Advances in Complex Systems

NECESSARY BACKBONE OF SUPER-HIGHWAYS FOR TRANSPORT ON GEOGRAPHICAL COMPLEX NETWORKS

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Many real networks have a common topological structure called scale-free (SF) that follows a power law degree distribution, and are embedded on an almost planar space which is suitable for wireless communication. However, the geographical constraints on local cycles cause more vulnerable connectivity against node removals, whose tolerance is reduced from the theoretical prediction under the assumption of uncorrelated locally tree-like structure. We consider a realistic generation of geographical networks with the SF property, and show the significant improvement of the robustness by adding a small fraction of shortcuts between randomly chosen nodes. Moreover, we quantitatively investigate the contribution of shortcuts to transport many packets on the shortest path for the spatially heterogeneous amount of communication requests. Such a shortcut strategy preserves topological properties and a backbone naturally emerges bridging isolated clusters.

Keywords: Robustness; shortcuts; planar triangulation; ad hoc networks; centrality of traffic flow.

1. Introduction

As the complexity of interactive human activities increases and becomes very important for socio-economical infrastructures, we require more robust communication systems against natural disasters and cyber-terrorism. In addition, we expect to realize efficient delivery of information packets on dynamically constructed future ad hoc networks through wireless communications, which adaptively works in an emergent situation. However, a universal mechanism to design both robust and efficient networks has nor been comprehended. For large complex networks, a challenging approach has been initiated before this decade.

With the groundbreaking science of complex networks [1, 2], common topological characteristics have been found in many real networks such as in the Internet, the World-Wide-Web, social acquaintance relationships, and biological metabolic systems. One of such characteristics is the small-world (SW) phenomenon according to which the length of a path between any two nodes becomes short as $O(\log N)$, even for a huge network size N, where N is the total number of nodes. Another characteristic is the scale-free (SF) structure that follows a power law degree distribution $P(k) \sim k^{-\gamma}$, $2 < \gamma < 3$. In other words, a SF network consists of many nodes with low degrees and a few hubs with high degrees. Since a path through hubs tends to be short for many possibilities to select proper mediators or a terminal node, the SW property is also maintained in the SF structures, whose fundamental generation rule is a preferential attachment with the grown of the network in a distributed manner. Thus, self-organized real complex networks have quite different topological properties from the conventional network models of a regular lattice and a random graph. In one of the most shocking finds, it has been shown that the tolerance of connectivity is extremely vulnerable to intentional attacks on hubs in SF networks [3, 4].

Moreover, many infrastructures such as power grids, airline networks, and the Internet, are **embedded in a metric space**, and **long-range links are restricted** [5, 6] for economical reasons. Although geographical SF network models have been gradually considered, unfortunately, it has been suggested [7, 8, 9] that **the tolerance of connectivity is more reduced in a spatial construction** than the theoretical prediction in large networks under the assumption of uncorrelated locally tree-like structure [10, 11, 12]. In an approximative analysis for the effect of cycles of a length [7, 8], the percolation threshold to maintain the whole connectivity increases (for the breaking of the connected component, equivalently, the critical fraction of randomly removed nodes decreases). Triangular cycles tend to be particularly constructed by a geographical constraint on local connections, and this case of the smallest-order of a cycle counted by hops is the worst for decreasing the fault tolerance in the approximative analysis.

If the geographically constrained links are randomly rewired, the random null model has a similar robustness to that in the theoretical prediction. Indeed, it has been numerically shown that the robustness can be significantly improved by fully random rewirings under the same degree distributions [9] in typical geographical networks: Delaunay triangulation (DT) [17] in computer science, random Apollonian (RA) [18, 19], and Delaunay-like scale-free (DLSF) networks [9, 13] in complex network science. The degree distributions follow a power law in RA networks, lognormal in DTs, and the intermediate power law with an exponential cutoff in DLSF networks, as shown later. Although the types of degree distribution are of primary importance, a something of randomness without geographical constraints may become more dominant on the robustness. Since a node tends to connect spatially further nodes by rewirings, the connectivity is extended more globally and tightly. Therefore, from a similar reason, we expect **the shortcut effect on the improvement of robustness** in the geographical SF networks. Adding shortcuts is practically more natural than rewirings, because the already constructed links are not wastefully discarded. Note that adding shortcuts by connecting pairs of nodes selected at random corresponds to complement the planar RA, DLSF, and DT graphs with an Erdös-Rényi graph [14] embedded in the plane. We also note that the triangle connections in the geographical networks contribute to increasing the clustering coefficient, therefore the vulnerability is also related to the fact that the controlled clustering in random SF networks can strongly affect some percolation properties [15, 16]. However, it is intractable beyond our current scope for investigating fundamental traffic backbones to comprehend the individual and mutual effects of topological clustering and geographical cycles on the robustness.

