i} = AIn(S_i)$. Suppose that $x \preceq^{b,A}_{r_i} y$. Then there is some $s \in S$ such that $r_i = rank(s) \land (\forall S_{eq} \in EQ(S).s \in S_{eq} \Rightarrow (x,y) \in AIn(S_{eq})$. From $S_i = \{s \in S | r_i = rank(s)\}$, $S_i \in EQ(S)$. Therefore, $(x,y) \in AIn(S_i)$. Suppose that $(x,y) \in AIn(S_i)$. Then there is some $s \in S$ such that $r_i = rank(s) \land (\forall S_i \in EQ(S).s \in S_i \Rightarrow (x,y) \in AIn(S_i)$. Therefore, $x \preceq^{b,A}_{r_i} y$. $\Box$

Proof of Proposition 5.10. Let $S = \bigcup S_i$, $\oplus^b_p(\mathcal{P}^b_{\mathcal{A}}) = (\preceq^b, l^b)$, $\preceq^{b\prime} = ARf(S)$, and $l^{b\prime} : \preceq^{b\prime} \to \mathcal{RANK}$ such that $l^{b\prime}((x,y)) = max\{rank(s) : x \leq^b_s y, s \in S\}$. It suffices to show that $\preceq^b = \preceq^{b\prime}$ and $l^b = l^{b\prime}$.

Suppose that $x \preceq^b y$. I show that $x \preceq^{b\prime} y$, i.e., there exists $s \in S$ such that $x <^b_s y$ and, for every $s' \sqsupset s \in S$, $x \approx^b_{s'} y$, and that $l^{b\prime}((x,y)) = l^b((x,y))$. Since $x \preceq^b y$, there exists A_i and r such that $x \preceq^{b,A_i}_r y$ and, for every $A_j \in \mathcal{A}$ and $r' > r \in \mathcal{RANK}.x \sim^{b,A_j}_{r'} y$. Suppose that $\mathcal{RANK}^{(x,y)}_i = \{r_1, ..., r_K\}$ is the set of $r_k \in \mathcal{RANK}$ such that $x \preceq^{b,A_i}_{r_k} y$ and, for every $A_j \in \mathcal{A}$ and $r' > r_k \in \mathcal{RANK}.x \sim_{r'}^{b,A_j} y$. We will discuss each of $r_k \in \mathcal{RANK}_i^{(x,y)}$. Since $x \preceq_{r_k}^{b,A_i} y$, there exists $s \in S_i$ such that $x <_{s_k}^b y$, $rank(s_k) = r_k$, and, for every $s' \sqsupset s_k \in S_i$, $x \approx_{s'}^b y$. $S_i \subseteq S$, so there exists $s_k \in S$ such that $x <_{s_k}^b y$. Now suppose that s' is a maximal rank source of S with $x <_{s'}^b y$ or $y <_{s'}^b x$. Such an s' exists since $x <_{s_k}^b y$. Since $\sqsupseteq$ is a partial preorder, it suffices to show that $s' \not\sqsupset s_k$. Suppose $s' \in S_j$. Since $S_j \subseteq S$, s' is also a maximal rank source of S_j with $x <_{s'}^b y$ or $y <_{s'}^b x$, so $x \preceq_{rank(s')}^{b,A_j} y$ or $y \preceq_{rank(s')}^{b,A_j} x$. But since $x \preceq_{r_k}^{b,A_j} y$, $rank(s') \not> rank(s_k) = r_k$, so $s' \not\sqsupset s_k$. Furthermore, $l^{b\prime}((x,y)) = \{rank(s_1), ..., rank(s_K)\} = \{r_1, ..., r_K\} = l^b((x,y))$.