On the other hand, for traffic flow, no packets are transferred between isolated areas by failures and/or intentional attacks. Thus, it is crucial to find such **a necessary backbone for maintaining the whole connectivity**. However, it has been known that the optimization is generally difficult, since the minimum dominating set problem is NP-hard [20] in algorithm theory. This criterion aims to reduce the number of base-stations with high load on which many packets are concentrated. Instead of the minimum, we consider a candidate of the necessary backbone for transport from the viewpoint of complex network science, e.g. by using a percolation analysis and the centrality of traffic flow.

The organization of this paper is as follows. In Section 2, we explain three properties: the degree distribution, the planarity suitable for a routing of packets without global information, and the spatial distribution of nodes in which the mixing of dense and sparse areas emerges from an iterative network construction. It is remarkable that the geographical networks belong to a family of SF networks designed on a space. In Section 3, we numerically investigate the robustness of connectivity, especially against targeted attacks on hubs, and propose a practical strategy to significantly improve it by adding a small fraction of shortcuts between randomly chosen nodes. For the delivery of packets, such shortcuts are expected to have high throughput, just like superhighways on overhead bridges between nodes which are embedded on a planar space in the original network. Thus, we consider some traffic properties measured by the frequency of used shortcuts on the shortest paths, and the link centrality defined by the number of passing through them. In the serious case of the breaking of the giant component against the attacks, we show that the shortcuts contribute toward accelerating traffic flow particularly between isolated areas by the attacks on the original one. In Section 4, we summarize these results and briefly discuss further research.

2. Geographical network models

Planar networks without crossing of links are suitable for efficient geographical routings in wired and/or wireless connections, since we can easily find the shortest distance path from a set of edges on the faces that intersect the straight line between source and terminal nodes. In computer science, online face routing algorithms [21] that **guarantee delivery of messages using only local information** about the positions of the source, the terminal, and the adjacent nodes to a current node are wellknown. In order to apply these good properties to SF networks, we consider two models called Delaunay triangulation (DT) and random Apollonian (RA) network.

A DT yields the optimal planar triangulation according to some geometric criteria [17], and the shortest path length is bounded by a constant factor to the direct Euclidean distance between any source and terminal nodes [22]. While a RA network belongs to both the SF and planar [18, 19], long-range links inevitably appear near the edges of an initial polygon. To reduce the long-range links, a model of Delaunay-like scale-free (DLSF) network has been proposed [9, 13] as a combination of DT and RA networks.

The generation procedures of these planar networks are as follows.

Step 0: Set an initial planar triangulation on a space.

Step 1: At each time step, choose a triangle at random and add a new node at the barycenter. For each model, different linking processes are applied.

- **RA:** Then, connect the new node to three nodes of the chosen triangle [18, 19], as shown in Fig. 1.
- **DLSF:** Moreover, by iteratively applying diagonal flips [17], connect the new node to the nearest node within a radius defined by the distance between the new node and the nearest node of the chosen triangle [9, 13], as shown in Fig. 2.
  If there is no node within the radius, this flipping is skipped, therefore the new node is connected to the three nodes.
- **DT:** After the temporal subdivision of the chosen triangle as shown in Fig. 1, expanding from the neighboring quadrilaterals, diagonal flips are globally applied to any pair of triangles until the minimum angle is not increased by exchanging diagonal links in a quadrilateral.

Step 2: The above process is repeated until the required size N is reached.

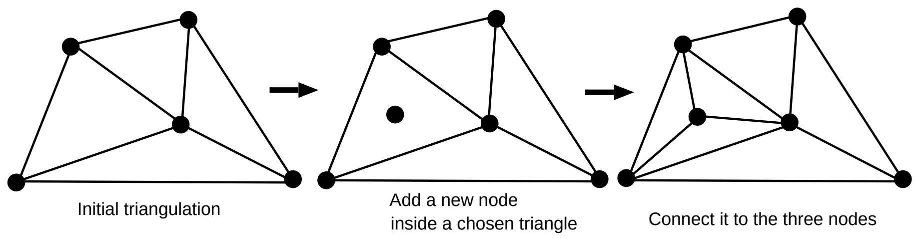


Fig. 1. Basic procedure of the subdivision of a chosen triangle in a RA network.

Figure 3 shows the typical structure of each network, whose spatial arrangement emerges with the **mixing of dense and sparse areas similar to a population density**. In other spatial models of SF networks [23], nodes are distributed uniformly at random or restricted on a two-dimensional lattice, therefore they are not realistic in the positioning of nodes.

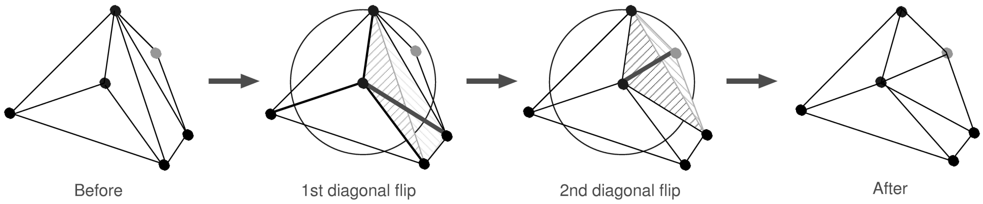


Fig. 2. Linking procedures in a DLSF network. The long-range links (two solid lines at left) are exchanged with the crossing ones (fat lines) in the shading triangles by diagonal flips in the middles. This results in five new triangles with contours at left.

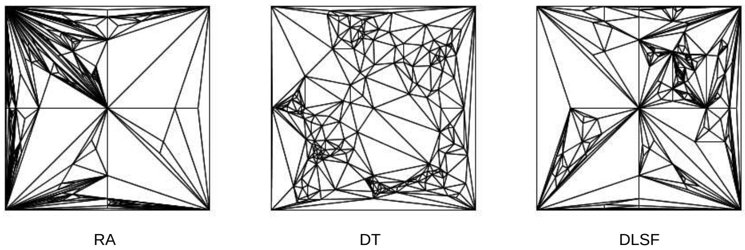


Fig. 3. Example of geographical networks grown from an initial triangulation of a square to a configuration with the mixing of dense and sparse areas similar to a population density. Note that the four corners and the center points tend to be hubs in RA and DLSF networks.

Figure 4 shows that the degree distributions follow a power law with the exponent nearly 3 in RA networks, a log-normal in DTs, and a power law with an exponential cutoff in DLSF networks [9]. Such a lognormal distribution has a unimodal shape similar to the one in classical Erdös-Rényi random networks, and a cutoff is rather natural in real networks [24]. The power law in RA networks is analytically derived [19] while the cutoff in DLSF networks is approximative [9]. Although the log-normal distribution is not verified in DTs, it numerically fits well. The estimated parameters are shown in Table 1.

model	estimated function	parameters
RA	$P(k) \sim k^{-\gamma_{RA}}$	$\gamma_{RA} = 2.79$
DT	$P(k) \sim \exp\left(-\frac{(\ln k - \mu)^2}{2\sigma^2}\right)$	$\mu = 1.533$, $\sigma = 0.32$
DLSF	$P(k) \sim k^{-\gamma} \exp(-ak)$	$\gamma = 1.37$, $a = 0.09$

Table 1. Estimated parameters for the degree distribution $P(k)$ for each network.