Now suppose that $x \preceq^{b\prime} y$, i.e., there exists A_i and r such that $x \preceq_r^{b,A_i} y$ and, for every $A_j \in \mathcal{A}$ and $r' > r \in \mathcal{RANK}$, $x \sim_{r'}^{b,A_j} y$, and that $l^b((x,y)) = l^{b\prime}((x,y))$. Since $x \preceq^{b\prime} y$, there exists $s \in S$ such that $x <_s^b y$ and, for every $s' \sqsupset s \in S.x \approx_{s'}^b y$. Suppose that $S^{(x,y)} = \{s_1, ..., s_K\}$ is the set of $s_k \in S$ such that $x <_{s_k}^b y$ and, for every $s' \sqsupset s_k \in S.x \approx_{s'}^b y$. We will discuss each of $s_k \in S^{(x,y)}$. Suppose $s_k \in S_i$. Since $S_i \subseteq S$, it is also the case that for every $s' \sqsupset s_k \in S_i$, $x \approx_{s'}^b y$, so $x \preceq_{rank(s_k)}^{b,A_i} y$. Now let A_j and r' be such that $x \preceq_{r'}^{b,A_j} y$ or $y \preceq_{r'}^{b,A_j} x$. It suffices to show that $r' \not> rank(s_k)$. By Proposition 5.2, there exists $s' \in S_j$ such that $x <_{s'}^b y$ or $y <_{s'}^b x$ and $rank(s') = r'$. But then $s' \not\sqsupset s_k$, so $rank(s') = r' \not> rank(s_k)$. Furthermore, $l^b((x,y)) = \{rank(s_1), ..., rank(s_K)\} = l^{b\prime}((x,y))$. $\square$

Proof of Proposition 5.12. Let $S = \bigcup S_i$. $\oplus_p^p(\mathcal{P}_{\mathcal{A}}^p) = (\preceq^p, l^p)$, $\preceq^{p\prime} = \cup_{S_{eq} \in EQ(S)} AIn(S_{eq})$ and $l^{p\prime} : \preceq^p \rightarrow \mathcal{RANK}$ such that $l^{p\prime}((x,y)) = \{rank(s) : x \leq_s^p y \wedge (\forall S_{eq} \in EQ(S).s \in S_{eq} \Rightarrow (x,y) \in AIn(S_{eq}), s \in S\}$. It suffices to show that $\preceq^p = \preceq^{p\prime}$ and $l^p = l^{p\prime}$.

Suppose that $x \preceq^p y$. I show that $x \preceq^{p\prime} y$, i.e., there exists $S_{eq} \in EQ(S)$ such that $(x,y) \in AIn(S_{eq})$ and $l^{p\prime}((x,y)) = l^p((x,y))$. Since $x \preceq^p y$, for some $r \in \mathcal{RANK}$, $x \preceq_r^{p,\mathcal{A}} y$, and then for some $r \in \mathcal{RANK}$, for all $A_i \in \mathcal{A}$, $x \preceq_r^{p,A_i} y$. Suppose that $\mathcal{RANK}_i^{(x,y)} = \{r_1, ..., r_K\}$ is the set of $r_k \in \mathcal{RANK}$ such that for all $A_i \in \mathcal{A}$, $x \preceq_{r_k}^{p,A_i} y$. We will discuss each of $r_k \in \mathcal{RANK}_i^{(x,y)}$. Then for some $r_k \in \mathcal{RANK}$, for all $A_i \in \mathcal{A}$, $x \preceq_{r_k}^{p,A_i} y$. Therefore, for some $r_k \in \mathcal{RANK}$, for all $A_i \in \mathcal{A}$, there is a $s \in S_i$ such that $r_k = rank(s)$ and $(\forall S_{eq} \in EQ(S_i).s \in S_{eq} \Rightarrow (x,y) \in AIn(S_{eq})$. We suppose that $S_i^k \subseteq S_i$ such that $s \in S_i^k$ and $S_i^k \in EQ(S_i)$. Also we suppose that $S^k = \bigcup S_i^k$. I will show that $S^k \in EQ(S)$. For all $A_i \in \mathcal{A}$, $S_i^k \subseteq S_i$, then $S^k = \bigcup S_i^k \subseteq \bigcup S_i = S$. For all $s, s' \in S^k$, $r_k = rank(s) = rank(s')$ is obvious. Let $s \in S$ such that $r_k = rank(s)$. Then for some i, $s \in S_i$. From $r_k = rank(s)$ and $s \in S_i$, $s \in S_i^k$. Hence, $s \in S^k$. Therefore, $S^k \in EQ(S)$. In addition, $AIn(S^k) = \cap_{s \in S^k} \leq_s^p = \cap_{s \in \cup S_i^k} \leq_s^p = \cap_{A_i \in \mathcal{A}} \cap_{s \in S_i^k} \leq_s^p = \cap_{A_i \in \mathcal{A}} AIn(S_i^k)$. Therefore, $(x,y) \in AIn(S_i^k)$. In addition, $l^p = \{r_1, ..., r_K\} = l^{p\prime}$.

Suppose that $x \preceq^{p\prime} y$. I show that $x \preceq^p y$, i.e., there exists $r \in \mathcal{RANK}$ such that $x \preceq_r^{p,\mathcal{A}} y$ and $l^p((x,y)) = l^{p\prime}((x,y))$. From $x \preceq^{p\prime} y$, there exists $S_{eq} \in EQ(S)$ such that $(x,y) \in AIn(S_{eq})$. Suppose that $S^{(x,y)} = \{s_1, ..., s_K\}$ is the set of $s_k \in S$ such that for all $S_{eq} \in EQ(S)$, if $s_k \in S_{eq}$, then $(x,y) \in AIn(S_{eq})$. I will discuss each of $s_k \in S^{(x,y)}$. Suppose $s_k \in S_i$, $S_i^k = \{s \in S_i | s \equiv s_k\}$, and $S^k = \{s \in S | s \equiv s_k\}$. $S_i^k \subseteq S^k$ is obvious. Therefore, $AIn(S^k) \subseteq AIn(S_i^k)$. Since $S^k \in EQ(S)$, $(x,y) \in AIn(S_i^k)$. Then $x \preceq y$, and if $r_k = rank(s_k)$, $r_k \in l((x,y))$. Hence $x \preceq_r^{p,A_i} y$. Since for all S_i, we can show $x \preceq_r^{p,A_i} y$,

$x \preceq_{r_k}^{p,\mathcal{A}} y$. In addition, $l^{p\prime} = \{r_1, ..., r_K\} = \{rank(s_1), ..., rank(s_K)\} = l^p$ $\Box$

A.5 Proofs of Chapter 6

Proof of Proposition 6.1. Suppose $\leq$ is the strict version of $\leq_A^b \otimes {}^b\leq_B^b$, From Proposition 3.16, $leq =<_A^b \otimes {}^b_< <_B^b$. It suffices to show that $\leq$ is the only one element of $PRF(<_A^b, <_B^b)$. At first, I use

$$\begin{aligned} \leq \ = \ & <_A^b \cup \\ & \{(w_a, w_b) : (w_a, w_b) \notin <_A^b \wedge \\ & \quad (w_b, w_a) \notin <_A^b \wedge \\ & \quad (w_a, w_b) \in <_B^b\} \end{aligned}$$

by Definition 4.2, and $\leq \in PRF(<_A^b, <_B^b)$ iff

$$\begin{aligned} & \leq \\ = \ & <_A^b \cup \\ & \{(w_a, w_b) : (w_a, w_b) \notin <_A^b \wedge \\ & \quad (w_b, w_a) \notin <_A^b \wedge \\ & \quad (w_a, w_b) \in <_B^b) \wedge \\ & \forall (w_b, w_a) \notin \leq^+\}, \end{aligned}$$

by Definition 2.3. It suffices to show that

$$\begin{aligned} & \leq \\ \subseteq \ & <_A^b \cup \\ & \{(w_a, w_b) : (w_a, w_b) \notin <_A^b \wedge \\ & \quad (w_b, w_a) \notin <_A^b \wedge \\ & \quad (w_a, w_b) \in <_B^b) \wedge \\ & \forall (w_b, w_a) \notin \leq^+\}, \end{aligned}$$

Suppose that $(w_a, w_b) \in \leq$ and $(w_b, w_a) \in \leq^+$. Then $(w_b, w_b) \in \leq^+ = \leq$, and it contradicts the fact that $\leq$ is irreflexive. Therefore, '$\subseteq$' is satisfied. $\Box$