On the other hand, we have found [9, 13] that the original RA and DLSF net-

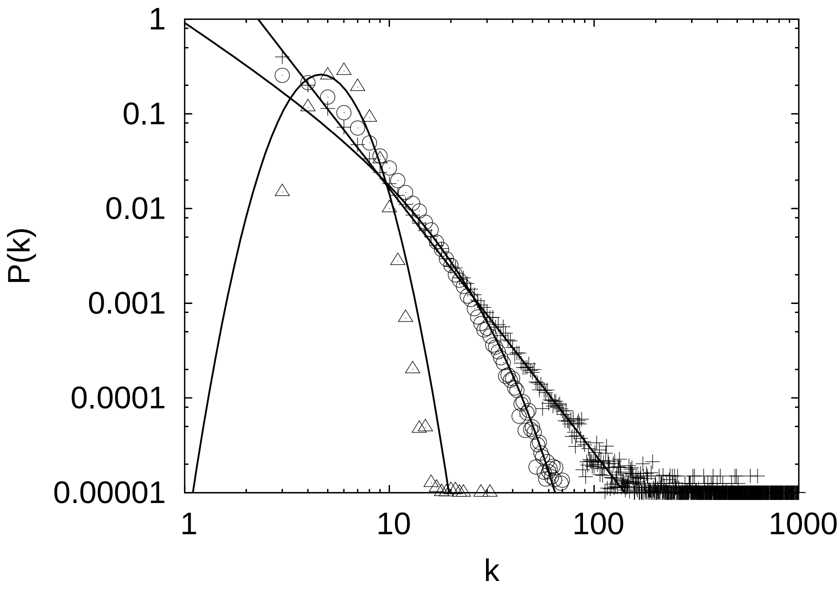


Fig. 4. Degree distribution $P(k)$ in each original network for a total number of nodes $N = 10^5$ and the links $M = 3(N - 5) + 8 = 299993$ generated from the initial triangulation of a square. The plus, circle, and triangle marks correspond to RA, DLSF, and DT, respectively. The solid lines are estimated by the nonlinear mean-square-error method for the functions in Table 1. These results are obtained over 100 realizations.

works without shortcuts are vulnerable because of double constraints of the planarity and of geographical distances on the linkings in the SF structure, but DTs are robust. Figure 5 shows typical examples of the damage by targeted attacks on hubs. Each network is generated from the initial planar triangulation in Step 0, which consists of the four-corner nodes of a square and of the center node connected to them. While RA and DLSF networks are drastically fragmented by removing the five nodes as hubs, in a DT, since there are no such large degree nodes, the damage in keeping the connected component after the attacks is weaker.

To improve the robustness of connectivity, just like superhighways on overhead bridges, we add shortcuts between randomly chosen pares of nodes excluding self-loops and multi-links after constructing the planar networks. For adding shortcuts, the routing algorithm can be extended [13] from that in [21]. In addition, we confirm that the degree distributions endamage only a small deviation from the original ones for added shortcut rates up to 30% of the total number of links, as shown in Fig 6(a)-(c). Thus, we can compare the effect of shortcuts on the robustness in the geographical networks under quasi-invariant degree distributions.

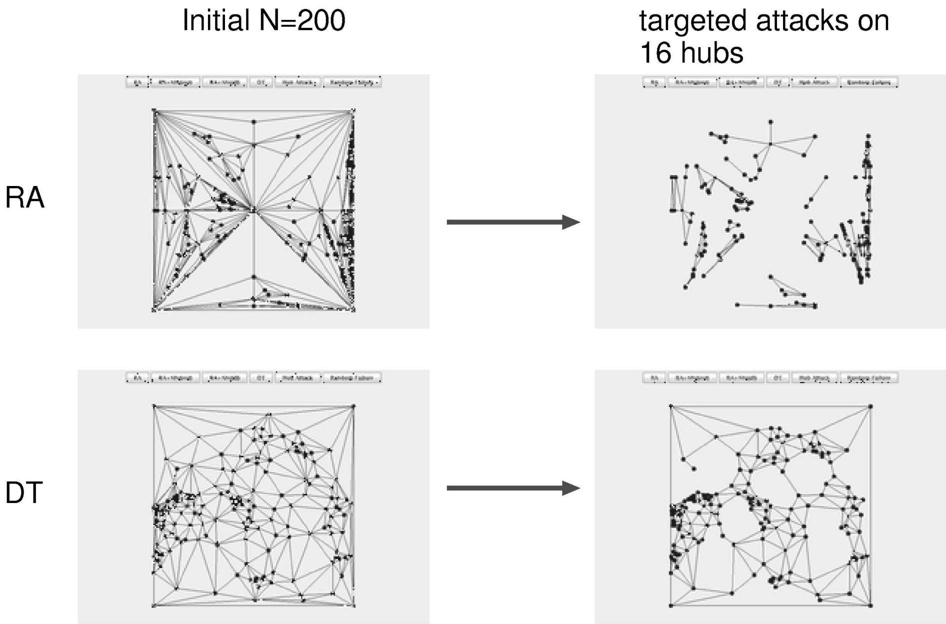


Fig. 5. Examples of isolated clusters which are not able to communicate with each other in a RA network, while the giant component still remains in a DT.