Proof of Proposition 6.4. Suppose $\leq$ is the strict version of $\leq_A^b \otimes {}^b\leq_B^b$, From Proposition 3.16, $\leq = <_A^b \otimes {}^b_< <_B^b$. It suffices to show that $\leq$ is equal to $<_A^b \otimes^{gp} <_B^b$. Then

$$\begin{aligned} \leq \ = \ & <_A^b \cup \\ & \{(w_a, w_b) : (w_a, w_b) \notin <_A^b \wedge \\ & \quad (w_b, w_a) \notin <_A^b \wedge \\ & \quad (w_a, w_b) \in <_B^b\} \end{aligned}$$

by Definition 4.2, and

$$\begin{aligned} & \leq \\ = \ & <_A^b \cup \\ & \{(w_a, w_b) : (w_a, w_b) \notin <_A^b \wedge \\ & \quad (w_b, w_a) \notin <_A^b \wedge \\ & \quad (w_a, w_b) \in <_B^b)\}, \end{aligned}$$

by Definition 2.3. $\Box$

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Bibliography

[1] E.Adams, "Probability and the logic of conditionals," In J. Hintikka and P. Suppes, editors, Aspects of Inductive Logic. North-Holland, Amsterdam (1966).

[2] E.Adams, "The Logic of Conditionals," Reidel, Dordrecht (1975).

[3] C.E.Alchourrón, P.Gärdenfors, and D.Makinson,"On the logic of theory change: partial meet contraction and revision functions, "Journal of Symbolic Logic 50,pp.510-530 (1985).

[4] G.Antoniou, "Nonmonotonic Reasoning," Cambridge, MA: MIT Press (1997).

[5] K.J.Arrow, "Social Choice and Individual Values (2nd ed.)," New York: Wiley (1963).

[6] G.Birkhoff, "Lattice Theory," Volume 25 of Colloquim Publications, American Society (1948).

[7] R.Booth and J.B.Paris,"A Note on the Rational Closure of Knowledge Bases with Both Positive and Negative Knowledge," Journal of Logic,Language, and Information 7,pp.165-190 (1998).

[8] C.Boutilier, "Iterated revision of iterated revision and minimal change of conditional beliefs," Jounal of Philosophical Logic 25,pp.263-305 (1996).

[9] A.Darwiche and J.Pearl,"On the logic of iterated belief revision,"Artificial Intelligence 89,pp.1-29 (1997).

[10] J.Doyle,"A truth maintenance system,"Artificial Intelligence 12,pp.231-272 (1979).

[11] R.Fagin, J.D.Ullman, and M.Y.Vardi,"On the semantics of updates in databases," In Proceedings of Second ACM SIGACT-SIGMOD,pp.352-365 (1983).

[12] N.Friedman and J.Y.Halpern,"Belief Revision: A Critique," Journal of Logic,Language, and Information 8,pp.401-420 (1999).

[13] D.Gabbay,"Theoretical foundations for nonmonotonic reasoning in expert systems," In Logics and Models of Concurrent Systems, K.Apt, ed, Berlin: Springer-Verlag (1985).

[14] P.Gärdenfors,"Knowledge In Flux: Modeling the Dynamic of Epistemic States," Cambridge: The MIT Press (1988).

[15] P.Gärdenfors and D.Makinson,"Nonmonotonic Inference Based on Expectations,"Artificial Intelligence 65,pp.197-245 (1994).

[16] P.Gärdenfors and H.Rott,"Belief Revision," pp.35-132 in Handbook of Logic in Artificial Intelligence and Logic Programming Vol. 4:Epistemic and Temporal Reasoning, D.Gabbay, C.J.Hogger, and J.A.Robinson, ed, Oxford: Oxford University Press (1995).

[17] G.Grahne,"Updates and Counterfactual," Journal of Logic and Computation Vol. 8 No. 1, pp.87-117 (1998).

[18] A.Grove,"Two modelings for theory change," Journal of Philosophical Logic 17, pp.157-170 (1988).

[19] J.K.K.Hintikka, "Knowledge and Belief," Ithaca,N.Y.: Cornell University Press (1962).

[20] H.Katsuno and A.Mendelzon, "Propositional knowledge base revision and minimal change," Artificial Intelligence 52,pp.263-294 (1991).

[21] H.Katsuno and A.Mendelzon, "On the Difference between Updating a Knowledge Base and Revising it," in Belief Revision, Gärdenfors, P., ed., Cambridge: Cambridge University Press, pp183–203 (1992).

[22] S.Kraus, D.Lehmann and M.Magidor,"Nonmonotonic reasoning, preferential models and cumulative logics," Artificial Intelligence 41,pp.167-207 (1990).

[23] S.A.Kripke,"A Completeness Theorem in Modal Logic," Journal of Symbolic Logic 24, 1-14 (1959).

[24] D.Lehmann and M.Magidor,"What does a conditional knowledge base entail?," Artificial Intelligence 55,pp.1-60 (1992).

[25] I.Levi,"Subjunctives, dispositions and chances," Synthese 34,pp.423-455 (1977).

[26] D.Lehmann and M.Magidor,"What does a conditional knowledge base entail?," Artificial Intelligence 55,pp.1-60 (1992).

[27] D.Lewis, "Counterfactual," Basil Blackwell, Oxford (1973).

[28] P.Mainard-Reid II,"Pedigreed Belief Change," Ph.D. Thesis, Stanford University (2001).

[29] P.Mainard-Reid II and Y.Shoham,"Belief Fusion: Aggregating Pedigreed Belief States," Journal of Logic,Language, and Information 10,pp.183-209 (2001).

[30] D.Makinson,"General theory of Cumulative Inference," pp.1-18 in Nonmonotonic reasoning, LNAI346, M.Reinfrank et al., ed, Berlin: Springer-Verlag (1989).

[31] D.Makinson,"General Patterns in Nonmonotonic Reasoning," D.Gabbay, C.J.Hogger, and J.A.Robinson, editors, Handbook of Logic in Artificial Intelligence and Logic Programming Vol. 3 ,pp.35-110 (1994).

[32] D.Makinson and P.Gärdenfors,“Relations between the logic of theory change and nonmonotonic logic,” pp.185-205 in The Logic of Theory Change, LNAI465, A.Fuhrmann and M.Morreau, ed, Berlin: Springer-Verlag (1991).

[33] J.McCarthy, “Circumscription - A Form of Non-Monotonic Reasoning,” Artificial Intelligence 13,pp.27-39 (1980).

[34] D.McDermott and J.Doyle, “Non-Monotonic logic I,” Artificial Intelligence 13,pp.41-72 (1980).

[35] R.C.Moore,“Semantical Considerations on Nonmonotonic Logic,” Artificial Intelligence 25,pp.75-94(1985).

[36] D.Poole,“A logical framework for default reasoning,” Artificial Intelligence 36,pp.27-47 (1988).

[37] F.P.Ramsey,“General propositions and causality ,” pp.52-88 in Foundations of Mathematics and Other Logical Essays, New York: Routledge & Kegan Paul (1931).

[38] R.Reiter,“A Logic of Default Reasoning,” Artificial Intelligence 13,pp.81-132 (1980).

[39] H.Rott, “Change, Choice and Inference : A Study of Belief Revision and Nonmonotonic Reasoning,” GB: Oxford (2001).

[40] H.Rott,“Coherence and Conservatism in the Dynamics of Belief II: Iterated Belief Change withou Dispositional Coherence,” Journal of Logic and Computation Vol. 13 No. 1, pp.111-145 (2003).

[41] A.Sen, “Social Choice Theory,” in Handbook of Mathematical Economics, Vol. III., Arrow and Intiligator, ed, North-Holland (1986).

[42] Y.Shoham, “A semantic approach to non-monotonic logics,” In Proceedings of the Tenth International Joint Conference on Artificial Intelligence (IJCAI) (1987).

[43] Y.Shoham, “Reasoning About Change,” MIT Press, Cambridge, USA (1988).

[44] Y.Suzuki and S.Tojo, “Fusion of Pedigreed Preferential Relations,” In 10th Workshop on Logic, Language, Information and Computation (WoLLIC ’03),(2003).

[45] D.Touretzky, “The Mathematics of Inheritance Systems,” Pitman and Morgan Kaufmann, London and Los Altos California (1986).

Publications

[1] Y.Suzuki and S.Tojo, "Fusion of Pedigreed Preferential Relations," In 10th Workshop on Logic, Language, Information and Computation (WoLLIC '03),(2003).

[2] Y.Suzuki and S.Tojo, "Fusion of Pedigreed Preferential Relations as Beliefs," Transactions of Japanese Society for Artificial Intelligence (In Japanese), (2004).

[3] Y.Suzuki and S.Tojo, "Fusion of Pedigreed Preferential Relations," THEORETICAL COMPUTER SCIENCE (guest edited by R. de Queiroz and D. Kozen). (Under Submission)