3. Improved robustness by adding shortcuts

3.1. *Simulation results*

The fault tolerance and the attack vulnerability are known as the typical properties of SF networks [3, 4, 12], which are further affected by geographical constraints [7, 8, 9, 13]. We investigate the tolerance of connectivity in the giant component (GC) of the geographical networks with shortcuts compared with that of the original networks without shortcuts. The size S of the GC and the average size $\langle s \rangle$ of isolated clusters are numerically obtained over 100 realizations for each network model for $N = 10^5$.

Figure 7 shows the effect of shortcuts on the robustness against targeted attacks on hubs whose nodes are removed in decreasing order of degree. The breaking points of the GC corresponding to the peaks of $\langle s \rangle$ are shifted to the righter as the shortcut rate increases. Around a shortcut rate of 10%, the extremely vulnerable RA and DLSF networks improve up to a similar level as the DTs on results of robustness. In the original DTs without shortcuts, the most robust connectivity may be related to the fact that the tolerance becomes higher when enhancing the cutoff under the same average degree $\langle k \rangle$ and size N, e.g. in evolving networks with a connection rule of local preferential attachment [25]. By adding shortcuts around 10% under the quasi- invariant degree distributions, the robustness against intentional attacks

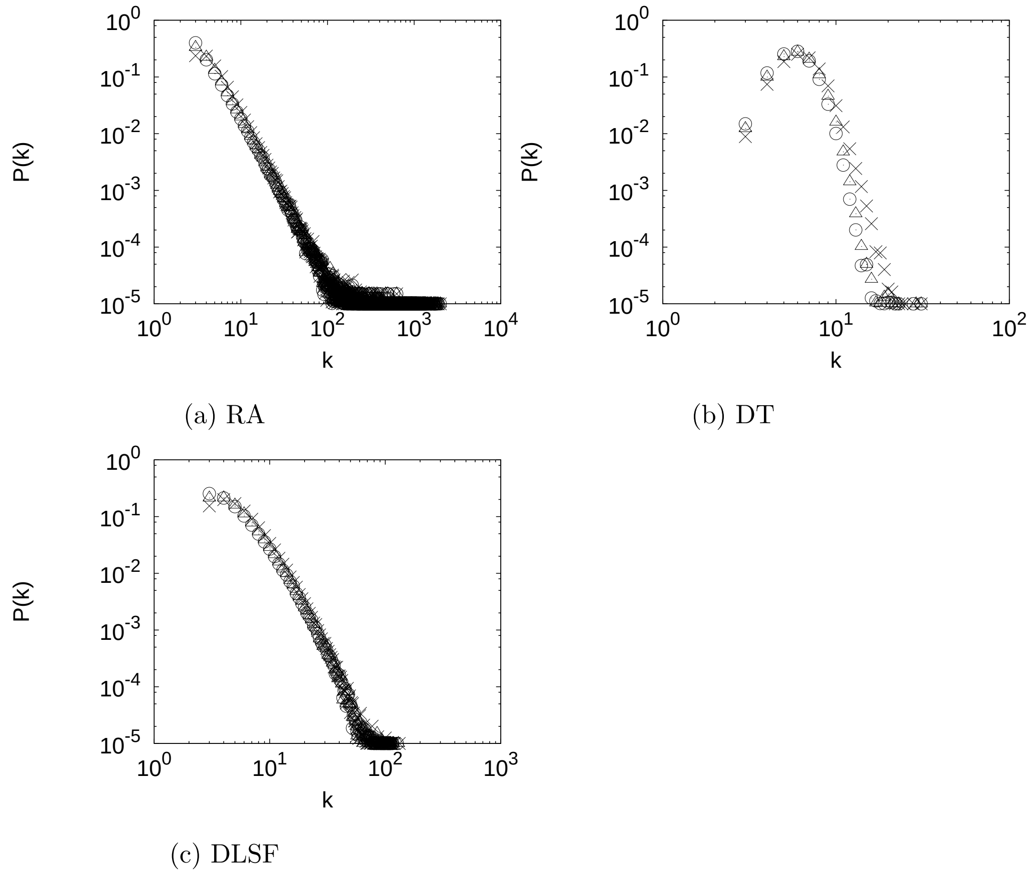


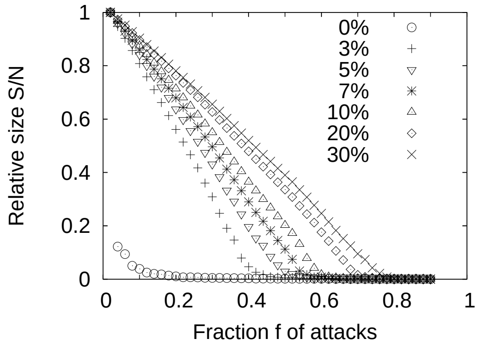
Fig. 6. Quasi-invariant degree distributions in the geographical networks with shortcuts. Each shortcut rate is denoted by different marks: circle(0%), upper-triangle(10%), and cross(30%). These results are obtained over 100 realizations of each original geographical network $\times$ 100 samples of random shortcuts for $N = 10^5$.

can be considerably improved up to a similar level as the fully rewired networks as a random null model by ignoring the geographical constraints [9], because the shortcuts consist of a necessary backbone to connect local areas, as illustrated in Fig. 8. More quantitative discussions are proceeded in the next subsection.

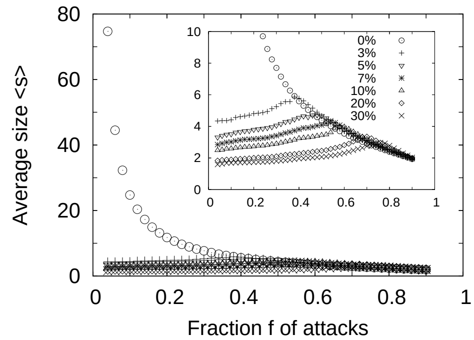
3.2. *Frequency and centrality of superhighways*

We investigate how many packets are transported passing through shortcut links on the shortest path. In order to study a necessary backbone, we discuss the recovery by adding shortcuts from the most serious case at the breaking of the GC in the original networks.

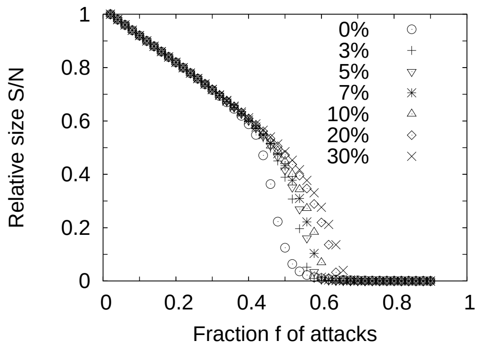
We define the cumulative distribution $P(u)$ for the frequency (or used rate) of



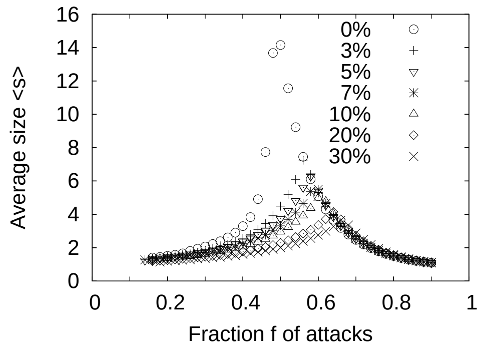
(a) S/N in RA



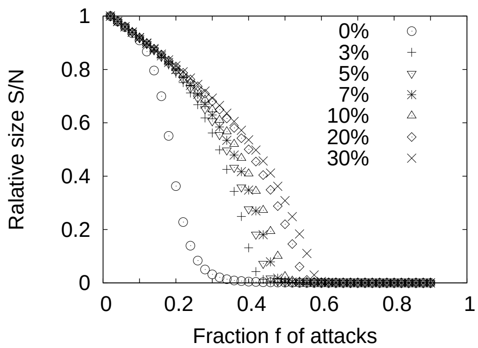
(d) $\langle s \rangle$ in RA



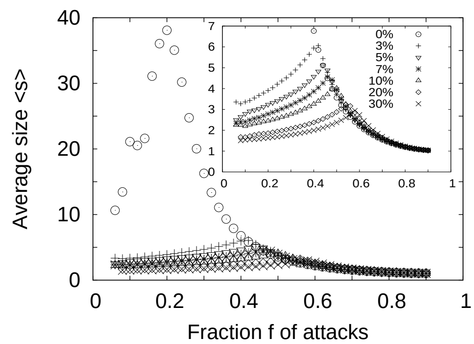
(b) S/N in DT



(e) $\langle s \rangle$ in DT



(c) S/N in DLSF



(f) $\langle s \rangle$ in DLSF

Fig. 7. Relative size S/N of the GC in (a)-(c), and average size $\langle s \rangle$ of isolated clusters except of the GC in (d)-(f) against attacks. Each shortcut rate is marked in the legend. The inset show the peaks magnified by using a scale of the vertical axis. The robustness increase for larger shortcut rates.

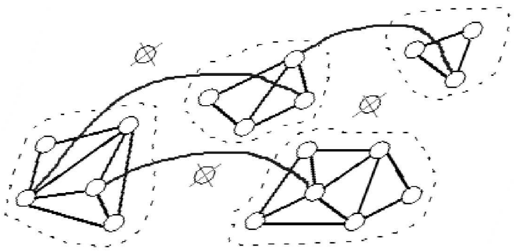


Fig. 8. Backbone of superhighways: The robustness of connectivity is increased by adding shortcuts, since shortcut links bridge isolated faces by attacks in the original planar network. The cross mark denotes the removed nodes.

superhighways

$$u \stackrel{\text{def}}{=} \frac{l_{super}}{l_{short}},$$

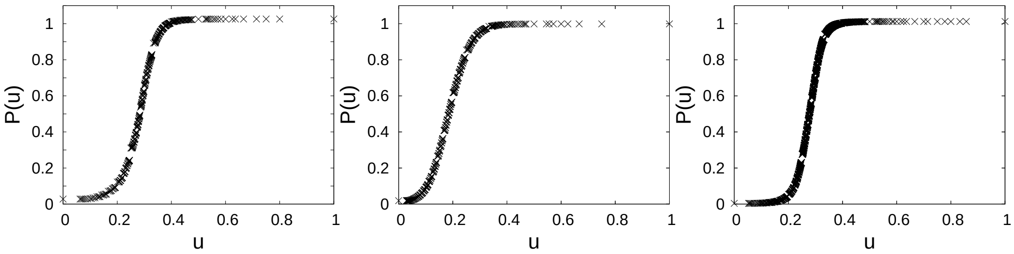
where l_{super} is the number of links belonging to shortcuts as superhighways in the shortest path of length l_{short} counted by hops. In other words, the frequency u refers to the occupation rate of shortcuts in the shortest path. This definition is given by a modification from a path based on the MST [26] to that on the shortest. In Fig. 9(a)-(c), the values of u concentrate on the range near the rapid increasing. These results show that shortcuts are frequently used on the shortest path in spite of the existing in only 10 % of the total links. The average $\langle u \rangle$ is about $2 \sim 3$ times larger than the shortcut rate q_s in the ordering of DT, DLSF, and RA networks.

Without loss of generality, we assume that both source and terminal nodes are chosen from all nodes uniformly at random. Thus, the packet generation and receiving seem to be homogeneous because of the equal selection probability for all nodes. However, if we consider the spatial distributions, as the number of generated or received packets per unit area, they are remarkably heterogeneous according to the node densities (remember Fig. 3). This situation depending on human activities is realistic, since packets are usually more generated and received as the population is larger in a dense area.

Let us return to the study of superhighways. For both shortcuts and the other planar links, we consider the normalized betweenness centrality of link l [27],

$$B_l \stackrel{\text{def}}{=} \frac{2}{(N-2)(N-3)} \sum_{k<j \neq l1, l2} \frac{b_k^j(l)}{b_k^j},$$

where $l1$ and $l2$ denote the end nodes of l, b_k^j is the number of shortest paths between nodes k and j, and $b_k^j(l)$ is the number of such paths passing through the link l. The centrality B_l corresponds to the load or throughput of link l for transferring



(a) RA (b) DT (c) DLSF

Fig. 9. Cumulative distribution $P(u)$ for the frequency u of a superhighway. The average $\langle u \rangle$ is (a):0.2968, (b):0.1865, (c):0.2753 for a shortcut rate $q_s = 0.1$ in each geographical network with shortcuts after intentional attacks on hubs at the breaking of the GC in the original network without shortcuts. Note that the accumulation absorbs noisy fluctuation of the distribution itself.

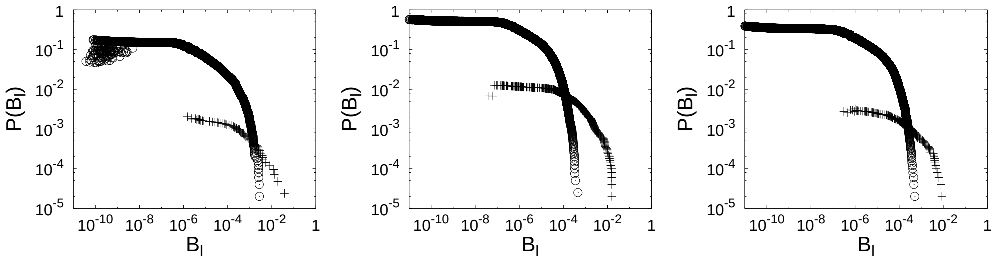
packets. As shown in the right-hand side of Fig. 10(a)-(c), there are relatively more shortcuts with high centralities $B_l > 10^{-4}$ than planar links. Two distributions for planar and shortcut links marked by circles and pluses are normalized by their sum to compare the frequency in the total links. We should note that each link is either embedded one on the planar space or added shortcut. On the shortest path, shortcuts act as necessary bridges between isolated clusters, however planar links are used on the paths only in each cluster (remember Fig. 8). Therefore, it is natural that some planar links have high centralities, while the majority of other planar links have low centralities. We confirm that these results do not depend on the initial configuration of networks. Similar results are obtained for the following normalized effective betweenness [28] in which $l1$ and $l2$ are included as the source and terminal nodes k, j;

$$\hat{B}_l \stackrel{\text{def}}{=} \frac{2}{N(N-1)} \sum_{k<j} \frac{b_k^j(l)}{b_k^j}.$$

Thus, we obtaine a relatively high contribution of shortcuts to passing many packets in terms of the frequency u and the centrality B_l on the shortest paths.

4. Conclusion

Regarding for the vulnerability caused by spatial constraints [7, 8], we have improved the tolerance of connectivity in geographical networks by adding only about 10 % of shortcuts between randomly chosen nodes. In particular, we have studied a class of planar networks called RA, DT, and DLSF [13], which are practically suitable for wireless communications without interference and for distributed routing algorithms with only local information. We also have considered a realistic situation of the spatially heterogeneous generation and receiving of packets which depend on a naturally emerged density of nodes through the growth of geographical networks.



(a) RA (b) DT (c) DLSF

Fig. 10. Distribution $P(B_l)$ of betweenness centrality B_l of planar (marked by circles) and shortcut (marked by pluses) links in each geographical network with shortcuts at the rate $q_s = 0.1$ after intentional attacks on hubs at the breaking of the GC in the original network without shortcuts. A high B_l concentrates on the shortcuts in the right-hand side of each figure, while the cloud of points for low values of B_l in the left-hand side of (a) shows the outliers away from the smooth curve, though the reason why they occur is unknown.

Moreover, considering transport properties, the frequency (used rate) and the link centrality on the shortest path have been investigated in comparison with shortcuts and the other planar links. Our results have shown that the shortcut is not only effective to avoid a serious breaking due to intentional attacks on hubs, but also highly contributes as a necessary backbone for the delivery of many packets such as superhighways to bridge isolated clusters. It should be noted that these quantitative measures are useful to extract important parts for traffic flow.

As a related mathematical topic, triangular embeddings on a hyperbolic surface by the elementary moves of diagonal flip and node insertion/removal have been discussed [29]. We remark that DT, RA, and DLSF networks are special cases of a triangulation embedded on a sphere surface, and the modified networks, by adding shortcuts, can be embedded on a hyperbolic surface with a genus more than one which corresponds to the number of bundles emanated from a sphere. To characterize our networks in a hierarchical ensemble classification of network-complexity may be extended in this direction of further research.

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